Issues in IFSs and GNs, Vol. 12, 2015/2016, 53–60

On Intuitionistic Fuzzy Negations, Law for Excluded Middle and De Morgan's Laws

Krassimir T. Atanassov[1,2], Nora A. Angelova[1]

[1] Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria
e-mails: krat@bas.bg, nora.angelova@biomed.bas.bg
[2] Intelligent Systems Laboratory
Prof. Asen Zlatarov University
Bougas 8010, Bulgaria

Abstract: The list of all intuitionistic fuzzy negations, introduced by the moment, is given. For them, the Law for Excluded Middle and the De Morgan's Laws are checked, and the lists of the negations satisfying both lawsare given.
Keywords and phrases: Intuitionistic fuzzy logic, intuitionistic fuzzy negation, Law for Excluded Middle, De Morgan's Laws.
2000 Mathematics Subject Classification: 03E72.

1 Introduction

In the paper, following and extending [3], we show which intuitionistic fuzzy negations satisfy the Law for Excluded Middle (LEM) and De Morgan's Laws (DMLs).

In the intuitionistic fuzzy logic, to the proposition p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p) + \nu(p) \leq 1.$$

Let this assignment be provided by an evaluation function V defined over a set of propositions, that here and below we will denote by $\mathcal{S}$, in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : S \to [0,1] \times [0,1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$ – a set of propositions.

Let for every real number x:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}.$$

For the needs of the discussion below, we define the notion of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [4, 5]) by:

x is an IFT if and only if for $V(x) = \langle a, b \rangle$ it holds that $a \geq b$,

while

x is a (classical) tautology if and only if $a = 1$ and $b = 0$.

The existing by the moment intuitionistic fuzzy implicatins generate the intuitionistic fuzzy negations using the formula

$$\neg \langle a, b \rangle = \langle a, b \rangle \to \langle 0, 1 \rangle,$$

where $\to$ is each one of the implications. All different negations are shown in the following Table.

Table: List of the intuitionistic fuzzy implications

$\neg_1$	$\langle b, a \rangle$
$\neg_2$	$\langle \overline{sg}(a), \mathrm{sg}(a) \rangle$
$\neg_3$	$\langle b, a.b + a^2 \rangle$
$\neg_4$	$\langle b, 1 - b \rangle$
$\neg_5$	$\langle \overline{sg}(1-b), \mathrm{sg}(1-b) \rangle$
$\neg_6$	$\langle \overline{sg}(1-b), \mathrm{sg}(a) \rangle$
$\neg_7$	$\langle \overline{sg}(1-b), a \rangle$
$\neg_8$	$\langle 1 - a, a \rangle$
$\neg_9$	$\langle \overline{sg}(a), a \rangle$
$\neg_{10}$	$\langle \overline{sg}(1-b), 1 - b \rangle$
$\neg_{11}$	$\langle \mathrm{sg}(b), \overline{sg}(b) \rangle$
$\neg_{12}$	$\langle b.(b+a), \min(1, a.(b^2 + a + b.a)) \rangle$
$\neg_{13}$	$\langle \mathrm{sg}(1-a), \overline{sg}(1-a) \rangle$
$\neg_{14}$	$\langle \mathrm{sg}(b), \overline{sg}(1-a) \rangle$

$\neg_{15}$	$\langle \overline{sg}(1-b), \overline{sg}(1-a) \rangle$
$\neg_{16}$	$\langle \overline{sg}(a), \overline{sg}(1-a) \rangle$
$\neg_{17}$	$\langle \overline{sg}(1-b), \overline{sg}(b) \rangle$
$\neg_{18}$	$\langle b.\mathrm{sg}(a), a.\mathrm{sg}(b) \rangle$
$\neg_{19}$	$\langle b.\mathrm{sg}(a), 0 \rangle$
$\neg_{20}$	$\langle b, 0 \rangle$
$\neg_{21}$	$\langle \min(1-a, \mathrm{sg}(a)), \min(a, \mathrm{sg}(1-a)) \rangle$
$\neg_{22}$	$\langle \min(1-a, \mathrm{sg}(a)), 0 \rangle$
$\neg_{23}$	$\langle 1-a, 0 \rangle$
$\neg_{24}$	$\langle \min(b, \mathrm{sg}(1-b)), \min(1-b, \mathrm{sg}(b)) \rangle$
$\neg_{25}$	$\langle \min(b, \mathrm{sg}(1-b)), 0 \rangle$
$\neg_{26}$	$\langle b, a.b + \overline{sg}(1-a) \rangle$
$\neg_{27}$	$\langle 1-a, a.(1-a) + \overline{sg}(1-a) \rangle$
$\neg_{28}$	$\langle b, (1-b).b + \overline{sg}(b) \rangle$
$\neg_{29}$	$\langle \max(0, b.a + \overline{sg}(1-b)), \min(1, a.(b.a + \overline{sg}(1-b)) + \overline{sg}(1-a)) \rangle$
$\neg_{30}$	$\langle a.b,\ a.(a.b + \overline{sg}(1-b)) + \overline{sg}(1-a) \rangle$
$\neg_{31}$	$\langle \max(0, (1-a).a + \overline{sg}(a)), \min(1, a.((1-a).a + \overline{sg}(a))$ $+ \overline{sg}(1-a)) \rangle$
$\neg_{32}$	$\langle (1-a).a,\ a.((1-a).a + \overline{sg}(a)) + \overline{sg}(1-a) \rangle$
$\neg_{33}$	$\langle b.(1-b) + \overline{sg}(1-b), (1-b).(b.(1-b) + \overline{sg}(1-b)) + \overline{sg}(b) \rangle$
$\neg_{34}$	$\langle b.(1-b),\ (1-b).(b.(1-b) + \overline{sg}(1-b)) + \overline{sg}(b) \rangle$
$\neg_{35}$	$\langle \frac{b}{2}, \frac{1+a}{2} \rangle$
$\neg_{36}$	$\langle \frac{b}{3}, \frac{2+a}{3} \rangle$
$\neg_{37}$	$\langle \frac{2b}{3}, \frac{2a+1}{3} \rangle$
$\neg_{38}$	$\langle \frac{1-a}{3}, \frac{2+a}{3} \rangle$
$\neg_{39}$	$\langle \frac{b}{3}, \frac{3-b}{3} \rangle$
$\neg_{40}$	$\langle \frac{2-2a}{3}, \frac{1+2a}{3} \rangle$
$\neg_{41}$	$\langle \frac{2b}{3}, \frac{3-2b}{3} \rangle$
$\neg_{42,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{43,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\neg_{44,\alpha,\beta}$	$\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, b+\varepsilon), \max(0, a-\eta) \rangle$, where $\varepsilon, \eta \in [0,1]$ and $\varepsilon \leq \eta < 1$
$\neg_{46,\lambda}$	$\langle \frac{\lambda-a}{2\lambda}, \frac{a+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{47,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{1-b+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{48,\gamma}$	$\langle \frac{1-a+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$

$\neg_{49,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{1-b+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\neg_{50,\alpha,\beta}$	$\langle \frac{b-1+\alpha}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1,\alpha]$
$\neg_{51,\alpha,\beta}$	$\langle \frac{b-1+\alpha}{\alpha+\beta}, \frac{1-b+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1,\alpha]$
$\neg_{52}$	$\langle \overline{sg}(a) + \text{sg}(a)b, a \rangle$
$\neg_{53}$	$\langle \overline{sg}(a) + \text{sg}(a)(1-a), a \rangle$
$\neg_{54}$	$\langle \overline{sg}(1-b) + \text{sg}(1-b)b, 1-a \rangle$

These negations are defined in a series of papers, starting with [1].

2 Main Results

First, following and extending [3], we give the Law for Excluded Middle (LEM) in the forms:

$$\langle a,b \rangle \vee \neg \langle a,b \rangle = \langle 1,0 \rangle$$

(tautology-form) and

$$\langle a,b \rangle \vee \neg \langle a,b \rangle = \langle p,q \rangle,$$

(IFT-form), where $1 \geq p \geq q \geq 0$ and $p+q \leq 1$.

Second, we give the Modified Law for Excluded Middle (MLEM) in the forms:

$$\neg\neg \langle a,b \rangle \vee \neg \langle a,b \rangle = \langle 1,0 \rangle$$

(tautology-form) and

$$\neg\neg \langle a,b \rangle \vee \neg \langle a,b \rangle = \langle p,q \rangle,$$

(IFT-form), where $1 \geq p \geq q \geq 0$ and $p+q \leq 1$ and $\vee$ is the disjunction from Section 1.1.

Theorem 1. Only negation $\neg_{13}$ satisfies the LEM in the tautological form.
Proof. Let $a, b, a+b \in [0,1]$. Then,

$$V(x \vee \neg_{13} x) = \langle a,b \rangle \vee \langle \text{sg}(1-a), \overline{sg}(1-a) \rangle$$

$$= \langle \max(a, \text{sg}(1-a)), \min(b, \overline{sg}(1-a)) \rangle.$$

Now, we see that

$$\max(a, \text{sg}(1-a)) = \begin{cases} \max(1, \text{sg}(0)), & \text{if } a = 1 \\ \max(a, \text{sg}(1-a)), & \text{if } a < 1 \end{cases}$$

$$= \begin{cases} \max(1,0), & \text{if } a = 1 \\ \max(a,1), & \text{if } a < 1 \end{cases} = 1$$

and

$$\min(b, \overline{sg}(1-a)) = \begin{cases} \min(0, \overline{sg}(0)), & \text{if } a = 1 \\ \min(0, \text{sg}(1-a)), & \text{if } a < 1 \end{cases}$$

$$= \begin{cases} \min(0,1), & \text{if } a = 1 \\ \min(0,1), & \text{if } a < 1 \end{cases} = 0.$$

Therefore,

$$V(x \vee \neg_{13} x) = \langle 1, 0 \rangle,$$

i.e., $x \vee \neg_{13} x$ is a tautology. □

Theorem 2. Only negations $\neg_2$, $\neg_5$, $\neg_9$, $\neg_{11}$, $\neg_{13}$, $\neg_{16}$ satisfy the MLEM in the tautological form.

Theorem 3. Only negations $\neg_2$, $\neg_5$, $\neg_6$, $\neg_{10}$, $\neg_{42}$, $\neg_{44}$, $\neg_{46}$, $\neg_{47}$, $\neg_{50}$, $\neg_{51}$ do not satisfy the LEM in the IFT form.

Theorem 4. Only negations $\neg_{10}$, $\neg_{42}$, $\neg_{44}$, $\neg_{46}$, $\neg_{47}$, $\neg_{50}$, $\neg_{51}$ do not satisfy the MLEM in the IFT form.

The checks of these assertions are similar to the above one. By this reason we will prove only Theorem 2 and Theorem 4 for the case of negation $\neg_5$.

$$\neg_5 \neg_5 \langle a, b \rangle \vee \neg_5 \langle a, b \rangle$$

$$= \langle 1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b), \text{sg}(1-b) \rangle \vee \langle 1 - \text{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b))$$
$$-\overline{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b)).\text{sg}(1 - \text{sg}(1-b)), \text{sg}(1 - \text{sg}(1-b)) \rangle$$
$$= \langle \max(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b), 1 - \text{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b))$$
$$-\overline{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b)).\text{sg}(1 - \text{sg}(1-b))), \min(\text{sg}(1-b),$$
$$\text{sg}(1 - \text{sg}(1-b))) \rangle$$

Let

$$X \equiv \max(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b), 1 - \text{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b))$$
$$-\overline{sg}(1 - \text{sg}(a) - \overline{sg}(a).\text{sg}(1-b)).\text{sg}(1 - \text{sg}(1-b)))$$

$$-\min(\text{sg}(1-b), \text{sg}(1-\text{sg}(1-b)))$$

Let $a = 0$. Then, $\text{sg}(a) = 0$, $\overline{sg}(a) = 1$ and

$$X = \max(1-\text{sg}(1-b), 1-\text{sg}(1-\text{sg}(1-b)) - \overline{sg}(1-\text{sg}(1-b))$$

$$.\text{sg}(1-\text{sg}(1-b))) - \min(\text{sg}(1-b), \text{sg}(1-\text{sg}(1-b))).$$

If $b = 1$, then $\text{sg}(1-b) = 0$ and

$$X = \max(1, 1-\text{sg}(1) - \overline{sg}(1).\text{sg}(1)) - \min(0, \text{sg}(1)) = \max(1,0) - \min(0,1)$$

$$= 1.$$

If $b < 1$, then $\text{sg}(1-b) = 1$ and

$$X = \max(1-1, 1-\text{sg}(1-1) - \overline{sg}(1-1).\text{sg}(1-1)) - \min(1, \text{sg}(1-1))$$

$$= \max(0,1) - \min(1,0) = 1.$$

Let $a > 0$. Then $\text{sg}(a) = 1$, $\overline{sg}(a) = 0$, $\text{sg}(1-b) = 1$ and

$$X \equiv \max(1-1, 1-\text{sg}(1-1) - \overline{sg}(1-1).\text{sg}(1-1)) - \min(1, \text{sg}(1-1))$$

$$= \max(0,1) - \min(1,0) = 1.$$

Therefore, negation $\neg_5$ satisfies the Modified LEM in the IFT-form. On the other hand, in all cases the evaluation of the expression is equal to $\langle 1, 0 \rangle$, i.e., this negation satisfies the Modified LEM in the tautological form.

Third, following and extending [2], we study which negations satisfy DMLs. Usually, they have the forms:

$$\neg x \wedge \neg y = \neg(x \vee y),$$

$$\neg x \vee \neg y = \neg(x \wedge y),$$

where $\wedge$ and $\vee$ are the conjunction and disjunction from Section 1.1.

Theorem 5. For every two propositional forms x and y:

$$\neg_i x \wedge \neg_i y = \neg_i(x \vee y),$$

$$\neg_i x \vee \neg_i y = \neg_i(x \wedge y)$$

for $i = 1,2,4,...,11,13,...,17,20,23,35,...,51,53$.

We shall illustrate only the fact that the DMLs are not valid for $i = 3$. For example, if $a = b = 0.5, c = 0.1, d = 0$, then

$$V(\neg_3 x \wedge \neg_3 y) = 0.5$$

$$V(\neg_3(x \vee y) = 0.25.$$

The above mentioned change of the Law for Excluded Middle inspire the idea to study the validity of De Morgan's Laws that the classical negation $\neg$ (here it is negation $\neg_1$) satisfies. Really, easy it can be proved that the expressions

$$\neg_1(\neg_1 x \vee \neg_1 y) = x \wedge y$$

and

$$\neg_1(\neg_1 x \wedge \neg_1 y) = x \vee y$$

are IFTs, but the other negations do not satisfy these equalities. For them the following assertion is valid.

Theorem 6. For every two propositional forms x and y:

$$\neg_i(\neg_i x \vee \neg_i y) = \neg_i\neg_i x \wedge \neg_i\neg_i y$$

$$\neg_i(\neg_i x \wedge \neg_i y) = \neg_i\neg_i x \vee \neg_i\neg_i y$$

for $i = 1,2,4,...,11,13,...,17,19,20,23,25,35,...,51,53$.

3 Conclusion

The results from the present paper give us possibility to classify negations as suitable in tautological form or in IFT-form. This will be a theme for a next research. This classification will give us the list of the negations that are proper for different decision making and intercriterial procedures.

Acknowledgements

The authors are thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DFNI-I-02-5 "InterCriteria Analysis: A New Approach to Decision Making".

References

[1] Atanassov, K. On some intuitionistic fuzzy negations. *Notes on Intuitionistic Fuzzy Sets*, Vol. 11, 2005, No. 6, 13–20.

[2] Atanassov, K. On intuitionistic fuzzy negations and De Morgan Laws. *Proc. of Eleventh International Conf. IPMU 2006*, Paris, July 2–7, 2006, 2399–2404.

[3] Atanassov, K. On intuitionistic fuzzy negations and Law of Excluded Middle, *Proc. of 2010 IEEE International Conference on Intelligent Systems (IS 2010)*, London, 7–9 July 2010, 266–269.

[4] Atanassov, K. *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[5] Atanassov K. On Intuitionistic Fuzzy Logics: Results and Problems. *Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics, Volume 1: Foundations*. (Atanassov, K., M. Baczynski, J. Drewniak, J. Kacprzyk, M. Krawczak, E. Szmidt, M. Wygralak, S. Zadrozny, eds.), SRI-PAS, Warsaw, 2014, 23–49.