Notes on Intuitionistic Fuzzy Sets
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Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras

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Abstract: In this paper, the concepts of intuitionistic fuzzy translation to intuitionistic fuzzy H-ideals in BCK/BCI-algebras are introduced. The notion of intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals with several related properties are investigated. Also the relationships between intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals are investigated.
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1 Introduction

After the introduction of fuzzy sets by Zadeh [25], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [2-4] is one among them. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degrees of membership and of nonmembership. Both degrees belong to the interval $[0, 1]$, and their sum should not exceed 1.

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [6, 7, 10]. It is known that the class of BCK-algebra is a proper subclass of

the class of BCI-algebras. In 1991, Xi [24] applied the concept of fuzzy sets to BCK-algebras. In 1993, Jun [11] and Ahmad [1] applied it to BCI-algebras. After that Jun, Meng, Liu and several researchers investigated further properties of fuzzy subalgebras and ideals in BCK/BCI-algebras (see [12-19]). In 1999, Khalid and Ahmad introduced fuzzy H-ideals in BCI-algebras. In [26, 27], Zhan and Tan discussed characterization of fuzzy H-ideals and doubt fuzzy H-ideals in BCK-algebras. Recently, Satyanarayana et al. [20, 21] introduced intuitionistic fuzzy H-ideals in BCK-algebras.

The concept of fuzzy translations in fuzzy subalgebras and ideals in BCK/BCI-algebras has been discussed respectively by Lee et al. [15] and Jun [13]. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. Motivated by this, in [22], the authors have studied fuzzy translations of fuzzy H-ideals in BCK/BCI-algebras. They also extend this study from fuzzy translations to intuitionistic fuzzy translations [23] in BCK/BCI-algebras.

In this paper, intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals in BCK/BCI-algebras are discussed. Relations among intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals in BCK/BCI-algebras are also investigated.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.
By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$

(ii) $(x * (x * y)) * y = 0$

(iii) $x * x = 0$

(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:

(1) $(x * y) * z = (x * z) * y$

(2) $((x * z) * (y * z)) * (x * y) = 0$

(3) $x * 0 = x$

(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

Throughout this paper, X always means a BCK/BCI-algebra without any specification.
A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A non-empty subset I of X is said to be a H-ideal [14] of X if it satisfies (I_1) and

(I_3) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

A BCI-algebra is said to be associative [5] if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.
A fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ in X is called a fuzzy subalgebra of X if it satisfies the inequality $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.
A fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ in X is called a fuzzy ideal [1, 24] of X if it satisfies

(F_1) $\mu_A(0) \geq \mu_A(x)$ and

(F_2) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$.

A fuzzy set $A = \{< x, \mu_A(x) >: x \in X\}$ in X is called a fuzzy H-ideal [14, 26] of X if it satisfies (F_1) and (F_3) $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$ for all $x, y, z \in X$.
An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X is called an intuitionistic fuzzy subalgebra [12] of X if it satisfies the following two conditions

(F_4) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and

(F_5) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X is called an intuitionistic fuzzy ideal [12] of X if it satisfies

(F_6) $\mu_A(0) \geq \mu_A(x)$, $\nu_A(0) \leq \nu_A(x)$,

(F_7) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$ and

(F_8) $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}$ for all $x, y \in X$.

An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X is called an intuitionistic fuzzy H-ideal [20] of X if it satisfies (F_6) and

(F_9) $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

(F_{10}) $\nu_A(x * z) \leq \max\{\nu_A(x * (y * z)), \nu_A(y)\}$ for all $x, y, z \in X$.

3 Main results

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Throughout this paper, we take $\mathfrak{T} := \inf\ \{\nu_A(x)|x \in X\}$ for any intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of X.

Definition 3.1 [23] Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and let $\alpha \in [0, \mathfrak{T}]$. An object having the form $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is called an intuitionistic fuzzy α-translation of A if $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$ and $(\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha$ for all $x \in X$.

Theorem 3.2 *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal of X for all $\alpha \in [0, \mathfrak{T}]$.*

Proof: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy H-ideal of X and $\alpha \in [0, \mathfrak{T}]$. Then $(\mu_A)^T_\alpha(0) = \mu_A(0) + \alpha \geq \mu_A(x) + \alpha = (\mu_A)^T_\alpha(x)$ and $(\nu_A)^T_\alpha(0) = \nu_A(0) - \alpha \leq \nu_A(x) - \alpha =$

$(\nu_A)^T_\alpha(x)$ for all $x \in X$. Now,

$$\begin{aligned}
(\mu_A)^T_\alpha(x*z) &= \mu_A(x*z)+\alpha \geq \min\{\mu_A(x*(y*z)),\mu_A(y)\}+\alpha \\
&= \min\{\mu_A(x*(y*z))+\alpha,\mu_A(y)+\alpha\} \\
&= \min\{(\mu_A)^T_\alpha(x*(y*z)),(\mu_A)^T_\alpha(y)\} \\
\text{and} \quad (\nu_A)^T_\alpha(x*z) &= \nu_A(x*z)-\alpha \leq \max\{\nu_A(x*(y*z)),\nu_A(y)\}-\alpha \\
&= \max\{\nu_A(x*(y*z))-\alpha,\nu_A(y)-\alpha\} \\
&= \max\{(\nu_A)^T_\alpha(x*(y*z)),(\nu_A)^T_\alpha(y)\}
\end{aligned}$$

for all $x, y, z \in X$. Hence, the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal of X. □

Theorem 3.3 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal of X for some $\alpha \in [0, \mathfrak{T}]$. Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X.*

Proof: Assume that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy H-ideal of X for some $\alpha \in [0, \mathfrak{T}]$. Let $x, y \in X$, we have

$$\begin{aligned}
\mu_A(0)+\alpha = (\mu_A)^T_\alpha(0) &\geq (\mu_A)^T_\alpha(x) = \mu_A(x)+\alpha \\
\nu_A(0)-\alpha = (\nu_A)^T_\alpha(0) &\leq (\nu_A)^T_\alpha(x) = \nu_A(x)-\alpha
\end{aligned}$$

which implies $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$. Now we have

$$\begin{aligned}
\mu_A(x*z)+\alpha &= (\mu_A)^T_\alpha(x*z) \geq \min\{(\mu_A)^T_\alpha(x*(y*z)),(\mu_A)^T_\alpha(y)\} \\
&= \min\{\mu_A(x*(y*z))+\alpha,\mu_A(y)+\alpha\} \\
&= \min\{\mu_A(x*(y*z)),\mu_A(y)\}+\alpha \\
\text{and} \quad \nu_A(x*z)-\alpha &= (\nu_A)^T_\alpha(x*z) \leq \max\{(\nu_A)^T_\alpha(x*(y*z)),(\nu_A)^T_\alpha(y)\} \\
&= \max\{\nu_A(x*(y*z))-\alpha,\nu_A(y)-\alpha\} \\
&= \max\{\nu_A(x*(y*z)),\nu_A(y)\}-\alpha
\end{aligned}$$

which implies that $\mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$ and $\nu_A(x*z) \leq \min\{\nu_A(x*(y*z)), \nu_A(y)\}$ for all $x, y, z \in X$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X. □

We now discuss the relation between intuitionistic fuzzy subalgebras and intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A for the intuitionistic fuzzy H-ideals.

Theorem 3.4 *If the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal of X for all $\alpha \in [0, \mathfrak{T}]$ then it must be an intuitionistic fuzzy subalgebra of X.*

Proof: Let the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A be an intuitionistic fuzzy H-ideal of X. Then we have $(\mu_A)^T_\alpha(x*z) \geq \min\{(\mu_A)^T_\alpha(x*(y*z)), (\mu_A)^T_\alpha(y)\}$ and

$(\nu_A)^T_\alpha(x*z) \leq \max\{(\nu_A)^T_\alpha(x*(y*z)), (\nu_A)^T_\alpha(y)\}$ for all $x, y, z \in X$. Substituting y for z we get

$$\begin{aligned}
(\mu_A)^T_\alpha(x*y) &\geq \min\{(\mu_A)^T_\alpha(x*(y*y)), (\mu_A)^T_\alpha(y)\} \\
&= \min\{(\mu_A)^T_\alpha(x*0), (\mu_A)^T_\alpha(y)\} \\
&= \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\} \\
\text{and} \quad (\nu_A)^T_\alpha(x*y) &\leq \max\{(\nu_A)^T_\alpha(x*(y*y)), (\nu_A)^T_\alpha(y)\} \\
&= \max\{(\nu_A)^T_\alpha(x*0), (\nu_A)^T_\alpha(y)\} \\
&= \max\{(\nu_A)^T_\alpha(x), (\nu_A)^T_\alpha(y)\}.
\end{aligned}$$

Therefore, A^T_α is an intuitionistic fuzzy subalgebra of X. □

Proposition 1 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal of X for $\alpha \in [0, \mathfrak{T}]$. If $(x*a)*b = 0$ for all $a, b, x \in X$, then $(\mu_A)^T_\alpha(x) \geq \min\{(\mu_A)^T_\alpha(a), (\mu_A)^T_\alpha(b)\}$ and $(\nu_A)^T_\alpha(x) \leq \max\{(\nu_A)^T_\alpha(a), (\nu_A)^T_\alpha(b)\}$.*

Proof: Let $a, b, x \in X$ be such that $(x*a)*b = 0$. Then

$$\begin{aligned}
(\mu_A)^T_\alpha(x) &\geq \min\{(\mu_A)^T_\alpha(x*a), (\mu_A)^T_\alpha(a)\} \\
&\geq \min\{\min\{(\mu_A)^T_\alpha((x*a)*b), (\mu_A)^T_\alpha(b)\}, (\mu_A)^T_\alpha(a)\} \\
&= \min\{\min\{(\mu_A)^T_\alpha(0), (\mu_A)^T_\alpha(b)\}, (\mu_A)^T_\alpha(a)\} \\
&= \min\{(\mu_A)^T_\alpha(b), (\mu_A)^T_\alpha(a)\} \text{ since } (\mu_{\mathrm{A}})^{\mathrm{T}}_\alpha(0) \geq (\mu_{\mathrm{A}})^{\mathrm{T}}_\alpha(\mathrm{b}) \\
&= \min\{(\mu_A)^T_\alpha(a), (\mu_A)^T_\alpha(b)\} \\
\text{and} \quad (\nu_A)^T_\alpha(x) &\leq \max\{(\nu_A)^T_\alpha(x*a), (\nu_A)^T_\alpha(a)\} \\
&\leq \max\{\max\{(\nu_A)^T_\alpha((x*a)*b), (\nu_A)^T_\alpha(b)\}, (\nu_A)^T_\alpha(a)\} \\
&= \max\{\max\{(\nu_A)^T_\alpha(0), (\nu_A)^T_\alpha(b)\}, (\nu_A)^T_\alpha(a)\} \\
&= \max\{(\nu_A)^T_\alpha(b), (\nu_A)^T_\alpha(a)\} \text{ since } (\nu_{\mathrm{A}})^{\mathrm{T}}_\alpha(0) \leq (\nu_{\mathrm{A}})^{\mathrm{T}}_\alpha(\mathrm{b}) \\
&= \max\{(\nu_A)^T_\alpha(a), (\nu_A)^T_\alpha(b)\}.
\end{aligned}$$

The proof is complete. □

The following can easily be proved by induction.

Corollary 1 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal of X for $\alpha \in [0, \mathfrak{T}]$. If $(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$ for all $x, a_1, a_2, \ldots, a_n \in X$, then $(\mu_A)^T_\alpha(x) \geq \min\{(\mu_A)^T_\alpha(a_1), (\mu_A)^T_\alpha(a_2), \ldots, (\mu_A)^T_\alpha(a_n)\}$ and $(\nu_A)^T_\alpha(x) \leq \max\{(\nu_A)^T_\alpha(a_1), (\nu_A)^T_\alpha(a_2), \ldots, (\nu_A)^T_\alpha(a_n)\}$.*

We now give a condition for intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A which is an intuitionistic fuzzy ideal of X to be an intuitionistic fuzzy H-ideal of X.

Theorem 3.5 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A is an intuitionistic fuzzy ideal of X for $\alpha \in [0, \mathfrak{T}]$. If it satisfies the condition $(\mu_A)_\alpha^T(x*y) \geq (\mu_A)_\alpha^T(x)$ and $(\nu_A)_\alpha^T(x*y) \leq (\nu_A)_\alpha^T(x)$ for all $x, y \in X$, then the intuitionistic fuzzy α-translation A_α^T of A is an intuitionistic fuzzy H-ideal of X.*

Proof: Let the intuitionistic fuzzy α-translation A_α^T of A be an intuitionistic fuzzy ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned}
(\mu_A)_\alpha^T(x*z) &\geq \min\{(\mu_A)_\alpha^T((x*z)*(y*z)), (\mu_A)_\alpha^T(y*z)\} \\
&= \min\{(\mu_A)_\alpha^T((x*(y*z))*z), (\mu_A)_\alpha^T(y*z)\} \\
&\geq \min\{(\mu_A)_\alpha^T(x*(y*z)), (\mu_A)_\alpha^T(y)\} \\
\text{and} \quad (\nu_A)_\alpha^T(x*z) &\leq \max\{(\nu_A)_\alpha^T((x*z)*(y*z)), (\nu_A)_\alpha^T(y*z)\} \\
&= \max\{(\nu_A)_\alpha^T((x*(y*z))*z), (\nu_A)_\alpha^T(y*z)\} \\
&\leq \max\{(\nu_A)_\alpha^T(x*(y*z)), (\nu_A)_\alpha^T(y)\}.
\end{aligned}$$

Hence, the intuitionistic fuzzy α-translation A_α^T of A is an intuitionistic fuzzy H-ideal of X for some $\alpha \in [0, \mathfrak{T}]$. □

Theorem 3.6 *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subset of associative BCK/BCI-algebra X such that the intuitionistic fuzzy α-translation $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A is an intuitionistic fuzzy ideal of X for $\alpha \in [0, \mathfrak{T}]$, then the intuitionistic fuzzy α-translation A_α^T of A is an intuitionistic fuzzy H-ideal of X.*

Proof: Let the intuitionistic fuzzy α-translation A_α^T of A be an intuitionistic fuzzy ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned}
(\mu_A)_\alpha^T(x*z) &\geq \min\{(\mu_A)_\alpha^T((x*z)*y), (\mu_A)_\alpha^T(y)\} \\
&= \min\{(\mu_A)_\alpha^T((x*y)*z), (\mu_A)_\alpha^T(y)\} \\
&= \min\{(\mu_A)_\alpha^T(x*(y*z)), (\mu_A)_\alpha^T(y)\} \\
\text{and} \quad (\nu_A)_\alpha^T(x*z) &\leq \max\{(\nu_A)_\alpha^T((x*z)*y), (\nu_A)_\alpha^T(y)\} \\
&= \max\{(\nu_A)_\alpha^T((x*y)*z), (\nu_A)_\alpha^T(y)\} \\
&= \max\{(\nu_A)_\alpha^T(x*(y*z)), (\nu_A)_\alpha^T(y)\}.
\end{aligned}$$

Hence, the intuitionistic fuzzy α-translation A_α^T of A is an intuitionistic fuzzy H-ideal of X. □

Theorem 3.7 *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A is an intuitionistic fuzzy H-ideal of X for $\alpha \in [0, \mathfrak{T}]$, then the sets $T_{\mu_A} = \{x \mid x \in X$ and $(\mu_A)_\alpha^T(x) = (\mu_A)_\alpha^T(0)\}$ and $T_{\nu_A} = \{x \mid x \in X$ and $(\nu_A)_\alpha^T(x) = (\nu_A)_\alpha^T(0)\}$ are H-ideals of X.*

Proof: Suppose that $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ is an intuitionistic fuzzy ideal of X. Then $(\mu_A)_\alpha^T$ and $(\nu_A)_\alpha^T$ are fuzzy H-ideal of X. Obviously $0 \in T_{\mu_A}, T_{\nu_A}$. Let $x, y, z \in X$ be such that $x*(y*z) \in T_{\mu_A}$ and $y \in T_{\mu_A}$. Then $(\mu_A)_\alpha^T(x*(y*z)) = (\mu_A)_\alpha^T(0) = (\mu_A)_\alpha^T(y)$ and so

$(\mu_A)^T_\alpha(x*z) \geq \min\{(\mu_A)^T_\alpha(x*(y*z)), (\mu_A)^T_\alpha(y)\} = (\mu_A)^T_\alpha(0)$. Since, $(\mu_A)^T_\alpha$ is a fuzzy H-ideal of X, we conclude that $(\mu_A)^T_\alpha(x*z) = (\mu_A)^T_\alpha(0)$. This implies $\mu_A(x*z) + \alpha = \mu_A(0) + \alpha$ or, $\mu_A(x*z) = \mu_A(0)$ so that $x*z \in T_{\mu_A}$. Therefore, T_{μ_A} is a H-ideal of X.

Again, let $a, b, c \in X$ be such that $a*(b*c) \in T_{\nu_A}$ and $b \in T_{\nu_A}$. Then $(\nu_A)^T_\alpha(a*(b*c)) = (\nu_A)^T_\alpha(0) = (\nu_A)^T_\alpha(b)$ and so $(\nu_A)^T_\alpha(a*c) \leq \max\{(\nu_A)^T_\alpha(a*(b*c)), (\nu_A)^T_\alpha(b)\} = (\nu_A)^T_\alpha(0)$. Since, $(\nu_A)^T_\alpha$ is a fuzzy H-ideal of X, we conclude that $(\nu_A)^T_\alpha(a*c) = (\nu_A)^T_\alpha(0)$. This implies $\nu_A(a*c) + \alpha = \nu_A(0) + \alpha$ or, $\nu_A(a*c) = \nu_A(0)$ so that $a*c \in T_{\nu_A}$. Therefore, T_{ν_A} is a H-ideal of X. □

Proposition 2 *Let the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A be an intuitionistic fuzzy H-ideal of X for $\alpha \in [0, \mathfrak{T}]$. If $x \leq y$ then $(\mu_A)^T_\alpha(x) \geq (\mu_A)^T_\alpha(y)$ and $(\nu_A)^T_\alpha(x) \leq (\nu_A)^T_\alpha(y)$, that is, $(\mu_A)^T_\alpha$ is order-reversing and $(\nu_A)^T_\alpha$ is order-preserving.*

Proof: Let $x, y \in X$ be such that $x \leq y$. Then $x*y = 0$ and hence

$$\begin{aligned}
(\mu_A)^T_\alpha(x) &= (\mu_A)^T_\alpha(x*0) \geq \min\{(\mu_A)^T_\alpha(x*(y*0)), (\mu_A)^T_\alpha(y)\} \\
&= \min\{(\mu_A)^T_\alpha(x*y), (\mu_A)^T_\alpha(y)\} = \min\{(\mu_A)^T_\alpha(0), (\mu_A)^T_\alpha(y)\} \\
&= (\mu_A)^T_\alpha(y) \\
\text{and} \qquad (\nu_A)^T_\alpha(x) &= (\nu_A)^T_\alpha(x*0) \leq \max\{(\nu_A)^T_\alpha(x*(y*0)), (\nu_A)^T_\alpha(y)\} \\
&= \max\{(\nu_A)^T_\alpha(x*y), (\nu_A)^T_\alpha(y)\} = \max\{(\nu_A)^T_\alpha(0), (\nu_A)^T_\alpha(y)\} \\
&= (\nu_A)^T_\alpha(y).
\end{aligned}$$

This completes the proof. □

The characterizations of intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A = (\mu_A, \nu_A)$ are given by the following theorem.

Theorem 3.8 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal of X for $\alpha \in [0, \mathfrak{T}]$, then the following assertions are equivalent:*

(i) A^T_α is an intuitionistic fuzzy H-ideal of X,

(ii) $(\mu_A)^T_\alpha(x*y) \geq (\mu_A)^T_\alpha(x*(0*y))$ and $(\nu_A)^T_\alpha(x*y) \leq (\nu_A)^T_\alpha(x*(0*y))$ for all $x, y \in X$,

(iii) $(\mu_A)^T_\alpha((x*y)*z) \geq (\mu_A)^T_\alpha(x*(y*z))$ and $(\nu_A)^T_\alpha((x*y)*z) \leq (\nu_A)^T_\alpha(x*(y*z))$ for all $x, y, z \in X$.

Proof: Assume that $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy ideal of X. Then $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy H-ideals of X.

$(i) \Rightarrow (ii)$ Let A^T_α is an intuitionistic fuzzy H-ideal of X. Then for all $x, y \in X$ we have $(\mu_A)^T_\alpha(x*y) \geq \min\{(\mu_A)^T_\alpha(x*(0*y)), (\mu_A)^T_\alpha(0)\} = (\mu_A)^T_\alpha(x*(0*y))$ and $(\nu_A)^T_\alpha(x*y) \leq \max\{(\nu_A)^T_\alpha(x*(0*y)), (\nu_A)^T_\alpha(0)\} = (\nu_A)^T_\alpha(x*(0*y))$. Therefore, the inequality (ii) is satisfied.

$(ii) \Rightarrow (iii)$ Assume that (ii) is satisfied. For all $x, y, z \in X$, we have $((x*y)*(0*z))*(x*(y*z)) = ((x*y)*(x*(y*z)))*(0*z) \leq ((y*z)*y)*(0*z) = ((y*y)*z)*(0*z) = (0*z)*(0*z) = 0$. It follows from Proposition 2 that $(\mu_A)^T_\alpha((x*y)*(0*z))*(x*(y*z)) \geq (\mu_A)^T_\alpha(0)$ and $(\nu_A)^T_\alpha((x*y)*(0*z))*(x*(y*z)) \leq (\nu_A)^T_\alpha(0)$. Since $(\mu_A)^T_\alpha$ and $(\nu_A)^T_\alpha$ are fuzzy

H-ideal of X, therefore, we have $(\mu_A)^T_\alpha((x*y)*(0*z))*(x*(y*z)) = (\mu_A)^T_\alpha(0)$ and $(\nu_A)^T_\alpha((x*y)*(0*z))*(x*(y*z)) = (\nu_A)^T_\alpha(0)$.
Using (ii) we get

$$\begin{aligned}
&(\mu_A)^T_\alpha((x*y)*z) \geq (\mu_A)^T_\alpha((x*y)*(0*z)) \\
&= \min\{(\mu_A)^T_\alpha(((x*y)*(0*z))*(x*(y*z))), (\mu_A)^T_\alpha(x*(y*z))\} \\
&= \min\{(\mu_A)^T_\alpha(0), (\mu_A)^T_\alpha(x*(y*z))\} = (\mu_A)^T_\alpha(x*(y*z))
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_A)^T_\alpha((x*y)*z) \leq (\nu_A)^T_\alpha((x*y)*(0*z)) \\
&= \max\{(\nu_A)^T_\alpha(((x*y)*(0*z))*(x*(y*z))), (\nu_A)^T_\alpha(x*(y*z))\} \\
&= \max\{(\nu_A)^T_\alpha(0), (\nu_A)^T_\alpha(x*(y*z))\} = (\nu_A)^T_\alpha(x*(y*z)).
\end{aligned}$$

Therefore, inequality (iii) is also satisfied.
$(iii) \Rightarrow (i)$ Assume that (iii) is valid. For all $x, y, z \in X$, we have

$$\begin{aligned}
&(\mu_A)^T_\alpha(x*z) \geq \min\{(\mu_A)^T_\alpha((x*z)*y), (\mu_A)^T_\alpha(y)\} \\
&= \min\{(\mu_A)^T_\alpha((x*y)*z), (\mu_A)^T_\alpha(y)\} \\
&\geq \min\{(\mu_A)^T_\alpha(x*(y*z)), (\mu_A)^T_\alpha(y)\}
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_A)^T_\alpha(x*z) \leq \max\{(\nu_A)^T_\alpha((x*z)*y), (\nu_A)^T_\alpha(y)\} \\
&= \max\{(\nu_A)^T_\alpha((x*y)*z), (\nu_A)^T_\alpha(y)\} \\
&\leq \max\{(\nu_A)^T_\alpha(x*(y*z)), (\nu_A)^T_\alpha(y)\}.
\end{aligned}$$

Therefore, $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ is an intuitionistic fuzzy H-ideal of X. Hence, the assertion (i) holds. The proof is complete. □

Next we give another characterizations of intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of $A = (\mu_A, \nu_A)$ in the following theorem.

Theorem 3.9 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy ideal of X, then the following assertions are equivalent:*

(i) *A^T_α is an intuitionistic fuzzy H-ideal of X,*

(ii) *$(\mu_A)^T_\alpha((x*z)*y) \geq (\mu_A)^T_\alpha((x*z)*(0*y))$ and $(\nu_A)^T_\alpha((x*z)*y) \leq (\nu_A)^T_\alpha((x*z)*(0*y))$ for all $x, y, z \in X$,*

(iii) *$(\mu_A)^T_\alpha(x*y) \geq \min\{(\mu_A)^T_\alpha((x*z)*(0*y)), (\mu_A)^T_\alpha(z)\}$ and $(\nu_A)^T_\alpha(x*y) \leq \max\{(\nu_A)^T_\alpha((x*z)*(0*y)), (\nu_A)^T_\alpha(z)\}$ for all $x, y, z \in X$.*

Proof: $(i) \Rightarrow (ii)$ Same as above theorem.
$(ii) \Rightarrow (iii)$ Assume that (ii) is valid. For all $x, y, z \in X$, we have $(\mu_A)^T_\alpha(x*y) \geq \min\{(\mu_A)^T_\alpha((x*y)*z), (\mu_A)^T_\alpha(z)\} = \min\{(\mu_A)^T_\alpha((x*z)*y), (\mu_A)^T_\alpha(z)\} \geq \min\{(\mu_A)^T_\alpha((x*z)*(0*y)), (\mu_A)^T_\alpha(z)\}$ and $(\nu_A)^T_\alpha(x*y) \leq \max\{(\nu_A)^T_\alpha((x*y)*z), (\nu_A)^T_\alpha(z)\} = \max\{(\nu_A)^T_\alpha((x*z)*y), (\nu_A)^T_\alpha(z)\} \leq \max\{(\nu_A)^T_\alpha((x*z)*(0*y)), (\nu_A)^T_\alpha(z)\}$. Therefore, (iii) is satisfied.
$(iii) \Rightarrow (i)$ Assume that (iii) is valid. Therefore, for all $x, y, z \in X$, we have $(\mu_A)^T_\alpha(x*y) \geq \min\{(\mu_A)^T_\alpha((x*z)*(0*y)), (\mu_A)^T_\alpha(z)\}$ and $(\nu_A)^T_\alpha(x*y) \leq \max\{(\nu_A)^T_\alpha((x*z)*(0*y)), (\nu_A)^T_\alpha(z)\}$. Putting $z = 0$ we get $(\mu_A)^T_\alpha(x*y) \geq \min\{(\mu_A)^T_\alpha((x*0)*(0*y)), (\mu_A)^T_\alpha(0)\} = \min\{(\mu_A)^T_\alpha(x*(0*y)), (\mu_A)^T_\alpha(0)\} = (\mu_A)^T_\alpha(x*(0*y))$ and $(\nu_A)^T_\alpha(x*y) \leq \max\{(\nu_A)^T_\alpha((x*0)*(0*y)), (\nu_A)^T_\alpha(0)\} = \max\{(\nu_A)^T_\alpha(x*(0*y)), (\nu_A)^T_\alpha(0)\} = (\nu_A)^T_\alpha(x*(0*y))$. It follows from Theorem 3.8 that A^T_α is an intuitionistic fuzzy H-ideal of X. □

Definition 3.10 [23] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy subsets of X. If $A \leq B$ i.e, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$, then we say that B is an intuitionistic fuzzy extension of A.

Definition 3.11 *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subsets of X. Then B is called an intuitionistic fuzzy H-ideal extension of A if the following assertions are valid:*

(i) B is an intuitionistic fuzzy extension of A.

(ii) If A is an intuitionistic fuzzy H-ideal of X, then B is an intuitionistic fuzzy H-ideal of X.

From the definition of intuitionistic fuzzy α-translation, we get $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$ and $(\nu_A)^T_\alpha(x) = \nu_A(x) - \alpha$ for all $x \in X$. Therefore, we have the following theorem.

Theorem 3.12 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy H-ideal of X and $\alpha \in [0, \mathfrak{T}]$. Then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal extension of A.*

An intuitionistic fuzzy H-ideal extension of an intuitionistic fuzzy H-ideal A may not be represented as an intuitionistic fuzzy α-translation of A, that is, the converse of Theorem 3.12 is not true in general as seen in the following example.

Example 1 *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.72	0.64	0.55	0.41	0.41
ν_A	0.27	0.32	0.40	0.55	0.55

Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.74	0.70	0.58	0.45	0.45
ν_B	0.25	0.28	0.37	0.51	0.51

Then B is an intuitionistic fuzzy H-ideal extension of A. But it is not the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A for all $\alpha \in [0, \mathfrak{T}]$.

Clearly, the intersection of intuitionistic fuzzy H-ideal extensions of an intuitionistic fuzzy H-ideal A of X is an intuitionistic fuzzy H-ideal extension of A. But the union of intuitionistic fuzzy H-ideal extensions of an intuitionistic fuzzy H-ideal A of X is not an intuitionistic fuzzy H-ideal extension of A as seen in the following example.

Example 2 *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.63	0.41	0.41	0.41	0.41
ν_A	0.35	0.58	0.58	0.58	0.58

Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X. Let $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be intuitionistic fuzzy subsets of X defined by

X	0	1	2	3	4
μ_B	0.76	0.66	0.43	0.43	0.66
ν_B	0.22	0.33	0.54	0.54	0.33

and

X	0	1	2	3	4
μ_C	0.78	0.51	0.57	0.51	0.51
ν_C	0.21	0.45	0.41	0.45	0.45

respectively. Then B and C are intuitionistic fuzzy H-ideal extensions of A. Obviously, the union $B \cup C$ is an intuitionistic fuzzy extension of A, but it is not an intuitionistic fuzzy ideal extension of A since $\mu_{B\cup C}(3*0) = \mu_{B\cup C}(3) = 0.51 \ngeq 0.57 = \min\{\mu_{B\cup C}(3*(2*0)), \mu_{B\cup C}(2)\}$ and $\nu_{B\cup C}(3*0) = \nu_{B\cup C}(3) = 0.45 \nleq 0.41 = \max\{\nu_{B\cup C}(1), \nu_{B\cup C}(2)\} = \max\{\nu_{B\cup C}(3*(2*0)), \nu_{B\cup C}(2)\}$.

For an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, $\alpha \in [0, \mathfrak{T}]$ and $t, s \in [0, 1]$ with $t \geq \alpha$, let

$$\begin{aligned} U_\alpha(\mu_A; t) &= \{x \mid x \in X \text{ and } \mu_A(x) \geq t - \alpha\} \\ \text{and} \quad L_\alpha(\nu_A; s) &= \{x \mid x \in X \text{ and } \nu_A(x) \leq s + \alpha\}. \end{aligned}$$

If A is an intuitionistic fuzzy H-ideal of X, then it is clear that $U_\alpha(\mu; t)$ and $L_\alpha(\nu_A; s)$ are H-ideals of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t \geq \alpha$. But, if we do not give a condition that A is an intuitionistic fuzzy H-ideal of X, then $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are not H-ideals of X as seen in the following example.

Example 3 *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in Example 2 and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by*

X	0	1	2	3	4
μ_A	0.66	0.49	0.31	0.49	0.49
ν_A	0.33	0.48	0.67	0.48	0.48

Since $\mu_A(3 * 1) = \mu_A(2) = 0.31 \ngeq 0.49 = \min\{\mu_A(3), \mu_A(0)\} = \min\{\mu_A(3 * (0 * 1)), \mu_A(0)\}$ and $\nu_A(3 * 1) = \nu_A(2) = 0.67 \nleq 0.48 = \max\{\nu_A(3), \nu_A(0)\} = \max\{\nu_A(3 * (0 * 1)), \nu_A(0)\}$, therefore, $A = (\mu_A, \nu_A)$ is not an intuitionistic fuzzy H-ideal of X.

For $\alpha = 0.16$, $t = 0.60$ and $s = 0.45$, we obtain $U_\alpha(\mu_A; t) = L_\alpha(\nu_A; s) = \{0, 1, 3, 4\}$ which are not H-ideals of X since $3 * (0 * 1) = 3 \in \{0, 1, 3, 4\}$, but $3 * 1 = 2 \notin \{0, 1, 3, 4\}$.

Theorem 3.13 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\alpha \in [0, \mathfrak{T}]$. Then the intuitionistic fuzzy α-translation of A is an intuitionistic fuzzy H-ideal of X if and only if $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are H-ideals of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geq \alpha$.*

Proof: Suppose that $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ is an intuitionistic fuzzy H-ideal of X. Then $(\mu_A)_\alpha^T$ and $(\nu_A)_\alpha^T$ are fuzzy H-ideals of X. Let $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geq \alpha$.

Since $(\mu_A)_\alpha^T(0) \geq (\mu_A)_\alpha^T(x)$ for all $x \in X$, we have $\mu_A(0) + \alpha = (\mu_A)_\alpha^T(0) \geq (\mu_A)_\alpha^T(x) = \mu_A(x) + \alpha \geq t$ for $x \in U_\alpha(\mu_A; t)$. Hence, $0 \in U_\alpha(\mu_A; t)$. Let $x, y, z \in X$ such that $x * (y * z), y \in U_\alpha(\mu_A; t)$. Then $\mu_A(x * (y * z)) \geq t - \alpha$ and $\mu_A(y) \geq t - \alpha$ i.e., $(\mu_A)_\alpha^T(x * (y * z)) = \mu_A(x * (y * z)) + \alpha \geq t$ and $(\mu_A)_\alpha^T(y) = \mu_A(y) + \alpha \geq t$. Since $(\mu_A)_\alpha^T$ is a fuzzy H-ideal, therefore, we have $\mu_A(x * z) + \alpha = (\mu_A)_\alpha^T(x * z) \geq \min\{(\mu_A)_\alpha^T(x * (y * z)), (\mu_A)_\alpha^T(y)\} \geq t$ that is, $\mu_A(x * z) \geq t - \alpha$ so that $x * z \in U_\alpha(\mu_A; t)$. Therefore, $U_\alpha(\mu_A; t)$ is a H-ideal of X.

Again, since $(\nu_A)_\alpha^T(0) \leq (\nu_A)_\alpha^T(x)$ for all $x \in X$, we have $\nu_A(0) - \alpha = (\nu_A)_\alpha^T(0) \leq (\nu_A)_\alpha^T(x) = \nu_A(x) - \alpha \leq s$ for $x \in L_\alpha(\nu_A; s)$. Hence, $0 \in L_\alpha(\nu_A; s)$. Let $x, y, z \in X$ such that $x * (y * z), y \in L_\alpha(\nu_A; s)$. Then $\nu_A(x * (y * z)) \leq s + \alpha$ and $\nu_A(y) \leq s + \alpha$ i.e., $(\nu_A)_\alpha^T(x*(y*z)) = \nu_A(x*(y*z)) - \alpha \leq s$ and $(\nu_A)_\alpha^T(y) = \nu_A(y) - \alpha \leq s$. Since $(\nu_A)_\alpha^T$ is a fuzzy H-ideal, therefore, we have $\nu_A(x*z) - \alpha = (\nu_A)_\alpha^T(x*z) \leq \max\{(\nu_A)_\alpha^T(x*(y*z)), (\nu_A)_\alpha^T(y)\} \leq s$ that is, $\nu_A(x * z) \leq s + \alpha$ so that $x * z \in L_\alpha(\nu_A; s)$. Therefore, $L_\alpha(\nu_A; s)$ is a H-ideal of X.

Conversely, suppose that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are H-ideals of X for $t \in Im(\mu_A)$, $s \in Im(\nu_A)$ with $t \geq \alpha$. If there exists $u \in X$ such that $(\mu_A)_\alpha^T(0) < \psi \leq (\mu_A)_\alpha^T(u)$, then $\mu_A(u) \geq$

$\psi - \alpha$ but $\mu_A(0) < \psi - \alpha$. This shows that $u \in U_\alpha(\mu_A; t)$ and $0 \notin U_\alpha(\mu_A; t)$. This is a contradiction, and $(\mu_A)^T_\alpha(0) \geq (\mu_A)^T_\alpha(x)$ for all $x \in X$. Again, if there exists $v \in X$ such that $(\nu_A)^T_\alpha(0) > \kappa \geq (\nu_A)^T_\alpha(v)$, then $\nu_A(v) \leq \kappa + \alpha$ but $\nu_A(0) > \kappa + \alpha$. This shows that $v \in L_\alpha(\nu_A; s)$ and $0 \notin L_\alpha(\nu_A; s)$. This is a contradiction, and $(\nu_A)^T_\alpha(0) \leq (\nu_A)^T_\alpha(x)$ for all $x \in X$.

Now we assume that there exist $a, b, c \in X$ such that

$$(\mu_A)^T_\alpha(a * c) < \zeta \leq \min\{(\mu_A)^T_\alpha(a * (b * c)), (\mu_A)^T_\alpha(b)\}.$$

Then $\mu_A(a*(b*c)) \geq \zeta - \alpha$ and $\mu_A(b) \geq \zeta - \alpha$ but $\mu_A(a*c) < \zeta - \alpha$. Hence, $a*(b*c) \in U_\alpha(\mu_A; t)$ and $b \in U_\alpha(\mu_A; t)$ but $a * c \notin U_\alpha(\mu_A; t)$ which is a contradiction, therefore, $(\mu_A)^T_\alpha(x * z) \geq \min\{(\mu_A)^T_\alpha(x * (y * z)), (\mu_A)^T_\alpha(y)\}$ for all $x, y, z \in X$.

Again, we assume that there exist $d, e, f \in X$ such that $(\nu_A)^T_\alpha(d * f) > \eta \leq \max\{(\nu_A)^T_\alpha(d * (e * f)), (\nu_A)^T_\alpha(e)\}$. Then $\nu_A(d * (e * f)) \leq \eta + \alpha$ and $\nu_A(e) \leq \eta + \alpha$ but $\nu_A(d * f) > \eta + \alpha$. Hence, $d * (e * f) \in L_\alpha(\nu_A; s)$ and $e \in L_\alpha(\nu_A; s)$ but $d * f \notin L_\alpha(\nu_A; s)$ which is a contradiction, therefore, $(\nu_A)^T_\alpha(x * z) \leq \max\{(\nu_A)^T_\alpha(x * (y * z)), (\nu_A)^T_\alpha(y)\}$ for all $x, y, z \in X$. Consequently, A^T_α is an intuitionistic fuzzy H-ideal of X. □

Theorem 3.14 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy H-ideal of X and let $\alpha, \beta \in [0, \mathfrak{T}]$. If $\alpha \geq \beta$, then the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy H-ideal extension of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A.*

Proof: Straightforward. □

For every intuitionistic fuzzy H-ideal $A = (\mu_A, \nu_A)$ of X and $\beta \in [0, \mathfrak{T}]$, the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is an intuitionistic fuzzy H-ideal of X. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy H-ideal extension of A^T_β, then there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geq \beta$ and $B \geq A^T_\alpha$ that is $\mu_B(x) \geq (\mu_A)^T_\alpha$ and $\nu_B(x) \leq (\nu_A)^T_\alpha$ for all $x \in X$. Hence, we have the following theorem.

Theorem 3.15 *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy H-ideal of X and let $\beta \in [0, \mathfrak{T}]$. For every intuitionistic fuzzy H-ideal extension $B = (\mu_B, \nu_B)$ of the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A, there exists $\alpha \in [0, \mathfrak{T}]$ such that $\alpha \geq \beta$ and B is an intuitionistic fuzzy H-ideal extension of the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A.*

Let us illustrate Theorem 3.15 using the following example.

Example 4 *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_A	0.55	0.42	0.34	0.42	0.34
ν_A	0.43	0.57	0.66	0.57	0.66

Then A is an intuitionistic fuzzy H-ideal of X and $\mathfrak{T} = 0.43$. If we take $\beta = 0.13$, then the intuitionistic fuzzy β-translation $A^T_\beta = ((\mu_A)^T_\beta, (\nu_A)^T_\beta)$ of A is given by

X	0	1	2	3	4
$(\mu_A)^T_\beta$	0.68	0.55	0.47	0.55	0.47
$(\nu_A)^T_\beta$	0.30	0.44	0.53	0.44	0.53

Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy subset of X defined by

X	0	1	2	3	4
μ_B	0.74	0.63	0.52	0.63	0.52
ν_B	0.26	0.36	0.48	0.36	0.48

Then B is clearly an intuitionistic fuzzy H-ideal of X which is an intuitionistic fuzzy H-ideal extension of the intuitionistic fuzzy β-translation A^T_β of A. But B is not an intuitionistic fuzzy α-translation of A for all $\alpha \in [0, \mathfrak{T}]$. If we take $\alpha = 0.16$ then $\alpha = 0.16 > 0.13 = \beta$ and the intuitionistic fuzzy α-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is given as follows:

X	0	1	2	3	4
$(\mu_A)^T_\alpha$	0.71	0.58	0.50	0.58	0.50
$(\nu_A)^T_\alpha$	0.27	0.41	0.52	0.41	0.52

Note that $B(x) \geq A^T_\alpha(x)$ that is $\mu_B(x) \geq (\mu_A)^T_\alpha$ and $\nu_B(x) \leq (\nu_A)^T_\alpha$ for all $x \in X$, and hence, B is an intuitionistic fuzzy H-ideal extension of the intuitionistic fuzzy α-translation A^T_α of A.

Definition 3.16 [23] Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\gamma \in [0, 1]$. An object having the form $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is called an intuitionistic fuzzy γ-multiplication of A if $(\mu_A)^m_\gamma(x) = \mu_A(x) \cdot \gamma$ and $(\nu_A)^m_\gamma(x) = \nu_A(x) \cdot \gamma$ for all $x \in X$.

For any intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, an intuitionistic fuzzy 0-multiplication $A^m_0 = ((\mu_A)^m_0, (\nu_A)^m_0)$ of A is an intuitionistic fuzzy ideal of X.

Theorem 3.17 *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X, then the intuitionistic fuzzy γ-multiplication of A is an intuitionistic fuzzy H-ideal of X for all $\gamma \in [0, 1]$.*

Proof: Straightforward. □

Theorem 3.18 *If $A = (\mu_A, \nu_A)$ is any intuitionistic fuzzy subset of X, then the following assertions are equivalent:*

(i) A is an intuitionistic fuzzy H-ideal of X.

(ii) for all $\gamma \in (0, 1]$, A^m_γ is an intuitionistic fuzzy H-ideal of X.

Proof: Necessity follows from Theorem 3.17. For sufficient part let $\gamma \in (0,1]$ be such that $A^m_\gamma = ((\mu_A)^m_\gamma, (\nu_A)^m_\gamma)$ is an intuitionistic fuzzy H-ideal of X. Then for all $x, y, z \in X$, we have

$$\begin{aligned}
\mu_A(x*z).\gamma &= (\mu_A)^m_\gamma(x*z) \geq \min\{(\mu_A)^m_\gamma(x*(y*z)), (\mu_A)^m_\gamma(y)\} \\
&= \min\{\mu_A(x*(y*z)).\gamma, \mu_A(y).\gamma\} \\
&= \min\{\mu_A(x*(y*z)), \mu_A(y)\}.\gamma \\
\text{and} \qquad \nu_A(x*z).\gamma &= (\nu_A)^m_\gamma(x*z) \leq \max\{(\nu_A)^m_\gamma(x*(y*z)), (\nu_A)^m_\gamma(y)\} \\
&= \max\{\nu_A(x*(y*z)).\gamma, \nu_A(y).\gamma\} \\
&= \max\{\nu_A(x*(y*z)), \nu_A(y)\}.\gamma.
\end{aligned}$$

Therefore, $\mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$ and $\nu_A(x*z) \leq \max\{\nu_A(x*(y*z)), \nu_A(y)\}$ for all $x, y, z \in X$ since $\gamma \neq 0$. Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy H-ideal of X. $\square$

4 Conclusions and future work

In this paper, translation of intuitionistic fuzzy H-ideals in BCK/BCI-algebras are introduced and investigated some of their useful properties. The relationships between intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals have been constructed.

It is our hope that this work would other foundations for further study of the theory of BCK/BCI-algebras. In our future study of fuzzy structure of BCK/BCI-algebras, may be the following topics should be considered: (i) to find translation of intuitionistic fuzzy a-ideals in BCK/BCI-algebras, (ii) to find translation of intuitionistic fuzzy p-ideals in BCK/BCI-algebras, (iii) to find the relationship between translations of intuitionistic fuzzy H-ideals, a-ideals and p-ideals in BCK/BCI-algebras.

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