Intuitionistic fuzzy implication $\rightarrow^C$ and negation $\neg^C$

Krassimir T. Atanassov

Bioinformatics and Mathematical Modelling Department
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria
e-mail: krat@bas.bg

Abstract: A new type of intuitionistic fuzzy implication $\rightarrow^C$ and negation $\neg^C$ are introduced in intuitionistic fuzzy logic and some of their properties are discussed. These new operations are extensions of the operations $\neg^{\varepsilon, \eta}$ and $\rightarrow^{\varepsilon, \eta}$.

Keywords: Intuitionistic fuzzy logic, Quantifier, Topological operator.

AMS Classification: 03E72.

1 Introduction

In [1, 2] a new type of negations and implications over an Intuitionistic Fuzzy Set (IFS; see [3, 4]) are introduced. Here, we continue to research in this area. We extend the introduced already negations and implications. The used notation for IFS is from [3, 4]. In the second book these operations are described in details.

In [2], the set of IF-negations that has the form

$$N = \{\neg^{\varepsilon, \eta} | 0 \leq \varepsilon < 1 & 0 \leq \eta < 1\},$$

where for each IFS $A$,

$$\neg^{\varepsilon, \eta}A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle| x \in E\}$$

is constructed and it is proved that the inequality $\varepsilon \leq \eta$ is necessity for correctness of $\neg^{\varepsilon, \eta}A$.

There, the implication, generated by the new negation, is constructed, as

$$A \rightarrow^{\varepsilon, \eta} B = \{\langle x, \max(\mu_B(x), \min(1, \nu_A(x) + \varepsilon)), \min(\nu_B(x), \max(0, \mu_A(x) - \eta))\rangle| x \in E\}$$
Now, we extend these definitions.

2 Main results

Let everywhere below $C$ be an Intuitionistic Fuzzy Topological Set (IFTS), i.e., for every $x \in E$: $\mu_A(x) \geq \nu_A(x)$.

We construct the negation on the basis of $C$, that has the form for each IFS $A$,

$$\neg^C A = \{ \langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, \mu_A(x) - \mu_C(x)) \rangle | x \in E \}. \quad (1)$$

**Theorem 1.** For every IFS $A$ and IFTS $C$, the set $\neg^C A$ is an IFS.

**Proof.** Let $A$ be an IFS and $C$ – an IFTS. Then for each $x \in E$, if $\mu_A(x) \leq \mu_C(x)$ then,

$$\min(1, \nu_A(x) + \nu_C(x)) + \max(0, \mu_A(x) - \mu_C(x)) = \min(1, \nu_A(x) + \nu_C(x)) \leq 1;$$

if $\mu_A(x) \geq \mu_C(x)$ then,

$$\min(1, \nu_A(x) + \nu_C(x)) + \max(0, \mu_A(x) - \mu_C(x)) = \min(1, \nu_A(x) + \nu_C(x)) + \mu_A(x) - \mu_C(x) \leq \nu_A(x) + \nu_C(x) + \mu_A(x) - \mu_C(x) \leq \nu_A(x) + \mu_A(x) \leq 1.$$

Obviously, in the partial case, when $C = \{ \langle x, \eta, \varepsilon \rangle | x \in E \}$ and $\varepsilon + \eta \leq 1$, we obtain the above negation and implication.

In Figure 1, $x$ and $\neg_1 x$ are shown (where, the classical (the first) negation defined over IFSs is marked by $\neg_1$), while in Figures 2 and 3, $y$ and $\neg^C y$ and $z$ and $\neg^C z$ are shown.

![Fig. 1.](image)

Now, by analogy with the above construction, we can construct a new implication, generated by the new negation as

$$A \rightarrow^C B = \{ \langle x, \max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) \rangle,$$
\[
\min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x))) | x \in E \} = \{(x, \min(1, \max(\mu_B(x), \nu_A(x) + \nu_C(x))), \\
\max(0, \min(\nu_B(x), \mu_A(x) - \mu_C(x))) | x \in E \}.
\]

**Fig. 2.**

\[
\begin{align*}
\nu_A(y) - \nu_C(y) & \geq \mu_A(x) - \mu_C(x) \\
\nu_A(z) + \nu_C(z) & \leq 1 - \mu_A(z) + \mu_C(z) \\
\end{align*}
\]

**Fig. 3.**

**Theorem 2.** For every two IFSs \(A, B\) and IFTS \(C\), the set \(A \rightarrow^C B\) is an IFS.

**Proof.** Let \(A\) and \(B\) be IFSs and \(C\) – an IFTS. Let for each \(x \in E\), \(\mu_A(x) - \mu_C(x) \geq \nu_B(x)\). Then,
\[
\nu_A(x) + \nu_C(x) \leq 1 - \mu_A(x) + \mu_C(x) \leq 1 - \nu_B(x)
\]
and
\[
\mu_B(x) \leq 1 - \nu_B(x).
\]

Hence
\[
\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))
\]
\[
\leq \max(\mu_B(x), \min(1, 1 - \nu_B(x))) + \min(\nu_B(x), \mu_A(x) - \mu_C(x))
\]
\[
\leq \max(\mu_B(x), 1 - \nu_B(x)) + \nu_B(x) = 1.
\]
Let for each \( x \in E \), \( \mu_A(x) - \mu_C(x) < \nu_B(x) \). If \( \mu_A(x) \leq \mu_C(x) \), then
\[
\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))
\leq \max(\mu_B(x), 1) + \min(\nu_B(x), 0) = 1 + 0 = 1.
\]

If \( \mu_A(x) > \mu_C(x) \), then, as above
\[
\nu_A(x) + \nu_C(x) \leq 1 - \mu_A(x) + \mu_C(x)
\]
and
\[
1 - \mu_B(x) \geq \nu_B(x) > \mu_A(x) - \mu_C(x),
\]
i.e.,
\[
\mu_B(x) \leq 1 - \mu_A(x) + \mu_C(x).
\]

Hence
\[
\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))
\leq \max(\mu_B(x), \min(1, 1 - \mu_A(x) + \mu_C(x))) + \mu_A(x) - \mu_C(x)
\]
\[
= \max(\mu_B(x), 1 - \mu_A(x) + \mu_C(x)) + \mu_A(x) - \mu_C(x)
\]
\[
1 - \mu_A(x) + \mu_C(x) + \mu_A(x) - \mu_C(x) = 1.
\]

Therefore, \( A \rightarrow^C B \) is an IFS.

In [5], George Klir and Bo Yuan introduced the following axioms for implications and negations.

**Axiom 1** \((\forall x, y) (x \leq y \rightarrow (\forall z) (I(x, z) \geq I(y, z)).\)

**Axiom 2** \((\forall x, y) (x \leq y \rightarrow (\forall z) (I(z, x) \leq I(z, y)).\)

**Axiom 3** \((\forall y)(I(0, y) = 1).\)

**Axiom 4** \((\forall y)(I(1, y) = y).\)

**Axiom 5** \((\forall x)(I(x, x) = 1).\)

**Axiom 6** \((\forall x, y, z) (I(x, I(y, z)) = I(y, I(x, z)).\)

**Axiom 7** \((\forall x, y)(I(x, y) = 1 \iff x \leq y).\)

**Axiom 8** \((\forall x, y)(I(x, y) = I(N(y), N(x))), \) where \( N \) is an operation for a negation.

**Axiom 9** \( I \) is a continuous function.

**Theorem 3.** Implication \( \rightarrow^C \) and negation \( \neg^C \):

(a) satisfy Axioms 1, 2, 3, 6 and 9;

(b) satisfy Axioms 4 and 5 as IFTs, but not as tautologies;

(c) satisfy Axiom 8 in the form

**Axiom 8’**: \((\forall x, y)(I(x, y) \leq I(N(y), N(x))).\)

**Theorem 4.** For each IFS \( A \):

(a) \( A \cup \neg^C A \) is an IFTS, but not always equal to \( E^* \);

(b) \( \neg^C A \cup \neg^C A \) is an IFTS, but not always equal to \( E^* \).

Usually, in set theory the De Morgan’s Laws have the forms:

\[
\neg A \cap \neg B = \neg (A \cup B),
\]

(3)
\[ \neg A \cup \neg B = \neg (A \cap B). \] 
\hspace{1cm} (4)

or

\[ \neg(\neg A \cap \neg B) = A \cup B, \] 
\hspace{1cm} (5)

\[ \neg(\neg A \cup \neg B) = A \cap B. \] 
\hspace{1cm} (6)

but, as we discussed in [4], they can also have the forms:

\[ \neg(\neg A \cap \neg B) = \neg\neg A \cup \neg\neg B, \] 
\hspace{1cm} (7)

\[ \neg(\neg A \cup \neg B) = \neg\neg A \cap \neg\neg B. \] 
\hspace{1cm} (8)

**Theorem 5.** For every two IFSs \(A\) and \(B\):

(a) the IFSs from (3) and (4) with negation \(\neg^C\) are IFTSs, but not always equal to \(E^*\);

(b) the IFSs from (5) – (8) with negation \(\neg^C\) are not always IFTSs or not always equal to \(E^*\).

**Theorem 6.** For every IFS \(A\):

\[ \neg^C \Box A \supset \Box \neg^C A, \]

\[ \neg^C \Diamond A \subset \Diamond \neg^C A. \]

Let us prove, for example, the second inclusion. The rest of the assertions can be proved analogously. Let \(C\) be an IFTS. Then,

\[ \neg^C \Diamond A = \neg^C \{ \langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in E \} \]

\[ = \{ \langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, 1 - \nu_A(x) - \mu_C(x)) \rangle | x \in E \}. \]

\[ \Diamond \neg^C A = \Diamond \{ \langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, \mu_A(x) - \mu_C(x)) \rangle | x \in E \} \]

\[ = \{ \langle x, 1 - \max(0, \mu_A(x) - \mu_C(x)), \max(0, \mu_A(x) - \mu_C(x)) \rangle | x \in E \}. \]

Let

\[ X \equiv 1 - \max(0, \mu_A(x) - \mu_C(x)) - \min(1, \nu_A(x) + \nu_C(x)). \]

If \(\nu_A(x) + \nu_C(x) \geq 1\), then

\[ \mu_A(x) - \mu_C(x) \leq 1 - \nu_A(x) - \mu_C(x) \leq \nu_C(x) - \mu_C(x) \leq 0 \]

and

\[ X = 1 - 1 - 0 = 0. \]

If \(\nu_A(x) + \nu_C(x) \leq 1\), then there are two subcases. If \(\mu_A(x) - \mu_C(x) \leq 0\), then

\[ X = 1 - (\nu_A(x) + \nu_C(x)) - 0 \geq 0 \]

and if \(\mu_A(x) - \mu_C(x) \geq 0\), then

\[ X = 1 - (\nu_A(x) + \nu_C(x)) - \mu_A(x) + \mu_C(x) = 1 - \mu_A(x) - \mu_A(x) + \mu_C(x) - \nu_C(x) \geq 0. \]
Therefore, the first component of the second term is higher than the first component of the first term, while the inequality
\[
\max(0, 1 - \nu_A(x) - \mu_C(x)) - \max(0, \mu_A(x) - \mu_C(x)) \geq 0
\]
is obvious. Therefore, the inclusion is valid.

**Theorem 7.** For every IFS $A$ and IFTS $C$, for every two real numbers $\alpha, \beta$, so that $0 \leq \alpha, \beta \leq 1$
(a) $\neg^C G_{\alpha,\beta}(A) \supseteq G_{\beta,\alpha}(\neg^C A)$,
(b) $\neg^C H_{\alpha,\beta}(A) \supseteq H_{\beta,\alpha}(\neg^C A)$,
(c) $\neg^C J_{\alpha,\beta}(A) \subseteq J_{\beta,\alpha}(\neg^C A)$,
(d) $\neg^C H^*_A \supseteq H^*_A \subseteq J^*_{\beta,\alpha}(\neg^C A)$,
(e) $\neg^C J^*_{\alpha,\beta}(A) \subseteq J^*_{\beta,\alpha}(\neg^C A)$,
(f) $\neg^C P_{\alpha,\beta}(A) \subseteq P_{\alpha,\beta}(\neg^C A)$,
(g) $\neg^C Q_{\alpha,\beta}(A) \supseteq Q_{\alpha,\beta}(\neg^C A)$.

**Theorem 8.** For every IFSs $A, B$ and for every IFTS $C$
(a) $\neg^C G_B(A) \supseteq G_B(\neg^C A)$,
(b) $\neg^C H_B(A) \supseteq H_B(\neg^C A)$,
(c) $\neg^C J_B(A) \subseteq J_B(\neg^C A)$,
(d) $\neg^C H^*_B \supseteq H^*_B \supseteq J^*_B(\neg^C A)$,
(e) $\neg^C J^*_B(A) \subseteq J^*_B(\neg^C A)$,
(f) $\neg^C P_B(A) \subseteq P_B(\neg^C A)$,
(g) $\neg^C Q_B(A) \supseteq Q_B(\neg^C A)$.

3 Conclusion

In the paper, two new operations: a negation and an implication, were introduced. The implication is based on the new negation, but in general, has the classical form. In a next author’s research some non-classical forms of the implication $\rightarrow^C$ will be discussed.

Acknowledgement

The author is grateful for the support provided by the National Science Fund of Bulgaria under grant BIn-2/09.
References


