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MATHEMATIQUES
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ON SOME INTUITIONISTIC FUZZY IMPLICATIONS

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Abstract

Intuitionistic fuzzy variants of the most popular fuzzy implications and some specific intuitionistic fuzzy implications are given and their properties are studied. The relations between them together with Modus Ponens are described. A new type of intuitionistic fuzzy implication is constructed and its properties are presented. Relations between all (15 in number) intuitionistic fuzzy implications are shown.

Key words: implication, intuitionistic fuzzy logic, negation

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1. Introduction. On some previous results. Variants of intuitionistic fuzzy implications are discussed in [3] using as a basis the book of KLIR and YUAN [8], where the conventional fuzzy implications are given.

Let us note below each of these implications by $I(x, y)$. In [3] analogues of these implications for the case of intuitionistic fuzzy logic are given (for intuitionistic fuzzy sets and logics see, e.g. [1,2]) and the axioms from [8] are checked for them.

In intuitionistic fuzzy logic, if x is a variable, its truth-value is represented by the ordered couple

$$V(x) = \langle a, b\rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of nonvalidity of x. Any other formula is estimated by analogy.

Everywhere below we shall assume that for the three variables x, y and z there hold the equalities: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a+b, c+d, e+f \in [0, 1])$.

For the needs of the discussion we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1,2]) by

$$x \text{ is an IFT if and only if } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$.

We shall define the following relation:

$$V(x) \leq V(y) \text{ iff } a \leq c \text{ and } b \geq d.$$

In some definitions we shall use functions sg and $\overline{sg}$:

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0. \end{cases}$$

For two variables x and y operation "conjunction" (&) is defined (see [1,2] by

$$V(x \& y) = \langle \min(a, c), \max(b, d) \rangle.$$

Extending [3], in Table 1 we will include not only the ten implications from [3], but also the implications introduced by the author in [1,4,5,7] with coauthors KOLEV (in [5]) and TRIFONOV (in [7]), and this one from the next section.

Table 1

List of intuitionistic fuzzy implications

Notation	Name	Form of implication
$\rightarrow_1$	Zadeh	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$
$\rightarrow_2$	Gaines-Rescher	$\langle 1 - sg(a - c), d.sg(a - c) \rangle$
$\rightarrow_3$	Gödel	$\langle 1 - (1 - c).sg(a - c), d.sg(a - c) \rangle$
$\rightarrow_4$	Kleene-Dienes	$\langle \max(b, c), \min(a, d) \rangle$
$\rightarrow_5$	Lukasiewicz	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$
$\rightarrow_6$	Reichenbach	$\langle b + ac, ad \rangle$
$\rightarrow_7$	Willmott	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d)) \rangle$
$\rightarrow_8$	Wu	$\langle 1 - (1 - \min(b, c)).sg(a - c),$ $\max(a, d).sg(a - c).sg(d - b) \rangle$
$\rightarrow_9$	Klir and Yuan 1	$\langle b + a^2 c, ab + a^2 d \rangle$
$\rightarrow_{10}$	Klir and Yuan 2	$\langle c.\overline{sg}(1 - a) + sg(1 - a).(\overline{sg}(1 - c) + b.sg(1 - c)),$ $d.\overline{sg}(1 - a) + a.sg(1 - a).sg(1 - c) \rangle$
$\rightarrow_{11}$	Atanassov 1	$\langle 1 - (1 - c).sg(a - c), d.sg(a - c).sg(d - b) \rangle$
$\rightarrow_{12}$	Atanassov 2	$\langle \max(b, c), 1 - \max(b, c) \rangle$
$\rightarrow_{13}$	Atanassov and Kolev	$\langle b + c - b.c, a.d \rangle$
$\rightarrow_{14}$	Atanassov and Trifonov	$\langle 1 - (1 - c).sg(a - c) - d.\overline{sg}(a - c).sg(d - b),$ $d.sg(d - b) \rangle$
$\rightarrow_{15}$	Atanassov 3 (see below)	$\langle 1 - (1 - \min(b, c)).sg(sg(a - c) + sg(d - b))$ $- \min(b, c).sg(a - c).sg(d - b),$ $1 - (1 - \max(a, d)).sg(\overline{sg}(a - c) + \overline{sg}(d - b))$ $- \max(a, d).\overline{sg}(a - c).\overline{sg}(d - b) \rangle$

The correctness of the above definitions of implications is directly checked.

2. Main results. First, we shall introduce a new implication inspired by the paper of TAKEUTI and TITANI [9] and its answer by T. Trifonov and the author [6].

In [9] Takeuti and Titani introduce the following implication for $p, q \in [0, 1]$:

$$p \rightarrow q = \bigvee \{ r \in [0, 1] \mid p \wedge r \leq q \} = \begin{cases} 1, & \text{if } p \leq q \\ q, & \text{if } p > q. \end{cases}$$

In [6] the intuitionistic fuzzy extension is given in the form

$$\langle a, b \rangle \rightarrow \langle c, d \rangle = \langle \max(c, sg^*(a - c)), \min(d, sg(a - c)) \rangle$$

$$= \begin{cases} \langle 1,0\rangle, & \text{if } a \le c \text{ and } b \ge d \\ \langle 1,0\rangle, & \text{if } a \le c \text{ and } b < d \\ \langle c,d\rangle, & \text{if } a > c \text{ and } b \ge d \\ \langle c,d\rangle, & \text{if } a > c \text{ and } b < d \end{cases}$$

Theorem 1. The latter implication coincides with Gödel's implication ($\rightarrow_3$).
Proof. Let

$$X \equiv 1 - (1-c).sg(a-c) - \max(c, \overline{sg}(a-c)),$$

$$Y \equiv d.sg(a-c) - \min(d, sg(a-c)).$$

If $a > c$:

$$X = 1 - (1-c) - \max(c,0) = c - c = 0,$$
$$Y = d.1 - \min(d,1) = d - d = 0.$$

If $a \le c$:

$$X = 1 - (1-c).0 - \max(c,1) = 1 - 1 = 0,$$
$$Y = d.0 - \min(d,0) = 0 - 0 = 0,$$

i.e.

$$\langle 1-(1-c).sg(a-c), d.sg(a-c)\rangle = \langle \max(c, sg^*(a-c)), \min(d, sg(a-c))\rangle.$$

All constructions from [9] are rewritten in [6] for intuitionistic fuzzy case in order to show that Takeuti and Titani's sets are a particular case of intuitionistic fuzzy sets in the sense of [2]. Practically, the constructed by them set is ordinary fuzzy set with elements satisfying intuitionistic logic axioms. In [6] we show that there are intuitionistic fuzzy sets with elements satisfying intuitionistic logic axioms.

In [7] the latter implication is extended to the form of implication $\rightarrow_{14}$ from Table 1.

Here, we shall modify the implications $\rightarrow_3$ and $\rightarrow_{14}$ to the form

$$\langle a,b\rangle \rightarrow_{15} \langle c,d\rangle = \langle 1-(1-\min(b,c)).sg(sg(a-c)+sg(d-b))$$

$$-\min(b,c).sg(a-c).sg(d-b),$$

$$1-(1-\max(a,d)).sg(sg^*(a-c)+sg^*(d-b)) - \max(a,d).sg^*(a-c).sg^*(d-b)\rangle$$

$$= \begin{cases} \langle 1,0\rangle, & \text{if } a \le c \text{ and } b \ge d \\ \langle \min(b,c)\max(a,d)\rangle, & \text{if } a > c \text{ and } b \ge d \\ & \text{or } a \le c \text{ and } b < d \\ \langle 0,1\rangle, & \text{or } a > c \text{ and } b < d. \end{cases}$$

The definition of the new implication is correct, because for every $a, c, c, d \in [0,1]$ such that $a + b \le 1$ and $c + d \le 1$ for the expression

$$X \equiv 1-(1-\min(b,c)).sg(sg(a-c)+sg(d-b)) - \min(b,c).sg(a-c).sg(d-b)$$

$$+1-(1-\max(a,d)).sg(sg^*(a-c)+sg^*(d-b)) - \max(a,d).sg^*(a-c).sg^*(d-b)$$

we obtain

if $a \le c$ and $b \ge d$ then

$$X = 1-(1-\min(b,c)).sg(0+0) - \min(b,c).0 + 1 - (\max(a,d)).sg(1+1) - \max(a,d).1$$
$$= 1 + 1 - (1-\max(a,d)) - \max(a,d) = 1;$$

$$\text{if } a \leq c \text{ and } b < d \text{ then}$$

$$X = 1 - (1 - \min(b,c)).sg(0+1) - \min(b,c).0.1$$
$$+1 - (1 - \max(a,d)).sg(1+0) - \max(a,d).1.0$$
$$= 1 - 1 + \min(b,c) + 1 - 1 + \max(a,d)$$
$$= \min(b,c) + \max(a,d) \leq 1;$$

$$\text{if } a > c \text{ and } b \geq d \text{ then}$$

$$X = 1-(1-\min(b,c)).sg(1+0)-\min(b,c).1.0+1-(1-\max(a,d)).sg(0+1)-\max(a,d).0.1$$
$$= 1 - 1 + \min(b,c) + 1 - 1 + \max(a,d)$$
$$= \min(b,c) + \max(a,d) \leq 1;$$

$$\text{if } a > c \text{ and } b < d \text{ then}$$

$$X = 1-(1-\min(b,c)).sg(1+1)-\min(b,c).1.1+1-(1-\max(a,d)).sg(0+0)-\max(a,d).0.0$$
$$= 1 - (1 - \min(b,c)).1 - \min(b,c) + 1 - (1 - \max(a,d)).0$$
$$= 1 - 1 + \min(b,c)) - \min(b,c) + 1 = 1.$$

In [3–5,7] analogues of the following three theorems are proved for the first fourteen implications.

Theorem 2. Implication $\rightarrow_{15}$ satisfies Modus Ponens in the case of tautology.

Theorem 3. Implication $\rightarrow_{15}$ does not satisfy Modus Ponens in the case of IFT.

Theorem 4. For implication $\rightarrow_{15}$ and for every two variables x and y, $I(x\&I(x,y),y)$ is IFT.

Second, we will construct a negation generated by the new implication

$$V(\neg x) = \langle a,b\rangle \rightarrow \langle 0,1\rangle$$

$$= \langle 1 - sg(sg(a) + sg(1-b)), 1 - \overline{sg}(a).\overline{sg}(1-b)\rangle.$$

As we mentioned above, nine axioms are introduced in [8]. They are the following:

Axiom 1. $(\forall x,y)(x \leq y \rightarrow (\forall z)(I(x,z) \geq I(y,z)))$.

Axiom 2. $(\forall x,y)(x \leq y \rightarrow (\forall z)(I(z,x) \leq I(z,y)))$.

Axiom 3. $(\forall y)(I(0,y) = 1)$.

Axiom 4. $(\forall y)(I(1,y) = y)$.

Axiom 5. $(\forall x)(I(x,x) = 1)$.

Axiom 6. $(\forall x,y,z)(I(x,I(y,z)) = I(y,I(x,z)))$.

Axiom 7. $(\forall x,y)(I(x,y) = 1 \text{ iff } x \leq y)$.

Axiom 8. $(\forall x,y)(I(x,y) = I(N(y),N(x)))$, where N is an operation for a negation.

Axiom 9. I is a continuous function.

Theorem 5. Implication $\rightarrow_{15}$ satisfies Axioms 1,2,3,5,7,8.

Theorem 6. Implication $\rightarrow_{15}$ satisfies Axiom 8 when N is a negation with the form $V(N(x)) = \langle b,a\rangle$, but it does not satisfy Axiom 8, when N is the above mentioned negation $\neg$ generated by the new implication.

Third, we shall discuss another property of the fifteen implications.

We shall introduce the expression

$$Z_i = x \rightarrow_i y,$$

Table 2

Relations between elements of set $\{Z_i | 1 \le i \le 15\}$

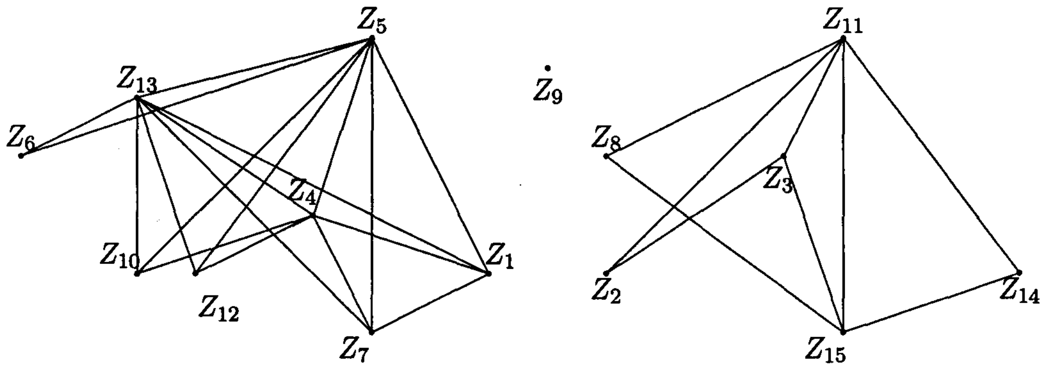
	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}	Z_{13}	Z_{14}	Z_{15}
Z_1	$=$	*	*	$\le$	$\le$	*	$\ge$	*	*	*	*	*	$\le$	*	*
Z_2	*	$=$	$\le$	*	*	*	*	*	*	*	$\le$	*	*	*	*
Z_3	*	$\ge$	$=$	*	*	*	*	*	*	*	$\le$	*	*	*	$\ge$
Z_4	$\ge$	*	*	$=$	$\le$	*	$\ge$	*	*	$\ge$	*	$\ge$	$\le$	*	*
Z_5	$\ge$	*	*	$\ge$	$=$	$\ge$	$\ge$	*	*	$\ge$	*	$\ge$	$\ge$	*	*
Z_6	*	*	*	*	$\le$	$=$	*	*	*	*	*	*	$\le$	*	*
Z_7	$\le$	*	*	$\le$	$\le$	*	$=$	*	*	*	*	*	$\le$	*	*
Z_8	*	*	*	*	*	*	*	$=$	*	*	$\le$	*	*	*	$\ge$
Z_9	*	*	*	*	*	*	*	*	$=$	*	*	*	*	*	*
Z_{10}	*	*	*	$\le$	$\le$	*	*	*	*	$=$	*	*	$\le$	*	*
Z_{11}	*	$\ge$	$\ge$	*	*	*	*	$\ge$	*	*	$=$	*	*	$\ge$	$\ge$
Z_{12}	*	*	*	$\le$	$\le$	*	*	*	*	*	*	$=$	$\le$	*	*
Z_{13}	$\ge$	*	*	$\ge$	$\le$	$\ge$	$\ge$	*	*	$\ge$	*	$\ge$	$=$	*	*
Z_{14}	*	*	*	*	*	*	*	*	*	*	$\le$	*	*	$=$	$\ge$
Z_{15}	*	*	$\le$	*	*	*	*	$\le$	*	*	$\le$	*	*	$\le$	$=$

where $1 \le i \le 15$. We shall call that Z_i is more powerful than Z_j for $1 \le i, j \le 15$, if

$$V(Z_i) \ge V(Z_j).$$

We can construct Table 2 in which the lack of relation between two implications is noted by "$*$".

Now, we can construct the following oriented graph where the arcs are directed top downwards.



3. Conclusion. Each of the above mentioned implications generates a respective operation negation. Their forms and properties will be an object of next author's research. The new negations will inspire changes in De Morgan's laws that will be discussed in another paper.

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