Atanassov’s intuitionistic fuzzy translations of intuitionistic fuzzy $H$-ideals in $BCK/BCI$-algebras

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Abstract: In this paper, the concepts of intuitionistic fuzzy translation to intuitionistic fuzzy $H$-ideals in $BCK/BCI$-algebras are introduced. The notion of intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy $H$-ideals with several related properties are investigated. Also the relationships between intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy $H$-ideals are investigated.

Keywords: Intuitionistic fuzzy ideal, Intuitionistic fuzzy $H$-ideal, Intuitionistic fuzzy translation, Intuitionistic fuzzy extension, Intuitionistic fuzzy multiplication.

AMS Classification: 06F35, 03G25, 08A72.

1 Introduction

After the introduction of fuzzy sets by Zadeh [25], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [2-4] is one among them. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degrees of membership and of nonmembership. Both degrees belong to the interval $[0, 1]$, and their sum should not exceed 1.

$BCK$-algebras and $BCI$-algebras are two important classes of logical algebras introduced by Imai and Iseki [6, 7, 10]. It is known that the class of $BCK$-algebra is a proper subclass of

The concept of fuzzy translations in fuzzy subalgebras and ideals in $BCK/BCI$-algebras has been discussed respectively by Lee et al. [15] and Jun [13]. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. Motivated by this, in [22], the authors have studied fuzzy translations of fuzzy $H$-ideals in $BCK/BCI$-algebras. They also extend this study from fuzzy translations to intuitionistic fuzzy translations [23] in $BCK/BCI$-algebras.

In this paper, intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy $H$-ideals in $BCK/BCI$-algebras are discussed. Relations among intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy $H$-ideals in $BCK/BCI$-algebras are also investigated.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

By a $BCI$-algebra we mean an algebra $(X, \ast, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

(i) $(x \ast y) \ast (x \ast z) = (z \ast y) \ast (x \ast y) = 0$

(ii) $(x \ast (x \ast y)) \ast y = 0$

(iii) $x \ast x = 0$

(iv) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$.

We can define a partial ordering “$\leq$” by $x \leq y$ if and only if $x \ast y = 0$.

If a $BCI$-algebra $X$ satisfies $0 \ast x = 0$ for all $x \in X$, then we say that $X$ is a $BCK$-algebra.

Any $BCK$-algebra $X$ satisfies the following axioms for all $x, y, z \in X$:

(1) $(x \ast y) \ast z = (x \ast z) \ast y$

(2) $((x \ast z) \ast (y \ast z)) \ast (x \ast y) = 0$

(3) $x \ast 0 = x$

(4) $x \ast y = 0 \Rightarrow (x \ast z) \ast (y \ast z) = 0, (z \ast y) \ast (z \ast x) = 0$.

Throughout this paper, $X$ always means a $BCK/BCI$-algebra without any specification.

A non-empty subset $S$ of $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for any $x, y \in S$. A nonempty subset $I$ of $X$ is called an ideal of $X$ if it satisfies

$(I_1)$ $0 \in I$ and

$(I_2)$ $x \ast y \in I$ and $y \in I$ imply $x \in I$.

A non-empty subset $I$ of $X$ is said to be a $H$-ideal [14] of $X$ if it satisfies $(I_1)$ and

$(I_3)$ $x \ast (y \ast z) \in I$ and $y \in I$ imply $x \ast z \in I$ for all $x, y, z \in X$. 

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A BCI-algebra is said to be associative [5] if \((x \ast y) \ast z = x \ast (y \ast z)\) for all \(x, y, z \in X\).

A fuzzy set \(A = \{x, \mu_A(x) \geq x \in X\}\) in \(X\) is called a fuzzy subalgebra of \(X\) if it satisfies the inequality \(\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}\) for all \(x, y \in X\).

A fuzzy set \(A = \{x, \mu_A(x) > x \in X\}\) in \(X\) is called a fuzzy ideal [1, 24] of \(X\) if it satisfies

\[
\begin{align*}
(F_1) & \quad \mu_A(0) \geq \mu_A(x) \\
(F_2) & \quad \mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(y)\}\end{align*}
\]

for all \(x, y \in X\).

A fuzzy set \(A = \{x, \mu_A(x) > x \in X\}\) in \(X\) is called a fuzzy \(H\)-ideal [14, 26] of \(X\) if it satisfies \((F_1)\) and \(\mu_A(x \ast z) \geq \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}\) for all \(x, y, z \in X\).

An intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}\) in \(X\) is called an intuitionistic fuzzy subalgebra [12] of \(X\) if it satisfies the following two conditions

\[
\begin{align*}
(F_4) & \quad \mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\} \\
(F_5) & \quad \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}\end{align*}
\]

for all \(x, y \in X\).

An intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}\) in \(X\) is called an intuitionistic fuzzy ideal [12] of \(X\) if it satisfies

\[
\begin{align*}
(F_6) & \quad \mu_A(0) \geq \mu_A(x), \nu_A(0) \leq \nu_A(x), \\
(F_7) & \quad \mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(y)\} \\
(F_8) & \quad \nu_A(x) \leq \max\{\nu_A(x \ast y), \nu_A(y)\}\end{align*}
\]

for all \(x, y \in X\).

An intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}\) in \(X\) is called an intuitionistic fuzzy \(H\)-ideal [20] of \(X\) if it satisfies \((F_6)\) and

\[
\begin{align*}
(F_9) & \quad \mu_A(x \ast z) \geq \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\} \\
(F_{10}) & \quad \nu_A(x \ast z) \leq \max\{\nu_A(x \ast (y \ast z)), \nu_A(y)\}\end{align*}
\]

for all \(x, y, z \in X\).

### 3 Main Results

For the sake of simplicity, we shall use the symbol \(A = (\mu_A, \nu_A)\) for the intuitionistic fuzzy subset \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}\). Throughout this paper, we take \(\mathfrak{F} := \inf\{\nu_A(x) | x \in X\}\) for any intuitionistic fuzzy set \(A = (\mu_A, \nu_A)\) of \(X\).

**Definition 3.1** [23] Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy subset of \(X\) and let \(\alpha \in [0, \mathfrak{F}]\). An object having the form \(A^\alpha_\alpha = ((\mu_A)_\alpha^\alpha, (\nu_A)_\alpha^\alpha)\) is called an intuitionistic fuzzy \(\alpha\)-translation of \(A\) if \((\mu_A)_\alpha^\alpha(x) = \mu_A(x) + \alpha\) and \((\nu_A)_\alpha^\alpha(x) = \nu_A(x) - \alpha\) for all \(x \in X\).

**Theorem 3.2** If \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy \(H\)-ideal of \(X\), then the intuitionistic fuzzy \(\alpha\)-translation \(A^\alpha_\alpha = ((\mu_A)_\alpha^\alpha, (\nu_A)_\alpha^\alpha)\) of \(A\) is an intuitionistic fuzzy \(H\)-ideal of \(X\) for all \(\alpha \in [0, \mathfrak{F}]\).

**Proof:** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy \(H\)-ideal of \(X\) and \(\alpha \in [0, \mathfrak{F}]\). Then

\[
(\mu_A)_\alpha^\alpha(0) = \mu_A(0) + \alpha \geq \mu_A(x) + \alpha = (\mu_A)_\alpha^\alpha(x)\]

and

\[
(\nu_A)_\alpha^\alpha(0) = \nu_A(0) - \alpha \leq \nu_A(x) - \alpha = (\nu_A)_\alpha^\alpha(x)\]
Theorem 3.3 \[ \text{Let } (\mu_A)_\alpha^T(x * z) \text{ for all } x \in X. \] Now,

\[
\begin{align*}
(\mu_A)_\alpha^T(x * z) &= \mu_A(x * z) + \alpha \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\} + \alpha \\
&= \min\{\mu_A(x * (y * z)) + \alpha, \mu_A(y) + \alpha\} \\
&= \min\{(\mu_A)_\alpha^T(x * (y * z)), (\mu_A)_\alpha^T(y)\}
\end{align*}
\]

and \(\begin{align*}
(\nu_A)_\alpha^T(x * z) &= \nu_A(x * z) - \alpha \leq \max\{\nu_A(x * (y * z)), \nu_A(y)\} - \alpha \\
&= \max\{\nu_A(x * (y * z)) - \alpha, \nu_A(y) - \alpha\} \\
&= \max\{(\nu_A)_\alpha^T(x * (y * z)), (\nu_A)_\alpha^T(y)\}
\end{align*}\)

for all \(x, y, z \in X\). Hence, the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) is an intuitionistic fuzzy \(H\)-ideal of \(X\).

\[ \blacksquare \]

Theorem 3.4 \textit{Let } \(A = (\mu_A, \nu_A)\text{ be an intuitionistic fuzzy subset of } X \text{ such that the intuitionistic fuzzy } \alpha\text{-translation } A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T) \text{ of } A \text{ is an intuitionistic fuzzy } H\text{-ideal of } X \text{ for some } \alpha \in [0, \mathfrak{T}]\). \textit{Then } A = (\mu_A, \nu_A) \text{ is an intuitionistic fuzzy } H\text{-ideal of } X. \]

\textbf{Proof:} Assume that \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) is an intuitionistic fuzzy \(H\)-ideal of \(X\) for some \(\alpha \in [0, \mathfrak{T}]\). Let \(x, y \in X\), we have

\[
\begin{align*}
\mu_A(0) + \alpha &= (\mu_A)_\alpha^T(0) \geq (\mu_A)_\alpha^T(x) = \mu_A(x) + \alpha \\
\nu_A(0) - \alpha &= (\nu_A)_\alpha^T(0) \leq (\nu_A)_\alpha^T(x) = \nu_A(x) - \alpha
\end{align*}
\]

which implies \(\mu_A(0) \geq \mu_A(x)\) and \(\nu_A(0) \leq \nu_A(x)\). Now we have

\[
\begin{align*}
\mu_A(x * z) + \alpha &= (\mu_A)_\alpha^T(x * z) \geq \min\{\mu_A(x * (y * z)), (\mu_A)_\alpha^T(y)\} \\
&= \min\{\mu_A(x * (y * z)) + \alpha, \mu_A(y) + \alpha\} \\
&= \min\{(\mu_A)_\alpha^T(x * (y * z)), \mu_A(y)\} + \alpha
\end{align*}
\]

and \(\begin{align*}
\nu_A(x * z) - \alpha &= (\nu_A)_\alpha^T(x * z) \leq \max\{\nu_A(x * (y * z)), (\nu_A)_\alpha^T(y)\} \\
&= \max\{\nu_A(x * (y * z)) - \alpha, \nu_A(y) - \alpha\} \\
&= \max\{(\nu_A)_\alpha^T(x * (y * z)), \nu_A(y)\} - \alpha
\end{align*}\)

which implies that \(\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}\) and \(\nu_A(x * z) \leq \min\{\nu_A(x * (y * z)), \nu_A(y)\}\) for all \(x, y, z \in X\). Hence, \(A = (\mu_A, \nu_A)\) is an intuitionistic fuzzy \(H\)-ideal of \(X\). \[ \blacksquare \]

We now discuss the relation between intuitionistic fuzzy subalgebras and intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) for the intuitionistic fuzzy \(H\)-ideals.

Theorem 3.4 \(\text{If the intuitionistic fuzzy } \alpha\text{-translation } A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T) \text{ of } A \text{ is an intuitionistic fuzzy } H\text{-ideal of } X \text{ for all } \alpha \in [0, \mathfrak{T}] \text{ then it must be an intuitionistic fuzzy subalgebra of } X. \)

\textbf{Proof:} Let the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) be an intuitionistic fuzzy \(H\)-ideal of \(X\). Then we have \(\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), (\mu_A)_\alpha^T(y)\}\) and

\[
\begin{align*}
(\nu_A)_\alpha^T(x * z) &= \nu_A(x * z) - \alpha \leq \max\{\nu_A(x * (y * z)), \nu_A(y)\} - \alpha \\
&= \max\{\nu_A(x * (y * z)) - \alpha, \nu_A(y) - \alpha\} \\
&= \max\{(\nu_A)_\alpha^T(x * (y * z)), (\nu_A)_\alpha^T(y)\}
\end{align*}
\]

for all \(x, y, z \in X\). Hence, the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) is an intuitionistic fuzzy \(H\)-ideal of \(X\). \[ \blacksquare \]
\[(\nu_A)_\alpha^T(x \ast z) \leq \max\{(\nu_A)_\alpha^T(x \ast (y \ast z)), (\nu_A)_\alpha^T(y)\} \text{ for all } x, y, z \in X. \text{ Substituting } y \text{ for } z \text{ we get}
\]
\[
(\mu_A)_\alpha^T(x \ast y) \geq \min\{(\mu_A)_\alpha^T(x \ast (y \ast y)), (\mu_A)_\alpha^T(y)\} = \min\{(\mu_A)_\alpha^T(x \ast 0), (\mu_A)_\alpha^T(y)\} = \min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}
\]
\[
\text{and } (\nu_A)_\alpha^T(x \ast y) \leq \max\{(\nu_A)_\alpha^T(x \ast (y \ast y)), (\nu_A)_\alpha^T(y)\} = \max\{(\nu_A)_\alpha^T(x \ast 0), (\nu_A)_\alpha^T(y)\} = \max\{(\nu_A)_\alpha^T(x), (\nu_A)_\alpha^T(y)\}.
\]

Therefore, \(A^T_\alpha\) is an intuitionistic fuzzy subalgebra of \(X\).

\[\square\]

**Proposition 1** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy subset of \(X\) such that the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) is an intuitionistic fuzzy ideal of \(X\) for \(\alpha \in [0, \Xi]\).

If \((x \ast a) \ast b = 0\) for all \(a, b, x \in X\), then \((\mu_A)_\alpha^T(x) \geq \min\{(\mu_A)_\alpha^T(a), (\mu_A)_\alpha^T(b)\}\) and \((\nu_A)_\alpha^T(x) \leq \max\{(\nu_A)_\alpha^T(a), (\nu_A)_\alpha^T(b)\}\).

\[\text{Proof: Let } a, b, x \in X \text{ be such that } (x \ast a) \ast b = 0. \text{ Then}
\]
\[
(\mu_A)_\alpha^T(x) \geq \min\{(\mu_A)_\alpha^T(x \ast a), (\mu_A)_\alpha^T(a)\} \\
\geq \min\{\min\{(\mu_A)_\alpha^T((x \ast a) \ast b), (\mu_A)_\alpha^T(b)\}, (\mu_A)_\alpha^T(a)\} \\
= \min\{\min\{(\mu_A)_\alpha^T(0), (\mu_A)_\alpha^T(b)\}, (\mu_A)_\alpha^T(a)\} \\
= \min\{(\mu_A)_\alpha^T(b), (\mu_A)_\alpha^T(a)\} \text{ since } (\mu_A)_\alpha^T(0) \geq (\mu_A)_\alpha^T(b) \\
= \min\{(\mu_A)_\alpha^T(a), (\mu_A)_\alpha^T(b)\}
\]
\[
\text{and } (\nu_A)_\alpha^T(x) \leq \max\{(\nu_A)_\alpha^T(x \ast a), (\nu_A)_\alpha^T(a)\} \\
\leq \max\{\max\{(\nu_A)_\alpha^T((x \ast a) \ast b), (\nu_A)_\alpha^T(b)\}, (\nu_A)_\alpha^T(a)\} \\
= \max\{\max\{(\nu_A)_\alpha^T(0), (\nu_A)_\alpha^T(b)\}, (\nu_A)_\alpha^T(a)\} \\
= \max\{(\nu_A)_\alpha^T(b), (\nu_A)_\alpha^T(a)\} \text{ since } (\nu_A)_\alpha^T(0) \leq (\nu_A)_\alpha^T(b) \\
= \max\{(\nu_A)_\alpha^T(a), (\nu_A)_\alpha^T(b)\}.
\]

The proof is complete. \[\square\]

The following can easily be proved by induction.

**Corollary 1** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy subset of \(X\) such that the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) is an intuitionistic fuzzy ideal of \(X\) for \(\alpha \in [0, \Xi]\). If \((\cdots ((x \ast a_1) \ast a_2) \ast \cdots) \ast a_n = 0\) for all \(x, a_1, a_2, \ldots, a_n \in X\), then \((\mu_A)_\alpha^T(x) \geq \min\{(\mu_A)_\alpha^T(a_1), (\mu_A)_\alpha^T(a_2), \ldots, (\mu_A)_\alpha^T(a_n)\}\) and \((\nu_A)_\alpha^T(x) \leq \max\{(\nu_A)_\alpha^T(a_1), (\nu_A)_\alpha^T(a_2), \ldots, (\nu_A)_\alpha^T(a_n)\}\).

We now give a condition for intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)\) of \(A\) which is an intuitionistic fuzzy ideal of \(X\) to be an intuitionistic fuzzy \(H\)-ideal of \(X\).
Theorem 3.5 Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A$ is an intuitionistic fuzzy ideal of $X$ for $\alpha \in [0, \Xi]$. If it satisfies the condition $(\mu_A)^T_\alpha(x+y) \geq (\mu_A)^T_\alpha(x)$ and $(\nu_A)^T_\alpha(x+y) \leq (\nu_A)^T_\alpha(x)$ for all $x, y \in X$, then the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$.

Proof: Let the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ be an intuitionistic fuzzy ideal of $X$. For any $x, y, z \in X$, we have

\[
(\mu_A)^T_\alpha(x * z) \geq \min\{((\mu_A)^T_\alpha((x * z) * (y * z)), (\mu_A)^T_\alpha(y * z))
= \min\{((\mu_A)^T_\alpha((x * y) * z), (\mu_A)^T_\alpha(y))
\geq \min\{((\mu_A)^T_\alpha(x * (y * z)), (\mu_A)^T_\alpha(y))
\]

and \[
(\nu_A)^T_\alpha(x * z) \leq \max\{((\nu_A)^T_\alpha((x * z) * (y * z)), (\nu_A)^T_\alpha(y * z))
= \max\{((\nu_A)^T_\alpha((x * y) * z), (\nu_A)^T_\alpha(y))
\leq \max\{((\nu_A)^T_\alpha(x * (y * z)), (\nu_A)^T_\alpha(y))
\}
\]

Hence, the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$ for some $\alpha \in [0, \Xi]$. □

Theorem 3.6 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subset of associative BCK/BCI-algebra $X$ such that the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A$ is an intuitionistic fuzzy ideal of $X$ for $\alpha \in [0, \Xi]$, then the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$.

Proof: Let the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ be an intuitionistic fuzzy ideal of $X$. For any $x, y, z \in X$, we have

\[
(\mu_A)^T_\alpha(x * z) \geq \min\{((\mu_A)^T_\alpha((x * z) * y), (\mu_A)^T_\alpha(y))
= \min\{((\mu_A)^T_\alpha((x * y) * z), (\mu_A)^T_\alpha(y))
\geq \min\{((\mu_A)^T_\alpha(x * (y * z)), (\mu_A)^T_\alpha(y))
\]

and \[
(\nu_A)^T_\alpha(x * z) \leq \max\{((\nu_A)^T_\alpha((x * z) * y), (\nu_A)^T_\alpha(y))
= \max\{((\nu_A)^T_\alpha((x * y) * z), (\nu_A)^T_\alpha(y))
\leq \max\{((\nu_A)^T_\alpha(x * (y * z)), (\nu_A)^T_\alpha(y))
\}
\]

Hence, the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$. □

Theorem 3.7 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$ for $\alpha \in [0, \Xi]$, then the sets $T_{\mu_A} = \{x \mid x \in X \text{ and } (\mu_A)^T_\alpha(x) = (\mu_A)^T_\alpha(0)\}$ and $T_{\nu_A} = \{x \mid x \in X \text{ and } (\nu_A)^T_\alpha(x) = (\nu_A)^T_\alpha(0)\}$ are $H$-ideals of $X$.

Proof: Suppose that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideal of $X$. Then $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy $H$-ideal of $X$. Obviously $0 \in T_{\mu_A}, T_{\nu_A}$. Let $x, y, z \in X$ be such that $x * (y * z) \in T_{\mu_A}$ and $y \in T_{\mu_A}$. Then $(\mu_A)^T_\alpha(x * (y * z)) = (\mu_A)^T_\alpha(0) = (\mu_A)^T_\alpha(y)$ and so
(μA)αT(x * z) ≥ min\{((μA)αT(x * (y * z)), (μA)αT(y)) = (μA)αT(0). Since, (μA)αT is a fuzzy H-ideal of X, we conclude that (μA)αT(x * z) = (μA)αT(0). This implies μA(x * z) + α = μA(0) + α or, μA(x * z) = μA(0) so that x * z ∈ TμA. Therefore, TμA is a H-ideal of X.

Again, let a, b, c ∈ X be such that a * (b * c) ∈ TνA and b ∈ TνA. Then (νA)αT(a * (b * c)) = (νA)αT(0) = (νA)αT(b) and so (νA)αT(a * c) ≤ max\{(νA)αT(a * (b * c)), (νA)αT(b)\} = (νA)αT(0). Since, (νA)αT is a fuzzy H-ideal of X, we conclude that (νA)αT(a * c) = (νA)αT(0). This implies νA(a * c) + α = νA(0) + α or, νA(a * c) = νA(0) so that a * c ∈ TνA. Therefore, TνA is a H-ideal of X.

Proposition 2 Let the intuitionistic fuzzy α-translation AαT = ((μA)αT, (νA)αT) of A be an intuitionistic fuzzy H-ideal of X for α ∈ [0, Σ]. If x ≤ y then (μA)αT(x) ≥ (μA)αT(y) and (νA)αT(x) ≤ (νA)αT(y), that is, (μA)αT is order-reversing and (νA)αT is order-preserving.

Proof: Let x, y ∈ X be such that x ≤ y. Then x * y = 0 and hence
\[
(μA)αT(x) = (μA)αT(x * 0) ≥ \min\{(μA)αT(x * (y * 0)), (μA)αT(y)) = \min\{(μA)αT(x * y), (μA)αT(y)\} = (μA)αT(y)
\]
and
\[
(νA)αT(x) = (νA)αT(x * 0) ≤ \max\{(νA)αT(x * (y * 0)), (νA)αT(y)) = \max\{(νA)αT(x * y), (νA)αT(y)\} = (νA)αT(y).
\]
This completes the proof.

The characterization of intuitionistic fuzzy α-translation AαT = ((μA)αT, (νA)αT) of A = (μA, νA) are given by the following theorem.

Theorem 3.8 Let A = (μA, νA) be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation AαT = ((μA)αT, (νA)αT) of A is an intuitionistic fuzzy ideal of X for α ∈ [0, Σ], then the following assertions are equivalent:

(i) AαT is an intuitionistic fuzzy H-ideal of X,
(ii) (μA)αT(x * y) ≥ (μA)αT(x * (0 * y)) and (νA)αT(x * y) ≤ (νA)αT(x * (0 * y)) for all x, y ∈ X,
(iii) (μA)αT((x * y) * z) ≥ (μA)αT(x * (y * z)) and (νA)αT((x * y) * z) ≤ (νA)αT(x * (y * z)) for all x, y, z ∈ X.

Proof: Assume that AαT = ((μA)αT, (νA)αT) is an intuitionistic fuzzy ideal of X. Then (μA)αT and (νA)αT are fuzzy H-ideals of X.

(i) ⇒ (ii) Let AαT is an intuitionistic fuzzy H-ideal of X. Then for all x, y ∈ X we have (μA)αT(x * y) ≥ \min\{(μA)αT(x * (0 * y)), (μA)αT(0)) = (μA)αT(x * (0 * y)) and (νA)αT(x * y) ≤ \max\{(νA)αT(x * (0 * y)), (νA)αT(0)) = (νA)αT(x * (0 * y)). Therefore, the inequality (ii) is satisfied.

(ii) ⇒ (iii) Assume that (ii) is satisfied. For all x, y, z ∈ X, we have (((x * y) * (0 * z)) * (x * (y * z))) * (0 * z) ≤ ((y * z) * (0 * z)) = (x * (y * z)) * (0 * z) = (0 * z) * (0 * z) = 0. It follows from Proposition 2 that (μA)αT(((x * y) * (0 * z)) * (x * (y * z))) ≥ (μA)αT((x * (0 * z)) and (νA)αT(((x * y) * (0 * z)) * (x * (y * z))) ≤ (νA)αT(0). Since (μA)αT and (νA)αT are fuzzy

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Therefore, we have \((\mu_A)^T_\alpha((x * y) * (0 * z)) * (x * (y * z)) = (\mu_A)^T_\alpha(0)\) and \((\nu_A)^T_\alpha((x * y) * (0 * z)) * (x * (y * z)) = (\nu_A)^T_\alpha(0)\).

Using (ii) we get

\[
(\mu_A)^T_\alpha((x * y) * z) \geq (\mu_A)^T_\alpha((x * y) * (0 * z)) = \min\{(\mu_A)^T_\alpha(((x * y) * (0 * z)) * (x * (y * z))), (\mu_A)^T_\alpha(x * (y * z))\} = \min\{(\mu_A)^T_\alpha(0), (\mu_A)^T_\alpha(x * (y * z))\} = (\mu_A)^T_\alpha(x * (y * z))
\]

and

\[
(\nu_A)^T_\alpha((x * y) * z) \leq (\nu_A)^T_\alpha((x * y) * (0 * z)) = \max\{(\nu_A)^T_\alpha(((x * y) * (0 * z)) * (x * (y * z))), (\nu_A)^T_\alpha(x * (y * z))\} = \max\{(\nu_A)^T_\alpha(0), (\nu_A)^T_\alpha(x * (y * z))\} = (\nu_A)^T_\alpha(x * (y * z)).
\]

Therefore, inequality (iii) is also satisfied.

(iii) \(\Rightarrow\) (i) Assume that (iii) is valid. For all \(x, y, z \in X\), we have

\[
(\mu_A)^T_\alpha(x * z) \geq \min\{(\mu_A)^T_\alpha((x * z) * y), (\mu_A)^T_\alpha(y)\} = \min\{(\mu_A)^T_\alpha((x * y) * z), (\mu_A)^T_\alpha(y)\} \geq \min\{(\mu_A)^T_\alpha(x * (y * z)), (\mu_A)^T_\alpha(y)\}
\]

and

\[
(\nu_A)^T_\alpha(x * z) \leq \max\{(\nu_A)^T_\alpha((x * z) * y), (\nu_A)^T_\alpha(y)\} = \max\{(\nu_A)^T_\alpha((x * y) * z), (\nu_A)^T_\alpha(y)\} \leq \max\{(\nu_A)^T_\alpha(x * (y * z)), (\nu_A)^T_\alpha(y)\}.
\]

Therefore, \(A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)\) is an intuitionistic fuzzy \(H\)-ideal of \(X\). Hence, the assertion (i) holds. The proof is complete.

Next we give another characterization of intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)\) of \(A = (\mu_A, \nu_A)\) in the following theorem.

**Theorem 3.9** Let \(A = (\mu_A, \nu_A)\) be an intuitionistic fuzzy subset of \(X\) such that the intuitionistic fuzzy \(\alpha\)-translation \(A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)\) of \(A\) is an intuitionistic fuzzy ideal of \(X\), then the following assertions are equivalent:

1. \(A^T_\alpha\) is an intuitionistic fuzzy \(H\)-ideal of \(X\),

2. \((\mu_A)^T_\alpha((x * z) * y) \geq (\mu_A)^T_\alpha((x * z) * (0 * y))\) and \((\nu_A)^T_\alpha((x * z) * y) \leq (\nu_A)^T_\alpha((x * z) * (0 * y))\) for all \(x, y, z \in X\),

3. \((\mu_A)^T_\alpha(x * y) \geq \min\{(\mu_A)^T_\alpha((x * z) * (0 * y)), (\mu_A)^T_\alpha(z)\}\) and \((\nu_A)^T_\alpha(x * y) \leq \max\{(\nu_A)^T_\alpha((x * z) * (0 * y)), (\nu_A)^T_\alpha(z)\}\) for all \(x, y, z \in X\).
Proof: (i) ⇒ (ii) Same as above theorem. 

(ii) ⇒ (iii) Assume that (ii) is valid. For all \( x, y, z \in X \), we have \( (\mu_A)^T(x*y) \geq \min\{(\mu_A)^T((x*y)*z), (\mu_A)^T(z)\} \) and \( (\nu_A)^T(x*y) \leq \max\{(\nu_A)^T((x*y)*z), (\nu_A)^T(z)\} \). 

Putting \( z = 0 \), \( x = y \), \( x = z \), we have \( (\mu_A)^T(x*y) = \min\{\mu_A(x*y), \mu_A(0)\} \) and \( (\nu_A)^T(x*y) = \max\{\nu_A(x*y), \nu_A(0)\} \) if the following assertions are valid:

(i) \( B \) is an intuitionistic fuzzy \( H \)-ideal extension of \( A \).

(ii) If \( A \) is an intuitionistic fuzzy \( H \)-ideal of \( X \), then \( B \) is an intuitionistic fuzzy \( H \)-ideal of \( X \).

From the definition of intuitionistic fuzzy \( \alpha \)-translation, we get \( (\mu_A)^T(x) = \mu_A(x) + \alpha \) and \( (\nu_A)^T(x) = \nu_A(x) - \alpha \) for all \( x \in X \). Therefore, we have the following theorem.

**Theorem 3.12** Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy \( H \)-ideal of \( X \) and \( \alpha \in [0, 3] \). Then the intuitionistic fuzzy \( \alpha \)-translation \( A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha}) \) of \( A \) is an intuitionistic fuzzy \( H \)-ideal extension of \( A \).

An intuitionistic fuzzy \( H \)-ideal extension of an intuitionistic fuzzy \( H \)-ideal \( A \) may not be represented as an intuitionistic fuzzy \( \alpha \)-translation of \( A \), that is, the converse of Theorem 3.12 is not true in general as seen in the following example.

**Example 1** Let \( X = \{0, 1, 2, 3, 4\} \) be a \( BCK \)-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<tr>
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<td>2</td>
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</tr>
<tr>
<td>3</td>
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<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy subset of \( X \) defined by

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_A )</td>
<td>0.72</td>
<td>0.64</td>
<td>0.55</td>
<td>0.41</td>
<td>0.41</td>
</tr>
<tr>
<td>( \nu_A )</td>
<td>0.27</td>
<td>0.32</td>
<td>0.40</td>
<td>0.55</td>
<td>0.55</td>
</tr>
</tbody>
</table>
Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy $H$-ideal of $X$. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of $X$ defined by

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_B$</td>
<td>0.74</td>
<td>0.70</td>
<td>0.58</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>$\nu_B$</td>
<td>0.25</td>
<td>0.28</td>
<td>0.37</td>
<td>0.51</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Then $B$ is an intuitionistic fuzzy $H$-ideal extension of $A$. But it is not the intuitionistic fuzzy $\alpha$-translation $A^\alpha_T = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A$ for all $\alpha \in [0, 1]$.

Clearly, the intersection of intuitionistic fuzzy $H$-ideal extensions of an intuitionistic fuzzy $H$-ideal $A$ of $X$ is an intuitionistic fuzzy $H$-ideal extension of $A$. But the union of intuitionistic fuzzy $H$-ideal extensions of an intuitionistic fuzzy $H$-ideal $A$ of $X$ is not an intuitionistic fuzzy $H$-ideal extension of $A$ as seen in the following example.

**Example 2** Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ defined by

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_A$</td>
<td>0.63</td>
<td>0.41</td>
<td>0.41</td>
<td>0.41</td>
<td>0.41</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td>0.35</td>
<td>0.58</td>
<td>0.58</td>
<td>0.58</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy $H$-ideal of $X$. Let $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be intuitionistic fuzzy subsets of $X$ defined by

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_B$</td>
<td>0.76</td>
<td>0.66</td>
<td>0.43</td>
<td>0.43</td>
<td>0.66</td>
</tr>
<tr>
<td>$\nu_B$</td>
<td>0.22</td>
<td>0.33</td>
<td>0.54</td>
<td>0.54</td>
<td>0.33</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_C$</td>
<td>0.78</td>
<td>0.51</td>
<td>0.57</td>
<td>0.51</td>
<td>0.51</td>
</tr>
<tr>
<td>$\nu_C$</td>
<td>0.21</td>
<td>0.45</td>
<td>0.41</td>
<td>0.45</td>
<td>0.45</td>
</tr>
</tbody>
</table>

respectively. Then $B$ and $C$ are intuitionistic fuzzy $H$-ideal extensions of $A$. Obviously, the union $B \cup C$ is an intuitionistic fuzzy extension of $A$, but it is not an intuitionistic fuzzy ideal extension of $A$ since $\mu_{B \cup C}(3 \ast 0) = \mu_{B \cup C}(3) = 0.51 \not\leq 0.57 = \min\{\mu_{B \cup C}(3 \ast (2 \ast 0)), \mu_{B \cup C}(2)\}$ and $\nu_{B \cup C}(3 \ast 0) = \nu_{B \cup C}(3) = 0.45 \not\leq 0.41 = \max\{\nu_{B \cup C}(1), \nu_{B \cup C}(2)\} = \max\{\nu_{B \cup C}(3 \ast (2 \ast 0)), \nu_{B \cup C}(2)\}$. 

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For an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of $X$, $\alpha \in [0, \Xi]$ and $t, s \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu_A; t) = \{ x \mid x \in X \text{ and } \mu_A(x) \geq t - \alpha \}$$
and
$$L_\alpha(\nu_A; s) = \{ x \mid x \in X \text{ and } \nu_A(x) \leq s + \alpha \}.$$

If $A$ is an intuitionistic fuzzy $H$-ideal of $X$, then it is clear that $U_\alpha(\mu; t)$ and $L_\alpha(\nu_A; s)$ are $H$-ideals of $X$ for all $t \in \text{Im}(\mu_A)$ and $s \in \text{Im}(\nu_A)$ with $t \geq \alpha$. But, if we do not give a condition that $A$ is an intuitionistic fuzzy $H$-ideal of $X$, then $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are not $H$-ideals of $X$ as seen in the following example.

**Example 3** Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in Example 2 and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ defined by

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_A$</td>
<td>0.66</td>
<td>0.49</td>
<td>0.31</td>
<td>0.49</td>
<td>0.49</td>
</tr>
<tr>
<td>$\nu_A$</td>
<td>0.33</td>
<td>0.48</td>
<td>0.67</td>
<td>0.48</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Since $\mu_A(3 \ast 1) = \mu_A(2) = 0.31 \nleq 0.49 = \min \{\mu_A(3), \mu_A(0)\} = \min \{\mu_A(3 \ast (0 \ast 1)), \mu_A(0)\}$ and $\nu_A(3 \ast 1) = \nu_A(2) = 0.67 \ngeq 0.48 = \max \{\nu_A(3), \nu_A(0)\} = \max \{\nu_A(3 \ast (0 \ast 1)), \nu_A(0)\}$, therefore, $A = (\mu_A, \nu_A)$ is not an intuitionistic fuzzy $H$-ideal of $X$.

For $\alpha = 0.16$, $t = 0.60$ and $s = 0.45$, we obtain $U_\alpha(\mu_A; t) = L_\alpha(\nu_A; s) = \{0, 1, 3, 4\}$ which are not $H$-ideals of $X$ since $3 \ast (0 \ast 1) = 3 \in \{0, 1, 3, 4\}$, but $3 \ast 1 = 2 \nleq \{0, 1, 3, 4\}$.

**Theorem 3.13** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ and $\alpha \in [0, \Xi]$. Then the intuitionistic fuzzy $\alpha$-translation of $A$ is an intuitionistic fuzzy $H$-ideal of $X$ if and only if $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are $H$-ideals of $X$ for $t \in \text{Im}(\mu_A)$, $s \in \text{Im}(\nu_A)$ with $t \geq \alpha$.

**Proof:** Suppose that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy $H$-ideal of $X$. Then $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy $H$-ideals of $X$. Let $t \in \text{Im}(\mu_A)$, $s \in \text{Im}(\nu_A)$ with $t \geq \alpha$.

Since $(\mu_A)^T_\alpha(0) \geq (\mu_A)^T_\alpha(x)$ for all $x \in X$, we have $\mu_A(0) + \alpha = (\mu_A)^T_\alpha(0) \geq (\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha \geq t$ for $x \in U_\alpha(\mu_A; t)$. Hence, $0 \in U_\alpha(\mu_A; t)$. Let $x, y, z \in X$ such that $x \ast (y \ast z), y \in U_\alpha(\mu_A; t)$. Then $\mu_A(x \ast (y \ast z)) \geq t - \alpha$ and $\mu_A(y) \geq t - \alpha$ i.e., $(\mu_A)^T_\alpha(x \ast (y \ast z)) = \mu_A(x \ast (y \ast z)) + \alpha \geq t$ and $(\mu_A)^T_\alpha(y) = \mu_A(y) + \alpha \geq t$. Since $(\mu_A)^T_\alpha$ is a fuzzy $H$-ideal, therefore, we have $\mu_A(x \ast z) + \alpha = (\mu_A)^T_\alpha(x \ast z) \geq \min \{(\mu_A)^T_\alpha(x \ast (y \ast z)), (\mu_A)^T_\alpha(y)\} \geq t$ that is, $\mu_A(x \ast z) \geq t - \alpha$ so that $x \ast z \in U_\alpha(\mu_A; t)$. Therefore, $U_\alpha(\mu_A; t)$ is a $H$-ideal of $X$.

Again, since $(\nu_A)^T_\alpha(0) \leq (\nu_A)^T_\alpha(x)$ for all $x \in X$, we have $\nu_A(0) - \alpha = (\nu_A)^T_\alpha(0) \leq (\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha \leq s$ for $x \in L_\alpha(\nu_A; s)$. Hence, $0 \in L_\alpha(\nu_A; s)$. Let $x, y, z \in X$ such that $x \ast (y \ast z), y \in L_\alpha(\nu_A; s)$. Then $\nu_A(x \ast (y \ast z)) \leq s + \alpha$ and $\nu_A(y) \leq s + \alpha$ i.e., $(\nu_A)^T_\alpha(x \ast (y \ast z)) = \nu_A(x \ast (y \ast z)) - \alpha \leq s$ and $(\nu_A)^T_\alpha(y) = \nu_A(y) - \alpha \leq s$. Since $(\nu_A)^T_\alpha$ is a fuzzy $H$-ideal, therefore, we have $\nu_A(x \ast z) - \alpha = (\nu_A)^T_\alpha(x \ast z) \leq \max \{(\nu_A)^T_\alpha(x \ast (y \ast z)), (\nu_A)^T_\alpha(y)\} \leq s$ that is, $\nu_A(x \ast z) \leq s + \alpha$ so that $x \ast z \in L_\alpha(\nu_A; s)$. Therefore, $L_\alpha(\nu_A; s)$ is a $H$-ideal of $X$.

Conversely, suppose that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are $H$-ideals of $X$ for $t \in \text{Im}(\mu_A)$, $s \in \text{Im}(\nu_A)$ with $t \geq \alpha$. If there exists $u \in X$ such that $(\mu_A)^T_\alpha(0) < \psi \leq (\mu_A)^T_\alpha(u)$, then $\mu_A(u) \geq$
$\psi - \alpha$ but $\mu_A(0) < \psi - \alpha$. This shows that $u \in U_\alpha(\mu_A; t)$ and $0 \notin U_\alpha(\mu_A; t)$. This is a contradiction, and $(\mu_A)_{\alpha}^{T}(0) \geq (\mu_A)_{\alpha}^{T}(x)$ for all $x \in X$. Again, if there exists $v \in X$ such that $(\nu_A)_{\alpha}^{T}(0) > \kappa \geq (\nu_A)_{\alpha}^{T}(v)$, then $\nu_A(v) \leq \kappa + \alpha$ but $\nu_A(0) > \kappa + \alpha$. This shows that $v \in L_\alpha(\nu_A; s)$ and $0 \notin L_\alpha(\nu_A; s)$. This is a contradiction, and $(\nu_A)_{\alpha}^{T}(0) \leq (\nu_A)_{\alpha}^{T}(x)$ for all $x \in X$.

Now we assume that there exist $a, b, c \in X$ such that

$$(\mu_A)_{\alpha}^{T}(a * c) \leq \zeta \leq \min\{(\mu_A)_{\alpha}^{T}(a * (b * c)), (\mu_A)_{\alpha}^{T}(b)\}.$$ 

Then $\mu_A(a * (b * c)) \geq \zeta - \alpha$ and $\mu_A(b) \geq \zeta - \alpha$ but $\mu_A(a * c) < \zeta - \alpha$. Hence, $a * (b * c) \in U_\alpha(\mu_A; t)$ and $b \in U_\alpha(\mu_A; t)$ but $a * c \notin U_\alpha(\mu_A; t)$ which is a contradiction, therefore, $(\mu_A)_{\alpha}^{T}(x * z) \geq \min\{(\mu_A)_{\alpha}^{T}(x * (y * z)), (\mu_A)_{\alpha}^{T}(y)\}$ for all $x, y, z \in X$.

Again, we assume that there exist $d, e, f \in X$ such that $(\nu_A)_{\alpha}^{T}(d * f) \geq \eta \leq \max\{(\nu_A)_{\alpha}^{T}(d * (e * f)), (\nu_A)_{\alpha}^{T}(e)\}$. Then $\nu_A(d * (e * f)) \leq \eta + \alpha$ and $\nu_A(e) \leq \eta + \alpha$ but $\nu_A(d * f) > \eta + \alpha$. Hence, $d * (e * f) \in L_\alpha(\nu_A; s)$ and $e \in L_\alpha(\nu_A; s)$ but $d * f \notin L_\alpha(\nu_A; s)$ which is a contradiction, therefore, $(\nu_A)_{\alpha}^{T}(x * z) \leq \max\{(\nu_A)_{\alpha}^{T}(x * (y * z)), (\nu_A)_{\alpha}^{T}(y)\}$ for all $x, y, z \in X$. Consequently, $A_{\alpha}^{T}$ is an intuitionistic fuzzy $H$-ideal of $X$.

**Theorem 3.14** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy $H$-ideal of $X$ and let $\alpha, \beta \in [0, \mathfrak{T}]$. If $\alpha \geq \beta$, then the intuitionistic fuzzy $\alpha$-translation $A_{\alpha}^{T} = ((\mu_A)_{\alpha}^{T}, (\nu_A)_{\alpha}^{T})$ of $A$ is an intuitionistic fuzzy $H$-ideal extension of the intuitionistic fuzzy $\beta$-translation $A_{\beta}^{T} = ((\mu_A)_{\beta}^{T}, (\nu_A)_{\beta}^{T})$ of $A$.

**Proof:** Straightforward. 

For every intuitionistic fuzzy $H$-ideal $A = (\mu_A, \nu_A)$ of $X$ and $\beta \in [0, \mathfrak{T}]$, the intuitionistic fuzzy $\beta$-translation $A_{\beta}^{T} = ((\mu_A)_{\beta}^{T}, (\nu_A)_{\beta}^{T})$ of $A$ is an intuitionistic fuzzy $H$-ideal of $X$. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy $H$-ideal extension of $A_{\beta}^{T}$, then there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geq \beta$ and $B \geq A_{\alpha}^{T}$ that is $\mu_B(x) \geq (\mu_A)_{\alpha}^{T}$ and $\nu_B(x) \leq (\nu_A)_{\alpha}^{T}$ for all $x \in X$. Hence, we have the following theorem.

**Theorem 3.15** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy $H$-ideal of $X$ and let $\beta \in [0, \mathfrak{T}]$. For every intuitionistic fuzzy $H$-ideal extension $B = (\mu_B, \nu_B)$ of the intuitionistic fuzzy $\beta$-translation $A_{\beta}^{T} = ((\mu_A)_{\beta}^{T}, (\nu_A)_{\beta}^{T})$ of $A$, there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geq \beta$ and $B$ is an intuitionistic fuzzy $H$-ideal extension of the intuitionistic fuzzy $\alpha$-translation $A_{\alpha}^{T} = ((\mu_A)_{\alpha}^{T}, (\nu_A)_{\alpha}^{T})$ of $A$.

Let us illustrate Theorem 3.15 using the following example.

**Example 4** Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy subset of \( X \) defined by

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
\hline
\mu_A & 0.55 & 0.42 & 0.34 & 0.42 \\
\nu_A & 0.43 & 0.57 & 0.66 & 0.67 \\
\end{array}
\]

Then \( A \) is an intuitionistic fuzzy \( H \)-ideal of \( X \) and \( \Xi = 0.43 \). If we take \( \beta = 0.13 \), then the intuitionistic fuzzy \( \beta \)-translation \( A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta) \) of \( A \) is given by

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
\hline
(\mu_A)^T_\beta & 0.68 & 0.55 & 0.47 & 0.55 \\
(\nu_A)^T_\beta & 0.30 & 0.44 & 0.53 & 0.44 \\
\end{array}
\]

Let \( B = (\mu_B, \nu_B) \) be an intuitionistic fuzzy subset of \( X \) defined by

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
\hline
\mu_B & 0.74 & 0.63 & 0.52 & 0.63 \\
\nu_B & 0.26 & 0.36 & 0.48 & 0.36 \\
\end{array}
\]

Then \( B \) is clearly an intuitionistic fuzzy \( H \)-ideal of \( X \) which is an intuitionistic fuzzy \( H \)-ideal extension of the intuitionistic fuzzy \( \beta \)-translation \( A^T_\beta \) of \( A \). But \( B \) is not an intuitionistic fuzzy \( \alpha \)-translation of \( A \) for all \( \alpha \in [0, \Xi] \). If we take \( \alpha = 0.16 \) then \( \alpha = 0.16 > 0.13 = \beta \) and the intuitionistic fuzzy \( \alpha \)-translation \( A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha) \) of \( A \) is given as follows:

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
\hline
(\mu_A)^T_\alpha & 0.71 & 0.58 & 0.50 & 0.58 \\
(\nu_A)^T_\alpha & 0.27 & 0.41 & 0.52 & 0.41 \\
\end{array}
\]

Note that \( B(x) \geq A^T_\alpha(x) \) that is \( \mu_B(x) \geq (\mu_A)^T_\alpha \) and \( \nu_B(x) \leq (\nu_A)^T_\alpha \) for all \( x \in X \), and hence, \( B \) is an intuitionistic fuzzy \( H \)-ideal extension of the intuitionistic fuzzy \( \alpha \)-translation \( A^T_\alpha \) of \( A \).

**Definition 3.16** [23] Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy subset of \( X \) and \( \gamma \in [0, 1] \). An object having the form \( A^m_\gamma = ((\mu_A)_\gamma^m, (\nu_A)_\gamma^m) \) is called an intuitionistic fuzzy \( \gamma \)-multiplication of \( A \) if \( (\mu_A)_\gamma^m(x) = \mu_A(x) \cdot \gamma \) and \( (\nu_A)_\gamma^m(x) = \nu_A(x) \cdot \gamma \) for all \( x \in X \).

For any intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of \( X \), an intuitionistic fuzzy 0-multiplication \( A^m_0 = ((\mu_A)_0^m, (\nu_A)_0^m) \) of \( A \) is an intuitionistic fuzzy ideal of \( X \).

**Theorem 3.17** If \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy \( H \)-ideal of \( X \), then the intuitionistic fuzzy \( \gamma \)-multiplication of \( A \) is an intuitionistic fuzzy \( H \)-ideal of \( X \) for all \( \gamma \in [0, 1] \).

**Proof:** Straightforward. \( \qed \)

**Theorem 3.18** If \( A = (\mu_A, \nu_A) \) is any intuitionistic fuzzy subset of \( X \), then the following assertions are equivalent:

(i) \( A \) is an intuitionistic fuzzy \( H \)-ideal of \( X \).
(ii) for all \( \gamma \in (0, 1] \), \( A^m_\gamma \) is an intuitionistic fuzzy \( H \)-ideal of \( X \).
\textbf{Proof:} Necessity follows from Theorem 3.17. For sufficient part let $\gamma \in (0, 1]$ be such that $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is an intuitionistic fuzzy $H$-ideal of $X$. Then for all $x, y, z \in X$, we have

$\mu_A(x \ast z)_\gamma = (\mu_A)^m_\gamma(x \ast (y \ast z)), (\mu_A)^m(y) \geq \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\} = \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}_\gamma$

and $\nu_A(x \ast z)_\gamma = (\nu_A)^m_\gamma(x \ast z) \leq \max\{(\nu_A)^m_\gamma(x \ast (y \ast z)), (\nu_A)^m(y)\} = \max\{\nu_A(x \ast (y \ast z)), \nu_A(y)\}_\gamma$

Therefore, $\mu_A(x \ast z) \geq \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}$ and $\nu_A(x \ast z) \leq \max\{\nu_A(x \ast (y \ast z)), \nu_A(y)\}$ for all $x, y, z \in X$ since $\gamma \neq 0$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy $H$-ideal of $X$. \Box

4 Conclusions and future work

In this paper, translation of intuitionistic fuzzy $H$-ideals in $BCK/BCI$-algebras are introduced and investigated some of their useful properties. The relationships between intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy $H$-ideals have been constructed.

It is our hope that this work would other foundations for further study of the theory of $BCK/BCI$-algebras. In our future study of fuzzy structure of $BCK/BCI$-algebras, may be the following topics should be considered: (i) to find translation of intuitionistic fuzzy $a$-ideals in $BCK/BCI$-algebras, (ii) to find translation of intuitionistic fuzzy $p$-ideals in $BCK/BCI$-algebras, (iii) to find the relationship between translations of intuitionistic fuzzy $H$-ideals, $a$-ideals and $p$-ideals in $BCK/BCI$-algebras.

References


