Notes on Intuitionistic Fuzzy Sets
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On IF-semistates

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Abstract: Semistates on a family $\mathcal{F}$ of IF-events are considered as functions $m : \mathcal{F} \to [0, 1]$, additive with respect to the Lukasiewicz disjunction $A \oplus B$ and conjunction $A \odot B$. The main result is an extension theorem extending m to an MV algebra $\bar{m} : \mathcal{M} \to [0, 1]$. The theorem generalizes the extension theorem of IF states from $\mathcal{F}$ to $\mathcal{M}$.
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1 Introduction

There are many results about probability on IF-sets. Of course, similarly as in quantum structures, another terminology is used. Instead of probability usualy the term state is used. Moreover, P. Grzegorzewski [6] introduced the term probability for mappings assigning to IF-sets some compact intervals in the set R of real numbers. On the other hand, the state is a mapping assigning to IF-sets real numbers. Of course, the Grzegorzewski concept of probability is in a one-to-one correspondence with the concept of the state.

The notion of a state has been defined axiomatically as an additive, continuous function with values in the unit interval, with value 0 in the least element and value 1 in the greatest element (see [4, 5, 9, 11, 12]).

One of the most important result of the theory is the theorem on embedding of the family of all IF-sets to a convenient MV-algebra together with the extension of a given state to a state on MV-algebra (see [10, 12]). In [13] a variant of the extension theorem was considered without continuity of given states – we had spoken about finitely additive states. In the paper we present some similar results. Of course, our assumptions are weaker, therefore instead of the term finitely additive state we use the term semistate.

In Section 2 we present some basic informations about IF-sets, in Section 3 about MV-algebras, in Section 4 the notion of semistate is studied, and Section 5 contains the embedding theorem.

2 IF-sets

An intuitionistic fuzzy set (see [1, 2, 12]) is a pair $A = (\mu_A, \nu_A)$ of functions $\mu_A, \nu_A : \Omega \to [0,1]$ such that

$$\mu_A + \nu_A \leq 1.$$

If $A = (\mu_A, \nu_A), B = (\mu_B = \nu_B)$, then we write

$$A \leq B$$

if and only if

$$\mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Here $(0_\Omega, 1_\Omega) \leq (\mu_A, \nu_A) \leq (1_\Omega, 0_\Omega)$ for all $A = (\mu_A, \nu_A)$. We shall write

$$A_n = (\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A) = A,$$

if and only if

$$\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A.$$

Denote by Δ the set

$$\Delta = \{(a, b) \in [0,1]^2; a + b \leq 1\}.$$

Then an IF set is a mapping $A : \Omega \to \Delta$. If we put $\nu_A = 1 - \mu_A$, then we obtain a fuzzy set $A : \Omega \to [0,1]$. If $A : \Omega \to \{0,1\}$, then we obtain a susbset $A_0 \subset \Omega$, where $\omega \in A_0$ if and only if $A(\omega) = 1$, hence A can be identified with the indicator χ_{A_0}.

In the paper, we work with a family $\mathcal{F}$ of mappings $A = (\mu_A, \nu_A) : \Omega \to \Delta$ closed with respect to the Łukasiewicz binary operations

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0).$$

and with respect to the unary operation

$$\neg A = (1 - \mu_A, 1 = \nu_A).$$

3 MV-algebras

A prototype of an MV-algebra is the unit interval $[0, 1]$ with two binary operations

$$a \oplus b = (a + b) \wedge 1,$$

$$a \odot b = (a + b - 1) \vee 0,$$

and one unary operation

$$\neg a = 1 - a,$$

and the usual ordering. The operation $\oplus$ correponds to the disjunction of statements (the union of sets), $\odot$ corresponds to the conjunction of statements (the product of sets), $\leq$ to the implication of statements (the inclusion of sets), $\neg a$ to the negation of a statement (the complement of a set).

Generally we shall use the Mundici characterization of MV-algebras (see [3, 7, 8, 14, 15]). It starts with the notion of an l-group. An l-group is an algebraic sructure $(G, +, \leq)$, where $(G, +)$ is a commutative group, $(G, \leq)$ is a lattice, and the implication $a \leq b \Longrightarrow a + c \leq b + c$ holds. An MV-algebra is an algebraic structure

$$(M, 0, u, \leq, \oplus, \odot),$$

where 0 is the neutral element in G, u is a positive element, $M = \{x \in G; 0 \leq x \leq u\}$, $\neg : M \to M$ is a unary operation given by the equality

$$\neg x = u - x,$$

and $\oplus, \odot$ are two binary operations given by

$$a \oplus b = (a + b) \wedge u,$$

$$a \odot b = (a + b - u) \vee 0.$$

Example 3.1. Consider $(R^2, +, \leq)$, where $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1 - (1 - y_1 + 1 - y_2)) = (x_1, y_1), y_1 + y_2 - 1)$, and $(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2, y_1 \geq y_2$. Then R^2 is an l-group. Put $u = (1, 0)$. Then

$$\mathcal{M} = \{(x, y) \in R^2; (0, 1) \leq (x, y) \leq u = (1, 0)\}$$

is an MV-algebra.

Theorem 3.1. Let $\mathcal{F}$ be a family of IF-sets closed with respect to $\oplus, \odot$, and $\neg$. Let $\mathcal{M}$ be the family of all $A = (\mu_A, \nu_A) : \Omega \to [0, 1]^2$ such that

$$(\mu_A, 0) = (\mu_A, \nu_A) \oplus (0, 1 - \nu_A).$$

Then $\mathcal{M}$ is an MV-algebra generated by $\mathcal{F}$.

Proof: Put

$$G = (R^2, +, \leq)$$

where

$$(a,b)+(c,d)=(a+c,b+d-1),$$

$$(a,b)\leq(c,d)\Longleftrightarrow a\leq c, b\geq d.$$

Then G is an l-group. Put $u=(1,0)$. We have

$$\mathcal{M}=\{A;(0,1)\leq(\mu_A,\nu_A)\leq(1,0)\},$$

and

$$(\mu_A,\nu_A)\oplus(0,1-\nu_A)=((\mu_A,\nu_A)+(0,1-\nu_A))\wedge(1,0)=$$

$$=(\mu_A,0)\wedge(1,0)=(\mu_A,0).$$

This completes the proof. □

4 IF-semistates

We shall consider a couple $(\Omega,\mathcal{S})$, where Ω is a non-empty set, and $\mathcal{S}$ is a σ-algebra of subsets of Ω, i.e.,

$(i)\Omega\in\mathcal{S}$,

$(ii)A_n\in\mathcal{S}(n=1,2,...)\Longrightarrow\bigcup_{n=1}^{\infty}A_n\in\mathcal{S}$,

$(iii)A\in\mathcal{S}\Longrightarrow\Omega-A\in\mathcal{S}$.

Definition 4.1. By an IF-event we shall consider any IF-set $A=(\mu_A,\nu_A)$ such that $\mu_A,\nu_A:\Omega\to[0,1]$ are $\mathcal{S}$-measurable, i.e.,

$I\subset R, I$ is an interval $\Longrightarrow\mu_A^{-1}(I)\in\mathcal{S},\nu_A^{-1}(I)\in\mathcal{S}$.

By $\mathcal{F}$ the family of all IF-events will be denoted.

Definition 4.2. A mapping $m:\mathcal{F}\to[0,1]$ is called IF-semistate, if

$$A,B\in\mathcal{F},A\odot B=(0_\Omega,1_\Omega)\Longrightarrow m(A\oplus B)=m(A)+m(B).$$

Definition 4.3. By $\mathcal{P}$ it will be denoted the family of all mappings $m:\mathcal{F}\to[0,1]$ satisfying the folloging condition: There exist probability measures $P,Q:\mathcal{S}\to[0,1]$ and $\alpha\in R$ such that

$$(\star)m(A)=\int\mu_A dP+\alpha(1-\int(\mu_A+\nu_A)dQ)$$

for any $A=(\mu_A,\nu_A)\in\mathcal{F}$.

Proposition 4.1. Any $m\in\mathcal{P}$ is an IF-semistate.

Proof: Let $A,B\in\mathcal{F},A=(\mu_A,\nu_A),B=(\nu_B,\nu_B),A\odot B=(0_\Omega,1_\Omega)$. Then

$$(\mu_A+\mu_B-1)\vee 0=0,(\nu_A+\nu_B)\wedge 1=1,$$

hence

$$\mu_A + \mu_B \leq 1, \nu_A + \nu_A \geq 1.$$

Therefore,

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0) =$$

$$= (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

Since $m \in \mathcal{P}$, we have

$$m(A \oplus B) = \int (\mu_A + \mu_B) dP + \alpha(1 - \int (\mu_A + \mu_B + \nu_A + \nu_B - 1) dQ =$$

$$= \int (\mu_A + \mu_B) dP + \alpha(2 - \int (\mu_A + \nu_A + \mu_B + \nu_B) dQ) =$$

$$= \int \mu_A dP + \alpha(1 - \int (\mu_A + \nu_A) dQ) + \int \mu_B dP + \alpha(1 - \int (\mu_B + \nu_B) dQ) =$$

$$= m(A) + m(B).$$

This completes the proof. □

Proposition 4.2. Let $P = Q$. Then the mapping $m : \mathcal{F} \to [0, 1]$ is a state, i.e., the following properties are satisfied:

(i) $m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1,$

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B_{=} m(A) + m(B),$

(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$

Proof: The property (ii) follows by the definition of IF-semistate, (i) is an easy consequence of $(\star)$:

$$m((0_\Omega, 1_\Omega)) = \int 0 dP + \alpha(1 - \int (0 + 1) dQ) = \alpha(1 - Q(\Omega)) = 0,$$

$$m((1_\Omega, 0_\Omega)) = \int 1 dP + \alpha(1 - \int (1 + 0) dQ) = P(\Omega) + \alpha.0 = 1.$$

Since $P = Q$, we obtain

$$m(A) = \int \mu_A dP + \alpha - \alpha \int \mu_A dP - \alpha \int \nu_A dP =$$

$$= (1 - \alpha) \int \mu_A dP + \alpha(1 - \int \nu_A dP).$$

Let $A_n \nearrow A$ i.e., $\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$. Since $1 - \alpha \geq 0, \alpha \geq 0$, we obtain by the monotone convergence theorem

$$\lim_{n \to \infty} m(A_n) = (1 - \alpha) \lim_{n \to \infty} \int \mu_{A_n} dP + \alpha(1 - \lim_{n \to \infty} \int \nu_{A_n} dP) =$$

$$= (1-\alpha)\int \mu_A dP + \alpha(1 - \int \nu_A dP) = m(A).$$

This completes the proof. □

Remark 4.1. If $\mu_{A_n} \to \mu_A, \nu_{A_n} \to \nu_A$, then by $(\star)$ and the Lebesgue convergence theorem $m(A_n) \to m(A)$, of course, we are not able to prove that the convergence is monotone.

Remark 4.2. By the Ciungu representation theorem (see [4]), any state $m : \mathcal{F} \to [0,1]$ belongs to the family $\mathcal{P}$.

5 Extension

Now we are able to formulate and prove the main result of the article.

Theorem 5.1. Let $\mathcal{F}$ be the family of all IF-events on $(\Omega, \mathcal{S})$, $\mathcal{M}$ be the MV-algebra generated by $\mathcal{F}$. Then to any $m \in \mathcal{P}$ there exists an IF-semistate $\bar{m} : \mathcal{M} \to [0,1]$ extending m.

Proof: Similarly as in [12], we define

$$\bar{m}((\mu_A, \nu_A)) = m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)).$$

If $A = (\mu_A, \nu_A) \in \mathcal{F}$, then

$$\bar{m}((\mu_A, \nu_A)) = \int \mu_A dP + \alpha(1 - \int (\mu_A + 0) dQ) - (\int 0 dP - \alpha(1 - \int (0 + 1 - \nu_A) dQ) =$$

$$= \int \mu_A dP + \alpha(1 - \int (\mu_A + \nu_A) dQ) = m((\mu_A, \nu_A)).$$

Let $A, B \in \mathcal{F}, A \odot B = (0_\Omega, 1_\Omega)$, i.e.,

$$((\mu_A + \nu_A - 1) \vee 0, (\nu_A + \nu_B) \wedge 1 = (0_\Omega, 1_\Omega),$$

hence

$$\mu_A + \mu_B \leq 1, \nu_A + \nu_B \geq 1.$$

Therefore,

$$A \oplus B = (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

We have

$$\bar{m}(A) = \int \mu_A dP + \alpha(1 - \int (\mu_A + \nu_A) dQ),$$

$$\bar{m}(B) = \int \mu_B dP + \alpha(1 - \int (\mu_B + \nu_B) dQ),$$

$$\bar{m}(A \oplus B) = \int (\mu_A + \mu_B) dP + \alpha(1 - \int (\mu_A + \mu_B + \nu_A + \nu_B - 1) dQ),$$

$$= \int \mu_A dP + \int \mu_B dP + \alpha(2 - \int (\mu_A + \nu_A) dQ - \int (\mu_B + \nu_B) dQ) =$$

$$= \int \mu_A dP + \alpha(1 - \int (\mu_A + \nu_A) dQ) + \int \mu_B dP + \alpha(1 - \int (\mu_B + \nu_B) dQ) =$$

$$= \bar{m}(A) + \bar{m}(B).$$

This completes the proof. □

Corollary 5.1. Let m be a state on $\mathcal{F}$. Then, there exists a state $\bar{m}$ on $\mathcal{M}$ extending m.

Proof: By the Ciungu representation theorem [4] we know that $m \in \mathcal{P}$, hence Theorem 5.1 is applicable. The only problem moreover is the monotonicity of $\bar{m}$. Let $A, B \in \mathcal{M}, A \leq B$, i.e., $\mu_A \leq \mu_B, \nu_A \geq \nu_B$. Consider $C \in \mathcal{S}$. We see that

$$\alpha = m((0_\Omega, 0_\Omega)) \leq m((\chi_C, 0_\Omega)) = \int \chi_C dP + \alpha(1 - \int (\chi_C + 0) dQ),$$

hence

$$\alpha \leq P(C) + \alpha - \alpha Q(C),$$

$$0 \leq P(C) - \alpha Q(C)$$

for any $C \in \mathcal{S}$. Therefore

$$0 \leq \int f dP - \alpha \int f dQ$$

for any non-negative f. Since $\mu_B \geq \mu_A$, we can put $f = \mu_B - \mu_A$. We obtain

$$\bar{m}(B) - \bar{m}(A) =$$

$$= \int (\mu_B - \mu_A) dP - \alpha \int (\mu_B - \mu_A) dQ + \alpha \int (\nu_A - \nu_B) dQ \geq$$

$$\geq \int f dP - \alpha \int f dQ \geq 0.$$

This completes the proof. □

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