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Intuitionistic Fuzzy Sets and Fuzzy Sets depending on $\mathbb{Q}$ - norms

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Abstract

In the paper the field of all rational numbers is considered as a universe set. A way of introducing Intuitionistic Fuzzy Sets and Fuzzy Sets with the help of some different norms on this field is proposed. Some modal operators are introduced and investigated too.

1 Keywords

Fuzzy Sets(FS), Intuitionistic Fuzzy Set (IFS), metric, norm, p-adic norm

2 Used denotations.

$\mathbb{Z}$ - the set of all integers, $\mathbb{Q}$ - the field of all rational numbers, E - the universe set, I - the interval [0,1], $\times$ - the Cartesian product of sets, FS - Fuzzy Set, IFS - Intuitionistic Fuzzy Set.

3 Introduction.

First we agree that our universe set (usually denoted by E) coincides with $\mathbb{Q}$. In the paper we shall consider a couple of mappings:

$$\mu : \mathbb{Q} \to \mathbb{Q} \cap I; \nu : \mathbb{Q} \to \mathbb{Q} \cap I$$

For $x \in \mathbb{Q}$, following Zadeh [1] and Atanassov [2] we call $\mu(x)$ a degree of membership of x and $\nu(x)$ a degree of non-membership of x. In [3] and [4] a different point of view on the notions of FS and IFS was proposed. It is based on some suitable metrics and norms. Several examples in this direction were provided. The present paper is a continuation of the ideas of the mentioned papers. The main difference lies in the fact that this time E is a countable set and some other norms are used.

4 Norms on $\mathbb{Q}$.

Let p be a fixed prime, i.e. $p \in \{2, 3, 5, 7, 11, 13...\}$. If $a \in \mathbb{Z}, a \neq 0$ we denote by $ord_p a$ the maximal non-negative m such that:

$$a \equiv 0 \pmod{p^m}$$

For example: $ord_2 12 = 2; ord_3 162 = 4; ord_7 15 = 0$.

It is easy to see that if $a, b \in \mathbb{Z}, a \neq 0, b \neq 0$, then

$$ord_p(a.b) = ord_p a + ord_p b \tag{1}$$

Or in other words $ord_p x$ and $log_p x$ share the same additive property.

Further we assume that $ord_p 0 = \infty$. Therefore (1) holds for all $a, b \in \mathbb{Z}$.
Let $x \in \mathbb{Q}$ and x admits the representation:

$$x = \frac{a}{b}$$

In this case we define

$$ord_p x = ord_p a - ord_p b \tag{2}$$

It is easy to establish that $ord_p x$ does not depend on the representation of x. Indeed if

$$x = \frac{a.c}{b.c}$$

then

$$ord_p x = ord_p(a.c) - ord_p(b.c) = \text{(due to (1))} =$$
$$= ord_p a + ord_p c - ord_p b - ord_p c = ord_p a - ord_p b$$

showing that the definition from (2) is correct.
Let us define:

$$\varphi_p(x) = \begin{cases} (\frac{1}{p})^{ord_p x}, x \neq 0; \\ 0, \ x = 0. \end{cases} \tag{3}$$

Then one may verify
Fact 1. φ_p is a norm on $\mathbb{Q}$ (this norm is called p-adic norm).
It is not difficult to establish that:

$$\varphi_p(x + y) \leq \max(\varphi_p(x), \varphi_p(y)), \text{ for all } x, y \in \mathbb{Q}$$

This means that φ_p is a non-archimedean norm on $\mathbb{Q}$.
Let us remind the following well-known fact (see [5]).
Lemma. Let F be a field and f and g be norms on F. Then f is equivalent to g on F if and only if there exists a positive real number α, such that for every $x \in F$ it is fulfilled:

$$f(x) = (g(x))^\alpha$$

Further, for an arbitrary $\alpha, 0 < \alpha \leq 1$ we define:

$$\varphi_\infty^{(\alpha)}(x) = (|x|)^\alpha$$

Then $\varphi_\infty^{(\alpha)}$ is an archimedean norm on $\mathbb{Q}$.

Fact 2. All norms $\varphi_\infty^{(\alpha)}, \alpha \in (0,1]$ are equivalent on $\mathbb{Q}$.

Let $\varrho \in (0,1)$. We consider φ_ϱ (with ϱ instead of $\frac{1}{p}$ in (3)).

Fact 3. φ_ϱ is an non-archimedean norm on $\mathbb{Q}$ that is equivalent to φ_p on $\mathbb{Q}$.

Fact 4. If p and q are different primes then the norms φ_p and φ_q are not equivalent on $\mathbb{Q}$.

A norm φ on $\mathbb{Q}$ is called trivial if $\varphi(x) = 1$ for every $x \in \mathbb{Q}$ (of course $\varphi(0) = 0$).

The following result of Ostrowski (see [5]) gives the full characterization of all nontrivial norms on $\mathbb{Q}$ (i.e. the norms that do not coincide with the trivial norm).

Theorem. For every nontrivial norm φ on $\mathbb{Q}$ the following is fulfilled:

If φ is a non-archimedean norm then there exists a prime p, such that φ is equivalent to φ_p on $\mathbb{Q}$.

If φ is an archimedean norm then φ is equivalent to $\varphi_\infty^{(1)}$ on $\mathbb{Q}$.

5 Intuitionistic Fuzzy Sets depending on $\mathbb{Q}$-norms.

Let φ be arbitrary nontrivial norm on $\mathbb{Q}$. The above Theorem describes φ very well in the case when φ is a non-archimedean norm just as in the case when φ is an archimedean norm. To understand that better one may reffer to the Lemma.

Considering the Cartesian product $\mathbb{Q} \times \mathbb{Q}$ and bearing in mind that φ makes $\mathbb{Q}$ a metric space with a metric d_φ that is given by:

$$d_\varphi(x, y) = \varphi(x - y), x, y \in \mathbb{Q},$$

we put the question how to introduce a metric in $\mathbb{Q} \times \mathbb{Q}$ that depends on φ. This can be done in infinitely many ways, relying on the fact that the same question but for the finite Cartesian product of an arbitrary metric spaces is answered in [6]. The situation with $\mathbb{Q} \times \mathbb{Q}$ is the same, so let $\hat{\varphi}$ be a norm on $\mathbb{Q} \times \mathbb{Q}$ that is expressed with the help of φ. For example we may put:

$$\hat{\varphi}(x, y) = \varphi(x) + \varphi(y), x, y \in \mathbb{Q} \tag{4}$$

or

$$\hat{\varphi}(x, y) = \max(\varphi(x), \varphi(y)), x, y \in \mathbb{Q}$$

or any other suitable explicit expression.

Since $\hat{\varphi}$ introduces a metric $d_{\hat{\varphi}}$ on $\mathbb{Q} \times \mathbb{Q}$, that is given by:

$$d_{\hat{\varphi}}((x_1, y_1), (x_2, y_2)) = \hat{\varphi}(x_1 - x_2, y_1 - y_2), (x_i, y_i) \in \mathbb{Q} \times \mathbb{Q}, (i = 1, 2),$$

then $\mathbb{Q} \times \mathbb{Q}$ is a metric space.

Remark. One may also consider $\mathbb{Q} \times \mathbb{Q}$ as a field, where the addition and multiplication are given by:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1),$$

where $x_i, y_i \in \mathbb{Q}, (i = 1, 2)$.

If we introduce in $\mathbb{Q} \times \mathbb{Q}$ the multiplication as:

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2 + 2y_1 y_2, x_1 y_2 + x_2 y_1),$$

and retain the defined above addition, then once again $\mathbb{Q} \times \mathbb{Q}$ is a field. An interesting fact is that we have

$$(0, 1) * (0, 1) = 2(1, 0).$$

Therefore, $(0, 1)$ is the $\mathbb{Q} \times \mathbb{Q}$ - representation of $\sqrt{2}$.

Now, following [3], §4, we are ready to introduce $d_{\hat{\varphi}}$-FS and $d_{\hat{\varphi}}$-IFS (i.e. $d_{\hat{\varphi}}$-Fuzzy sets and $d_{\hat{\varphi}}$-Intuitionistic Fuzzy sets).

Namely, let μ, ν be the mappings mentioned in §3 (from the present paper). Then for $x \in \mathbb{Q}, \mu(x)$ and $\nu(x)$ belong to $\mathbb{Q}$ too .

Definition1. We call the set

$$\{(\mu(x), \nu(x)) | x \in \mathbb{Q}\}$$

$d_{\hat{\varphi}}$-Fuzzy set (or $d_{\hat{\varphi}}$-FS), if it is fulfilled

$$\forall x \in \mathbb{Q}, d_{\hat{\varphi}}((\mu(x), \nu(x)), (0, 0)) = 1,$$

i.e.

$$\forall x \in \mathbb{Q}, \hat{\varphi}(\mu(x), \nu(x)) = 1$$

Definition2. We call the set

$$\{(\mu(x), \nu(x)) | x \in \mathbb{Q}\}$$

$d_{\hat{\varphi}}$-Intuitionistic Fuzzy set (or $d_{\hat{\varphi}}$-IFS), if it is fulfilled

$$\forall x \in \mathbb{Q}, d_{\hat{\varphi}}((\mu(x), \nu(x)), (0, 0)) \leq 1$$

i.e.

$$\forall x \in \mathbb{Q}, \hat{\varphi}(\mu(x), \nu(x)) \leq 1$$

It is easy to see that the usual FS (from [1]) and IFS from ([2]) are obtained as a particular case of the above definitions, when we take (4) with $\varphi = \varphi_\infty^{(1)}$. Also if we take (4) with $\varphi = \varphi_\infty^{(\alpha)}$, where $\alpha \in (0, 1]$, we obtain other types of FS and IFS. Some of them are considered and investigated in [2], [3] and [4].

All these sets correspond to the case when φ is an archimedean norm on $\mathbb{Q}$.

Below we propose an open problem.

Open problem 1. Using a non-archimedean norm φ on $\mathbb{Q}$, introduce a non-archimedean norm $\hat{\varphi}$ on $\mathbb{Q} \times \mathbb{Q}$ and investigate the properties of the generated $d_{\hat{\varphi}}$-FS and $d_{\hat{\varphi}}$-IFS in this case.

6 On $d_{\hat{\varphi}}$-IFS modal operators necessity and possibility

In the case of the usual IFS (see [2]) the modal operators possibility and necessity are defined as

$$\diamond A = \{(1 - \nu(x), \nu(x)) | x \in E\}$$

and

$$\square A = \{(\mu(x), 1 - \mu(x)) | x \in E\},$$

where $A = \{(\mu(x), \nu(x)) | x \in E\}$.

The following relations are fulfilled:

1.
$$\neg(\square \neg A) = \diamond A$$

2.
$$\neg(\diamond \neg A) = \square A$$

3.
$$\square A \subset A \subset \diamond A$$

4.
$$\square \square A = \square A$$

5.
$$\square \diamond A = \diamond A$$

6.
$$\diamond \square A = \square A$$

7.
$$\diamond \diamond A = \diamond A$$

where $\neg A := \{\nu(x), \mu(x) | x \in E\}$

Let $x_1, x_2, y_1, y_2 \in \mathbb{Q}$ and it is fulfilled:

$$0 \leq x_1 \leq x_2; 0 \leq y_1 \leq y_2; x_1 + x_2 < y_1 + y_2$$

If φ is an archimedean norm on $\mathbb{Q}$ and $\hat{\varphi}$ is given by (4), then it is easy to see that the inequality:

$$\hat{\varphi}(x_1, y_1) < \hat{\varphi}(x_2, y_2) \tag{5}$$

holds. That is clear from the Ostrowski's theorem, from the Lemma and from Fact 2. Based on (5) it is possible to introduce the modal operators $\square_{\hat{\varphi}}$ and $\diamond_{\hat{\varphi}}$ that act on $d_{\hat{\varphi}}$-IFS . These modal operators could be introduced using the method proposed in [3] for the introduction of the modal operators $\square_{\varphi}$ and $\diamond_{\varphi}$. One need only substitute φ from ([3]) with $\hat{\varphi}$. In the same manner the operator $\neg$ may be introduced here.

For the new modal operators the basic seven relations remain valid if we substitute $\square$ with $\square_{\hat{\varphi}}$, $\diamond$ with $\diamond_{\hat{\varphi}}$ and E with $\mathbb{Q}$.

Finally we put another open problem:

Open Problem 2. Introduce modal operators: $\square_{\hat{\varphi}}$, $\diamond_{\hat{\varphi}}$ acting on $d_{\hat{\varphi}}$-IFS, when φ is a non-archimedean norm on $\mathbb{Q}$.

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