On intuitionistic fuzzy level operators

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The concept of an Intuitionistic Fuzzy Set (IFS) is defined (see, e.g., [1]) as follows. Let a set \( E \) be fixed. An IFS \( A \) in \( E \) is an object of the following form:

\[
A = \{ (x, \mu_A(x), \nu_A(x)) | x \in E \},
\]

where the functions \( \mu_A : E \to [0, 1] \) and \( \nu_A : E \to [0, 1] \) determine the degree of membership and the degree of non-membership of the element \( x \in E \), respectively, and for every \( x \in E \):

\[
0 \leq \mu_A(x) + \nu_A(x) \leq 1.
\]

For every two IFSs \( A \) and \( B \) a variety of relations and operations have been defined (see, e.g. [1]). The most important of them are:

\[
\begin{align*}
A \subset B & \iff (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x)); \\
A = B & \iff (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)); \\
A \cap B & = \{ (x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) | x \in E \}; \\
A \cup B & = \{ (x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) | x \in E \}; \\
\overline{A} & = \{ (x, \nu_A(x), \mu_A(x)) | x \in E \}.
\end{align*}
\]

Let \( A \) be an IFS. The level operator \( N_{\alpha,\beta} \) is defined by (see [1]):

\[
N_{\alpha,\beta}(A) = \{ (x, \mu_A(x), \nu_A(x)) | (x \in E) \& (\mu_A(x) \geq \alpha) \& (\nu_A(x) \leq \beta) \}.
\]

We shall extend this definition, following the idea from [2], where some extended modal operators are defined. Four of them, that we will use below, are

\[
\begin{align*}
H_B(A) & = \{ (x, \mu_B(x), \mu_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)) | x \in E \}, \\
H_B^*(A) & = \{ (x, \mu_B(x), \mu_A(x), \nu_A(x) + \nu_B(x).(1 - \mu_B(x).\mu_A(x) - \nu_A(x))) | x \in E \}, \\
J_B(A) & = \{ (x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\nu_A(x)) | x \in E \}, \\
J_B^*(A) & = \{ (x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\nu_A(x)) | x \in E \},
\end{align*}
\]

Let \( A \) and \( B \) be two IFSs. Then

\[
\begin{align*}
N_B(A) & = \{ (x, \mu_A(x), \nu_A(x)) | (x \in E) \& (\mu_A(x) \geq \mu_B(x)) \& (\nu_A(x) \leq \nu_B(x)) \}, \\
N_B^*(A) & = \{ (x, \mu_A(x), \nu_A(x)) | (x \in E) \& (\mu_A(x) \leq \mu_B(x)) \& (\nu_A(x) \geq \nu_B(x)) \}.
\end{align*}
\]
It is very important to note that sets $N_{\alpha\beta}(A), N_B(A)$ and $N^*_B(A)$ are IFSs, but now over new universes, that are subsets of universe $E$.

Obviously, for each IFS $A$:

$$A = N_{O^*}(A) = N^*_E(A),$$

where

$$O^* = \{\langle x, 0, 1 \rangle | x \in E\}, \text{ and } E^* = \{\langle x, 1, 0 \rangle | x \in E\}.$$

**Theorem 1:** For every two IFSs $A$ and $B$:

a) $A = N_B(A)$ iff $B \subset A$,

b) $A = N^*_B(A)$ iff $A \subset B$,

c) $N_B(A) = \emptyset$ iff $(\forall x \in E)(\mu_A(x) < \mu_B(x)) \cup (\nu_A(x) > \nu_B(x))$,

d) $N^*_B(A) = \emptyset$ iff $(\forall x \in E)(\mu_A(x) > \mu_B(x)) \cup (\nu_A(x) < \nu_B(x))$,

e) $A \subset N_B(A) \cup N^*_B(A)$.

Following [3] we shall introduce the operators

$$P_B(A) = \{\langle x, \max(\mu_B(x), \mu_A(x)), \min(\nu_B(x), \nu_A(x)) \rangle | x \in E\},$$

$$Q_B(A) = \{\langle x, \min(\mu_B(x), \mu_A(x)), \max(\nu_B(x), \nu_A(x)) \rangle | x \in E\},$$

$$C(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y) \rangle | x \in E\},$$

$$\mathcal{I}(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y) \rangle | x \in E\}.$$

**Theorem 2:** For every three IFSs $A, B, C$:

a) $N_C(N_B(A)) = N_{B\cup C}(A)$,

b) $N^*_C(N_B(A)) = N^*_{B\cup C}(A)$,

c) $N_C(N_B(A)) = N_{P_C(B)}(A)$,

d) $N^*_C(N_B(A)) = N^*_{P_C(B)}(A)$,

e) $N_C(N^*_B(A)) = N_{Q_C(B)}(A)$,

f) $N^*_C(N^*_B(A)) = N^*_{Q_C(B)}(A)$.

**Proof:** a) Let the three IFSs $A, B, C$ be given. Then

$$N_C(N_B(A)) = N_C(\{\langle x, \mu_A(x), \nu_A(x) \rangle | (x \in E) \& (\mu_A(x) \geq \mu_B(x)) \& (\nu_A(x) \leq \nu_B(x))\})$$

$$= \{\langle x, \mu_A(x), \nu_A(x) \rangle | (x \in E) \& (\mu_A(x) \geq \mu_B(x))$$

$$\& (\nu_A(x) \geq \mu_C(x)) \& (\nu_A(x) \leq \nu_B(x)) \& (\nu_A(x) \leq \nu_C(x))\}$$

$$= \{\langle x, \mu_A(x), \nu_A(x) \rangle | (x \in E) \& (\mu_A(x) \geq \max(\mu_B(x), \mu_C(x))$$

$$\& (\nu_A(x) \leq \min(\nu_B(x), \nu_C(x)))\} = N_{B\cup C}(A).$$

The next assertions are proved analogously and we will omit their proofs.

**Theorem 3:** For every three IFSs $A, B, C$:

a) $N_C(A \cap B) = N_C(A) \cap^* N_C(B)$,

b) $N^*_C(A \cup B) = N^*_C(A) \cup^* N^*_C(B)$,

where here and below $\cup^*$ and $\cap^*$ are set theoretic operations “union” and “intersection”.

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Theorem 4: For every two IFSs $A$ and $B$: $N_B^*(A) = \overline{N_B(A)}$.

Theorem 5: For every two IFSs $A$ and $B$:

a) $N_C(B)(A) = \bigcap_{C \subset C(B)}^* N_C(A)$,

b) $N_{\mathcal{I}(B)}^*(A) = \bigcup_{\mathcal{I}(B) \subset C}^* N_B^*(A)$.

Theorem 6: For every two IFSs $A$ and $B$:

a) $N_C(\mathcal{I}(A)) = \bigcup_{B \subset \mathcal{I}(A)}^* N_C(B)$,

b) $N_C^*(\mathcal{I}(A)) = \bigcap_{C \subset C(A)} N_C(B)$.

Theorem 7: For every three IFSs $A, B, C$:

a) $N_{\mathcal{I}(B)}^*(A) \subset^* N_B^*(A)$,

b) $N_{\mathcal{I}(B)}^*(A) \supset^* N_B^*(A)$,

c) $N_{\mathcal{I}(B)}^*(A) \subset^* N_B^*(A)$,

d) $N_{\mathcal{I}(B)}^*(A) \supset^* N_B^*(A)$,

e) $N_{\mathcal{I}(B)}^*(A) \subset^* N_B^*(A)$,

f) $N_{\mathcal{I}(B)}^*(A) \supset^* N_B^*(A)$,

g) $N_{\mathcal{I}(B)}^*(A) \subset^* N_B^*(A)$,

h) $N_{\mathcal{I}(B)}^*(A) \supset^* N_B^*(A)$,

where $\subset^*$ and $\supset^*$ are set theoretical relations “inclusion”.

Theorem 8: For every two IFSs $A$ and $B$:

a) $N_C(B)(A) = \inf_{y \in E} N_{\mu_C(y), \inf_{y \in E} \nu_C(y)}(A)$,

b) $N_{\mathcal{I}(B)}^*(A) = \inf_{y \in E} N_{\mu_C(y), \sup_{y \in E} \nu_C(y)}(A)$.

Theorem 9: For every three IFSs $A, B, C$:

a) $N_{\mathcal{I}(B)}^*(A) = N_C(A) \cap^* N_B(A)$,

b) $N_{\mathcal{I}(B)}^*(A) = N_C(A) \cup^* N_B(A)$.

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References

