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On the Lebesgue IF–measure

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Abstract: An IF-state on the family of IF-subsets of the unit interval is constructed invariant under shifts.

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1 Introduction

In the classical measure theory, it is known theory about Haar measure ([4]) stating that in every compact Abelian group there exists a probability measure invariant under shifts. Here, we shall consider a special case, the group $(R, +)$, and the shift $T_a : [0, 1) \to [0, 1)$ given by the prescription $T_a(x) = x + a(mod 1)$. In the first section, there will be presented some basic facts about probabilities on σ-algebras of sets, in the second section we will discuss states in IF-sets. The third section contains the theorem about the existence of invariant IF-states on real numbers.

2 Sets

We shall work with the σ-algebra of all Borel subsets of the unit interval $[0, 1) \subset R$, i.e. with the σ-algebra $\mathcal{S} = \sigma(\mathcal{J})$ generated by the family $\mathcal{J} = \{I; I \subset [0, 1), I \text{ is an interval}\}$. Let $a \in [0, 1)$. Then we define the transformation $T_a : [0, 1) \to [0, 1)$ by the formula

$$T_a(x) = x + a(mod 1),$$

i.e.,

$$T_a(x) = x + a, \text{if } x + a < 1,$$

$$T_a(x) = x + a - 1, \text{if } x + a \geq 1.$$

It is well known the assertion ([4]) that there exists exactly one probability measure

$$P : \mathcal{S} \to [0,1],$$

such that

$$P([0,1)) = 1,$$

and

$$P(T_a^{-1}(A)) = P(A)$$

for any $a \in [0,1]$, and any $A \in \mathcal{S}$. We want to extend the result from the family of sets to the family of IF-sets.

3 IF-sets

Again, let $\mathcal{S} = \sigma(\mathcal{J})$, where $\mathcal{J}$ is the family of all subintervals of the unit interval $[0,1)$. By an IF-event ([1]) we consider any pair

$$A = (\mu_A, \nu_A)$$

of $\mathcal{S}$-measurable functions $\mu_A, \nu_A : [0,1] \to [0,1]$ such that

$$\mu_A + \nu_A \leq 1.$$

Denote by $\mathcal{F}$ the family of all IF-events. Let $a \in [0,1)$. By the shift τ_a we shall consider the mapping

$$\tau_a : \mathcal{F} \to \mathcal{F},$$

defined by

$$\tau_a(A) = (\mu_A \circ T_a, \nu_A \circ T_a).$$

In the IF-measure theory instead of the notion probability we use the notion of a state. For to define a state we need the following binary operations

$$A \odot B = \{(\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1\},$$

$$A \oplus B = \{(\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0\}.$$

and the partial ordering $A \leq B$ if and only if

$$\mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Hence,

$$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A.$$

By a state on $\mathcal{F}$ we consider a mapping $m : \mathcal{F} \to [0,1]$ satisfying the following conditions:

(i) $m((0_{[0,1)}, 1_{[0,1)})) = 0, m((1_{[0,1)}, 0_{[0,1)})) = 1;$

(ii) $A \odot B = (0_{[0,1)}, 1_{[0,1)}) \Longrightarrow m(A \oplus B) = m(A) + m(B);$

(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$

4 Invariant IF-states

Theorem. For any $\beta \in [0,1]$, and any $a \in [0,1)$ and to shift function τ_a, there exists exactly one state $m : \mathcal{F} \to [0,1]$, such that

$$m(\tau_a(A)) = m(A)$$

for any $A \in \mathcal{F}$, and such that

$$m((0_{[0,1)}, 0_{[0,1)})) = \beta.$$

Proof. Let $A = (\mu_A, \nu_A) \in \mathcal{F}$. Let $P : \sigma(\mathcal{C}) \to [0,1]$ be the invariant probability measure, i.e., $P(B + a) = P(B)$ for any $B \in \sigma(\mathcal{C})$ and any $a \in [0,1)$. Put

$$m(A) = (1-\beta)\int \mu_A dP + \beta(1 - \int \nu_A dP).$$

Then

$$m(\tau_a(A)) = (1-\beta)\int \mu_A \circ T_a dP + \beta(1 - \int \nu_A \circ T_a dP) =$$

$$= (1-\beta)\int \mu_A dP + \beta(1 - \int \nu_A dP) = m(A)$$

for any $A \in \mathcal{F}$. We have proved the existence of an invariant state $m : \mathcal{F} \to [0,1]$. Evidently $m((0_{[0,1)}, 0_{[0,1)})) = \beta$.

We shall prove the uniqueness. Let $\lambda : \mathcal{F} \to [0,1]$ be any invariant state. Then by the representation theorem ([2, 3, 5, 6]) there exists $\alpha \in [0,1]$ and a probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$\lambda(A) = \int_{[0,1)} \mu_A dP + \alpha(1 - \int_{[0,1)} (\mu_A + \nu_A) dP))$$

for any $A \in \mathcal{F}$.

First let $\alpha = 0$. Then,

$$\lambda(A) = \int_{[0,1)} \mu_A dP.$$

Of course, also

$$\lambda(\tau_a(A)) = \int_{[0,1)} \mu_A \circ T_a dP,$$

hence,

$$\int_{[0,1)} \mu_A dP = \int_{[0,1)} \mu_A \circ T_a dP$$

for any $A \in \mathcal{F}, a \in [0,1)$. For any $B \in \sigma(\mathcal{C})$ put $\mu_A = \chi_B$. It follows

$$P(B) = \int_{[0,1)} \mu_A dP = \int_{[0,1)} \mu_A \circ T_a dP = \int_{[0,1)} \chi_{\tau_a^{-1}(B)} dP = P(\tau_a^{-1}(B)),$$

hence, $P : \sigma(\mathcal{C})) \to [0,1]$ is invariant. Moreover,

$$P([0,1)) = \int_{[0,1)} 1_{[0,1)} dP = \lambda((1_{[0,1)}, 0_{[0,1)})) = 1,$$

hence P is an invariant probability measure, and it is determined uniquely. $\square$

Let now $\alpha \in (0,1]$. Then,

$$\lambda(A) = \int_{[0,1)} \mu_A dP + \alpha(1 - \int_{[0,1)} (\mu_A + \nu_A) dP).$$

Evidently,

$$\lambda((0_{[0,1)}, 0_{[0,1)})) = \alpha(1-0),$$

hence,

$$\alpha = \lambda((0_{[0,1)}, 0_{[0,1)})).$$

Moreover,

$$\lambda(\tau_a(A)) = \int_{[0,1)} \mu_A \circ T_a dP + \alpha(1 - \int_{[0,1)} (\mu_A \circ T_a + \nu_A \circ T_a) dP).$$

Put $A = (0_{[0,1)}, \nu_A)$. Then,

$$0 + \alpha(1 - \int_{[0,1)} (0 + \nu_A \circ T_a) dP) = \lambda(\tau_a(A)) =$$

$$= \lambda(A) = 0 + \alpha(1 - \int_{[0,1)} (0 + \nu_A) dP),$$

hence

$$\int_{[0,1)} \nu_A \circ T_a dP = \int_{[0,1)} \nu_A dP$$

for any $A \in \mathcal{F}$ and any $a \in [0,1)$. It is clear that $P : \sigma(\mathcal{C})) \to [0,1]$ is an invariant measure. Moreover,

$$0 = \lambda((0_{[0,1)}, 1_{[0,1)})) = \alpha(1 - \int_{[0,1)} 1_{[0,1)} dP).$$

Since $\alpha > 0$, we have

$$P([0,1)) = \int_{[0,1)} 1_{[0,1)} dP = 1,$$

hence $P : \sigma(\mathcal{C}) \to [0,1]$ is the unique invariant probability measure.

5 Conclusion

We have proved for any real number $\alpha \in [0,1]$ the existence of a unique state $m : \mathcal{F} \to [0,1]$ invariant with respect to the group transformations

$$\tau_a(\mu_A, \nu_A)(\omega) = (\mu_A(\omega + a), \nu_A(\omega + a)),$$

and such that

$$m(0_{[0,1)}, 0_{[0,1)}) = \alpha.$$

Of course, for different numbers α we can obtained different states m.

On the other hand, for fuzzy sets ([7, 8]) we have $\nu_A = 1 - \mu_A$, hence

$$m(A) = \int_{[0,1)} \mu_A dP + \alpha(1 - \int_{[0,1)} (\mu_A + \nu_A) dP) = \int_{[0,1)} \mu_A dP,$$

and

$$m(\tau_a(A)) = \int_{[0,1)} \mu_A \circ T_a dP = \int_{[0,1)} \mu_A dP = m(A).$$

We have obtained the existence of an invariant fuzzy state m, and even unique, it does not depend on α.

So from IF-invariant theory one can obtain the fuzzy invariant theory, but the oposite direction is not possible, the family of IF states is more rich. Hence, the result for IF sets is not a corollary of the existence of fuzzy invariant state.

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