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On Fodor's type of intuitionistic fuzzy implication and negation

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Abstract: A Fodor's type of intuitionistic fuzzy implication is constructed. Its relation with some forms of Klir and Yuan's axioms are studied. Some open problems, related to the operations of intuitionistic fuzzy propositional calculus, are formulated.
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1 Introduction

The concept of *"intuitionistic fuzzy propositional calculus"* has been introduced about 20 years ago (see, e.g., [1, 2]). Initially, it contained only one form of conjunction, disjunction and two forms of implication. In a series of papers, e.g., [3–10], other forms of these three operations were defined. Now, there are 4 forms of operations conjunction and disjunction, 154 forms of operation implication and 45 forms of operation negation.

Here, we introduce a new operation implication, study some of its properties and formulate open problems, related to the operations of intuitionistic fuzzy propositional calculus.

It is based on Janos Fodor's fuzzy implication, that for $a, c \in [0, 1]$ is defined by

$$a \to c = \begin{cases} 1, & \text{if } a \leq c \\ \max(1 - a, c), & \text{otherwise} \end{cases}$$

In intuitionistic fuzzy propositional calculus, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. In [5], we called this couple an *"intuitionistic fuzzy pair" (IFP).*

Below we shall assume that for the two variables x and y the equalities: $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1]) by:

$$x \text{ is an IFT if and only if for } V(x) = \langle a, b \rangle \text{ holds: } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0 \rangle$.

2 Main results

We start with the definition of the new implication.

$$V(x \to y) = \langle a, b \rangle \to \langle c, d \rangle = \langle \overline{\text{sg}}(a - c) + \text{sg}(a - c)\max(b, c), \text{sg}(a - c)\min(a, d) \rangle,$$

where we use functions sg and $\overline{\text{sg}}$ defined by,

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

It is based on Janus Fodor's fuzzy implication.

2.1. First, let

$$X \equiv \overline{\text{sg}}(a - c) + \text{sg}(a - c)\max(b, c) + \text{sg}(a - c)\min(a, d).$$

If $a \leq c$, then

$$X = 1 + 0 + 0 = 1.$$

If $a > c$, then

$$X \equiv 0 + \max(b, c) + \min(a, d) \leq \max(1 - a, 1 - d) + \min(a, d) = 1 - \min(a, d) + \min(a, d) = 1,$$

i.e., definition is correct.

When $b = 1 - a$ and $d = 1 - c$, we obtain:

$$Z \equiv \langle a, 1-a\rangle \to \langle c, 1-d\rangle = \langle \overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(1-a,c), \mathrm{sg}(a-c)\min(a,1-c)\rangle.$$

Let $a \le c$. Then:

$$Z = \langle \overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(1-a,c), \mathrm{sg}(a-c)\min(a,1-c)\rangle = \langle 1+0, 0\rangle = \langle 1, 0\rangle.$$

Let $a > c$. Then:

$$Z = \langle \overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(1-a,c), \mathrm{sg}(a-c)\min(a,1-c)\rangle$$

$$= \langle 0 + \max(1-a,c), \min(a,1-c)\rangle = \langle \max(1-a,c), \min(a,1-c)\rangle.$$

Therefore, the first component of each one of the two IF-pairs corresponds to the respective value of Fodor's implication, while the second component is 1 minus the first one. Hence, our implication as a partial case give Fodor's one

2.2. Second, we check that

$$\langle 0,0\rangle \to \langle 0,0\rangle = \langle 1,0\rangle,$$
$$\langle 0,0\rangle \to \langle 1,0\rangle = \langle 1,0\rangle,$$
$$\langle 1,0\rangle \to \langle 0,0\rangle = \langle 0,1\rangle,$$
$$\langle 1,0\rangle \to \langle 1,0\rangle = \langle 1,0\rangle,$$

i.e., the new implication has the beheviour of the standard classical logic implication.

2.3. Having in mind that in classical logic the negation is obtained by formula

$$\neg x = x \to \overline{0},$$

we obtain the negation, related to the new implication:

$$\neg\langle a,b\rangle = \langle a,b\rangle \to \langle 0,1\rangle = \langle \overline{\mathrm{sg}}(a-0) + \mathrm{sg}(a-0)\max(b,0), \mathrm{sg}(a-0)\min(a,1)$$

$$= \langle \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, \mathrm{sg}(a)a\rangle = \langle \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, a\rangle.$$

The so obtained negation does not analogous amoung the existing intuitionistic fuzzy negations (see, e.g., [4]) and we call it *Fodor's intuitionistic fuzzy negation.*

For the new negation we check the validity of the following three properties

Property P1: $A \to \neg\neg A$ is a tautology (an IFT),

Property P2: $\neg\neg A \to A$ is a tautology (an IFT),

Property P3: $\neg\neg\neg A = \neg A$.

First, we check Property P1:

$$A \to \neg\neg A = \langle a,b\rangle \to \neg\langle \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, a\rangle$$

$$= \langle a,b\rangle \to \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) + \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a, os(a) + \mathrm{sg}(a)b\rangle$$

$$= \langle \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) - \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a) + \mathrm{sg}(a - \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) - \mathrm{sg}(\overline{\mathrm{sg}}(a)$$

$$+\mathrm{sg}(a)b)a).\max(b,\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a),$$

$$\mathrm{sg}(a-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)-\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a).\min(a,\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)\rangle.$$

Let

$$X\equiv\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)-\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a)+\mathrm{sg}(a-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)-\mathrm{sg}(\overline{\mathrm{sg}}(a)$$

$$+\mathrm{sg}(a)b)a).\max(b,\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a).$$

If $a=0$, then

$$X=\overline{\mathrm{sg}}(0-\overline{\mathrm{sg}}(1)-\mathrm{sg}(1)a)-\mathrm{sg}(0-\overline{\mathrm{sg}}(1)-\mathrm{sg}(1)a).\max(b,\overline{\mathrm{sg}}(1)+\mathrm{sg}(1)a)=1-0.\max(b,a)=1.$$

If $a>0$, then

$$X=\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(b)-\mathrm{sg}(b)a)+\mathrm{sg}(a-\overline{\mathrm{sg}}(b)-\mathrm{sg}(b)a).\max(b,\overline{\mathrm{sg}}(b)+\mathrm{sg}(b)a).$$

If $b=0$, then

$$X=\overline{\mathrm{sg}}(a-1)+\mathrm{sg}(a-1).\max(0,1)=1+0=1.$$

If $b>0$, then

$$X=\overline{\mathrm{sg}}(a-a)+\mathrm{sg}(a-a).\max(b,a)=1.$$

Let

$$Y\equiv\mathrm{sg}(a-\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)-\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a).\min(a,\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b).$$

If $a=0$, then

$$Y=\mathrm{sg}(-\overline{\mathrm{sg}}(1)).\min(0,1)=0.$$

If $a>0$, then

$$Y=\mathrm{sg}(a-\overline{\mathrm{sg}}(b)-\mathrm{sg}(b)a).\min(a,b).$$

If $b=0$, then

$$Y=\mathrm{sg}(a-1).\min(a,0)=0.$$

If $b>0$, then

$$Y=\mathrm{sg}(a-a).\min(a,b)=0.$$

Therefore, $A\to\neg\neg A$ is a tautology and therefore, an IFT.

Second, we check Property 2:

$$\neg\neg A\to A=\langle\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a,\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b\rangle\to\langle a,b\rangle$$

$$=\langle\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a-a)+\mathrm{sg}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)$$

$$+\mathrm{sg}(a)b)a-a).\max(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b,a),\mathrm{sg}(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a-a)$$

$$.\min(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)+\mathrm{sg}(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b)a,b)\rangle.$$

Let

$$X \equiv \overline{\text{sg}}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a - a) + \text{sg}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a)$$

$$+\text{sg}(a)b)a - a). \max(\overline{\text{sg}}(a) + \text{sg}(a)b, a).$$

If $a = 0$, then

$$X = \overline{\text{sg}}(\overline{\text{sg}}(1) + 0) + \text{sg}(\overline{\text{sg}}(1) + \text{sg}(1)a - a). \max(1, 0) = 0 + \text{sg}(a - a). \max(1, 0) = 0.$$

If $a > 0$, then

$$X = \overline{\text{sg}}(\overline{\text{sg}}(b) + \text{sg}(b)a - a) + \text{sg}(\overline{\text{sg}}(b) + \text{sg}(b)a - a). \max(b, a).$$

If $b = 0$, then

$$X = \overline{\text{sg}}(1 - a) + \text{sg}(1 - a)a.$$

If $a = 1$, then

$$X = 1,$$

while, if $a < 1$, then

$$X = a.$$

Therefore, there are values of a for which Property 2 is not valid.

Third, we check Property 3:

$$\neg\neg\neg\langle a, b\rangle = \neg\neg\langle\overline{\text{sg}}(a) + \text{sg}(a)b, a\rangle$$

$$= \neg\langle\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a, \overline{\text{sg}}(a) + \text{sg}(a)b\rangle$$

$$= \langle\overline{\text{sg}}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a) + \text{sg}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a)$$

$$+\text{sg}(a)b)a)(\overline{\text{sg}}(a) + \text{sg}(a)b), \overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a\rangle.$$

Let

$$X \equiv \overline{\text{sg}}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a)$$

$$+\text{sg}(\overline{\text{sg}}(\overline{\text{sg}}(a) + \text{sg}(a)b) + \text{sg}(\overline{\text{sg}}(a) + \text{sg}(a)b)a)(\overline{\text{sg}}(a) + \text{sg}(a)b).$$

If $a = 0$, then

$$X = \overline{\text{sg}}(\overline{\text{sg}}(1)) + \text{sg}(\overline{\text{sg}}(1) + \text{sg}(1))(1 + 0) = 1 + \text{sg}(\overline{\text{sg}}(1) + \text{sg}(1))(1 + 0) = 1 = \overline{\text{sg}}(a) + \text{sg}(a)b.$$

If $a > 0$, then

$$X = \overline{\text{sg}}(\overline{\text{sg}}(b) + \text{sg}(b)a) + \text{sg}(\overline{\text{sg}}(b) + \text{sg}(b)a)b.$$

If $b = 0$, then

$$X = \overline{\text{sg}}(1) = 0 = \overline{\text{sg}}(a) + \text{sg}(a)b.$$

If $b > 0$, then

$$X = \overline{\text{sg}}(a) + \text{sg}(a)b = b = \overline{\text{sg}}(a) + \text{sg}(a)b.$$

Therefore, in all cases the first component of the IFPs $\neg\neg\neg\langle a, b\rangle$ and $\neg\langle a, b\rangle$ coincide.

Let

$$Y \equiv \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) + \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a.$$

If $a = 0$, then

$$Y = \overline{\mathrm{sg}}(1) + \mathrm{sg}(1)a = 0 = a.$$

If $a > 0$, then

$$Y = \overline{\mathrm{sg}}(b) + \mathrm{sg}(b)a.$$

If $b > 0$, then

$$Y = a,$$

but if $b = 0$, then

$$Y = 1.$$

Therefore, if $a < 1$, the second component of the IFPs $\neg\neg\neg\langle a, b\rangle$ and $\neg\langle a, b\rangle$ do not coincide in all cases, i.e., Property P3 is not valid always.

It is interesting to mention that the classical intuitionistic fuzzy negation that has the form $\neg\langle a, b\rangle = \langle b, a\rangle$ satisfy De Morgan's laws in the form

$$x \wedge y = \neg(\neg x \vee \neg y) \text{ and } x \vee y = \neg(\neg x \wedge \neg y),$$

while a part of non-classical intuitionistic fuzzy negations do not satisfy them in this form and they satisfy these laws in the forms (see [3])

$$\neg\neg x \wedge \neg\neg y = \neg(\neg x \vee \neg y),$$

$$\neg\neg x \vee \neg\neg y = \neg(\neg x \wedge \neg y).$$

Now, we see that the Fodor's intuitionistic fuzzy negation does not satisfy these equalities. Really,

$$\neg\neg\langle a, b\rangle \wedge \neg\neg\langle c, d\rangle = \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) + \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a, \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b\rangle$$

$$= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d) + \mathrm{sg}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d)c, \overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d\rangle$$

$$= \langle \min(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) + \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a, \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d) + \mathrm{sg}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d)c),$$

$$\max(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, \overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d)\rangle$$

and

$$\neg(\neg\langle a, b\rangle \vee \neg\langle c, d\rangle) = \neg(\langle \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, a\rangle \vee \langle \overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d, c\rangle$$

$$= \neg(\langle \max(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, \overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d), \min(a, c)\rangle$$

$$= \langle \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b, \overline{\mathrm{sg}}(c)+\mathrm{sg}(c)d)) + \mathrm{sg}(\max(\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)b, \overline{\mathrm{sg}}(c)+\mathrm{sg}(c)d)) \min(a, c),$$

$$\max(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, \overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d)\rangle.$$

Let

$$X \equiv \min(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b) + \mathrm{sg}(\overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b)a, \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d) + \mathrm{sg}(\overline{\mathrm{sg}}(c) + \mathrm{sg}(c)d)c)$$

$-\overline{\text{sg}}(\max(\overline{\text{sg}}(a)+\text{sg}(a)b, \overline{\text{sg}}(c)+\text{sg}(c)d)) - \text{sg}(\max(\overline{\text{sg}}(a)+\text{sg}(a)b, \overline{\text{sg}}(c)+\text{sg}(c)d))\min(a,c)$.

If $a = 0$, then

$$X = \min(\overline{\text{sg}}(1), \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) + \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)c) - \overline{\text{sg}}(\max(1, \overline{\text{sg}}(c)+\text{sg}(c)d))$$

$$-\text{sg}(\max(1, \overline{\text{sg}}(c)+\text{sg}(c)d))\min(0,c)$$

$$= \min(0, \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) + \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)c) - \overline{\text{sg}}(1) - \text{sg}(1)\min(0,c) = 0 - 0 - 0 = 0.$$

If $a > 0$, then

$$X = \min(\overline{\text{sg}}(b)+\text{sg}(b)a, \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) + \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)c) - \overline{\text{sg}}(\max(b, \overline{\text{sg}}(c)+\text{sg}(c)d))$$

$$-\text{sg}(\max(b, \overline{\text{sg}}(c)+\text{sg}(c)d))\min(a,c).$$

If $b = 0$, then

$$X = \min(1, \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) + \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)c) - \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) - \text{sg}(\overline{\text{sg}}(c)$$

$$+\text{sg}(c)d)\min(a,c)$$

$$= \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) + \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)c - \overline{\text{sg}}(\overline{\text{sg}}(c)+\text{sg}(c)d) - \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)\min(a,c)$$

$$= \text{sg}(\overline{\text{sg}}(c)+\text{sg}(c)d)(c - \min(a,c)).$$

If $a \geq c$, then we obtain that $X = 0$, i.e., the first members of the two intuitionistic fuzzy pairs coincide, but if $a < c$, then this it will be not valid and therefore, in this case the new form of De Morgan's law will be not valid. For the second law the check is the same.

2.4. Now, we discuss other properties of our new intuitionistic fuzzy implication taking into account the classic Georg Klir and Bo Yuan's book [7] that it is convenient for our purposes. However, a similar, if not practically identical, analyses can be performed in the new settings and views related to fuzzy implications, notably included in the Baczynksi and Jayaram's book [6].

Some variants of fuzzy implications (marked by $I(x,y)$) are described in [7] and the following nine axioms are discussed, where

$$I(x,y) \equiv x \to y.$$

Axiom 1 $(\forall x,y)(x \leq y \to (\forall z)(I(x,z) \geq I(y,z)))$.
Axiom 2 $(\forall x,y)(x \leq y \to (\forall z)(I(z,x) \leq I(z,y)))$.
Axiom 3 $(\forall y)(I(0,y) = 1)$.
Axiom 4 $(\forall y)(I(1,y) = y)$.
Axiom 5 $(\forall x)(I(x,x) = 1)$.
Axiom 6 $(\forall x,y,z)(I(x,I(y,z)) = I(y,I(x,z)))$.
Axiom 7 $(\forall x,y)(I(x,y) = 1$ iff $x \leq y)$.
Axiom 8 $(\forall x,y)(I(x,y) = I(N(y),N(x)))$, where N is a variant of the operation negation.
Axiom 9 I is a continuous function.

Below, we study which axioms are valid for the new implication and in which form (as tautologies or as IFTs).

Let $a \leq c$ and $b \geq d$. Then for Axiom 1 we obtain:

$$V(x \to z) = \langle \overline{\text{sg}}(a-e) + \text{sg}(a-e)\max(b,e), \text{sg}(a-e)\min(a,f)\rangle$$

and

$$V(y \to z) = \langle \overline{\text{sg}}(c-e) + \text{sg}(c-e)\max(d,e), \text{sg}(c-e)\min(c,f)\rangle.$$

Then $\overline{\text{sg}}(a-e) \geq \overline{\text{sg}}(c-e)$, $\text{sg}(a-e) \geq \text{sg}(c-e)$ and

$$\overline{\text{sg}}(a-e) + \text{sg}(a-e)\max(b,e) - \overline{\text{sg}}(c-e) - \text{sg}(c-e)\max(d,e) \geq \text{sg}(a-e)(\max(b,e)$$
$$- \max(d,e)) \geq 0,$$

because $b \geq d$.

On the other hand, from $a \leq c$ it follows that

$$\text{sg}(c-e)\min(c,f) - \text{sg}(a-e)\min(a,f) \geq \text{sg}(a-e)(\min(c,f) - \min(a,f)) \geq 0.$$

Hence, Axiom 1 is valid.

The validity of Axiom 2 is checked analogously.

For Axiom 3 we obtain:

$$\langle 0,1\rangle \to \langle c,d\rangle = \langle \overline{\text{sg}}(0-c) + \text{sg}(0-c)\max(1,c), \text{sg}(0-c)\min(0,d)\rangle = \langle 1,0\rangle.$$

Now, we must mention that Axiom 4 is not valid, because

$$X \equiv \langle 1,0\rangle \to \langle c,d\rangle = \langle \overline{\text{sg}}(1-c) + \text{sg}(1-c)\max(0,c), \text{sg}(1-c)\min(1,d)\rangle$$
$$= \langle \overline{\text{sg}}(1-c) + \text{sg}(1-c)\max(0,c), \text{sg}(1-c)d\rangle.$$

If $c = 1$, then $d = 0$ and

$$X = \langle 1,0\rangle = \langle c,d\rangle.$$

If $c < 1$, then

$$X = \langle c,d\rangle.$$

Hence, Axiom 4 is valid.

For Axion 5 we obtain:

$$\langle a,b\rangle \to \langle a,b\rangle = \langle \overline{\text{sg}}(a-a) + \text{sg}(a-a)\max(b,a), \text{sg}(a-a)\min(a,b)\rangle = \langle 1,0\rangle,$$

i.e., it is valid.

By analogy, we can see that Axiom 6 is not valid. We searched different its modifications by analogy with the modified above De Morgan's laws, e.g.,

$$(\forall x,y,z)(I(N(N(x)), I(N(N(y)), N(N(z)))) = I(N(N(y)), I(N(N(x)), N(N(z)))))$$

and

$$(\forall x,y,z)(I(N(N(x)), I(N(N(y)), N(N(z)))) = N(N(I(N(N(y)), I(N(N(x)), N(N(z)))))),$$

but without success.

For the check of Axion 7, first we see that if $\langle a, b\rangle \le \langle c, d\rangle$, i.e., $a \le c$ and $d \le b$, then

$$\langle a, b\rangle \to \langle c, d\rangle = \langle \overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(b,c), \mathrm{sg}(a-c)\min(a,d)\rangle = \langle 1, 0\rangle.$$

The opposite direction is not valid, because if

$$\langle a, b\rangle \to \langle c, d\rangle = \langle \overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(b,c), \mathrm{sg}(a-c)\min(a,d)\rangle\langle 1, 0\rangle,$$

then

$$\overline{\mathrm{sg}}(a-c) + \mathrm{sg}(a-c)\max(b,c) = 1,$$

$$\mathrm{sg}(a-c)\min(a,d) = 0.$$

If we assume that $a > c$, i.e., $c < 1$, then from the first equality we obtain that $\max(b, c) = 1$, but this is possible only in $b = 1$, from where we obtain that $a = 0$. But this is impossible, because $a > c \ge 0$. Hence, $a \le c$. Unfortunately, we cannot determine the relation between b and d.

Therefore, it is valid Axion 7 in the form

Axion 7* $(\forall x, y)$(if $x \le y$ then $I(x, y)$).

For Axiom 8, as Axiom 6, we see that it is not valid in the original and in some modified forms as

$$(\forall x, y)(N(N(I(N(N(x)), N(N(y))))) = I(N(y), N(x)))$$

and

$$(\forall x, y)(N(N(I(N(N(x)), N(N(y))))) = N(N(I(N(y), N(x))))).$$

Obviously, Axiom 9 is not valid, because of existing of the non-continuous opertions sg and $\overline{\mathrm{sg}}$ in the definition of the new implication.

Finally, on the basis of the above discussion, we can formulate the following

Theorem. The new implication satisfies Axioms 1, 2, 3, 4, 5, 7* as tautologies (and therefore, as IFTs).

3 Conclusion

In next research other new implications will be introduced and studied. All they show that intuitonistc fuzzy sets and logics in the sense, described in [2, 4] correspond to the ideas of Brouwer's intuitionism.

It is important to note that the search of new implications and negations is important for constructing of rules for multicriteria and intercriteria analyses and theie evaluations.

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