Notes on Intuitionistic Fuzzy Sets
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Radical structures of intuitionistic fuzzy polynomial ideals of a ring

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Abstract: In this paper we investigate the radical structure of an intuitionistic fuzzy polynomial ideal A_x induced by an intuitionistic fuzzy ideal A of a ring and study its properties. Given an intuitionistic fuzzy ideal B of a ring R' and a homomorphism $f : R \to R'$, we show that if $f_x : R[x] \to R'[x]$ is the induced homomorphism of f, that is, $f_x(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n}(f(a_i))x^i$, then $f_x^{-1}[(\sqrt{B})_x] = (\sqrt{f^{-1}(B)})_x$.
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1 Introduction

One of the remarkable generalizations of the fuzzy sets [14] is the intuitionistic fuzzy sets which was introduced by Atanassov [1, 2]. Biswas was the first one to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Later on Hur and others in [7] and [6] defined and studied intuitionistic fuzzy subrings and ideals of a ring. With a different approach Banerjee and Basnet in [4] also studied intuitionistic fuzzy subrings and ideals of a ring. Jun and other in [8] introduced and study the notion of intuitionistic nil radicals of intuitionistic fuzzy ideals and Euclidean intuitionistic fuzzy ideals in rings. Translate of intuitionistic fuzzy subring and ideal was studied by Sharma in [12]. Meena and Thomas in [11] studied the concept of intuitionistic fuzzy subring to lattice setting and introduced the notion of intuitionistic L-fuzzy subring. The concept of Characteristic intuitionistic fuzzy subrings of an intuitionistic fuzzy ring was introduced by Meena in [10]. The present authors [13] introduced the notion of an intuitionistic fuzzy polynomial ideal A_x of a polynomial ring $R[x]$ induced by an intuitionistic fuzzy ideal A of a ring R and obtained an isomorphism theorem of a ring of an intuitionistic fuzzy cosets of A_x. It was shown that an intuitionistic fuzzy ideal A of a ring is an intuitionistic fuzzy prime if and only if A_x is an intuitionistic fuzzy prime ideal of $R[x]$. Moreover, it was shown that if A_x is an intuitionistic fuzzy maximal ideal of $R[x]$, then A is an intuitionistic fuzzy ideal of R'.

In this paper, we investigate the radical structure of intuitionistic fuzzy polynomial ideal induced by an intuitionistic fuzzy ideal of a ring and study its properties.

2 Preliminaries

Definition 2.1. ([1]) Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively and $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.2. (i) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in X$. Then A is called a fuzzy set.
(ii) We denote the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.3. ([2, 11]) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$ be any two IFSs of X, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$

(iii) $A^c = \{\langle x, \mu_{A^c}(x), \nu_{A^c}(x)\rangle \mid x \in X\}$, where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$

(iv) $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\}$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x)$

(v) $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle \mid x \in X\}$, where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.

Definition 2.4. ([3, 4, 6, 11]) Let R be a ring. An IFS $A = (\mu_A, \nu_A)$ of R is said to be an intuitionistic fuzzy ideal (IFI) of R if

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y), \forall x, y \in R$.

If A is an intuitionistic fuzzy ideal of R, then
(a) $\mu_A(0) \geq \mu_A(x) \geq \mu_A(1)$ and $\nu_A(0) \leq \nu_A(x) \leq \nu_A(1), \forall x \in R$.
(b) $\mu_A(x-y) = \mu_A(0)$ and $\nu_A(x-y) = \nu_A(0) \Leftrightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.
(c) The (α, β)-cut set of A, i.e., the set $C_{(\alpha,\beta)}(A) = \{x \in R \mid \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is an ideal of R, where $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$.
(d) If A and B are two IFIs of the ring R, then sum $A+B$ and the product AB are defined as: $\mu_{A+B}(x) = \vee_{x=y+z}\{\mu_A(y) \wedge \mu_B(z)\}$ and $\nu_{A+B}(x) = \wedge_{x=y+z}\{\nu_A(y) \vee \nu_B(z)\}, \forall x \in R$ and $\mu_{AB}(x) = \vee_{x=yz}\{\mu_A(y) \wedge \mu_B(z)\}$ and $\nu_{AB}(x) = \wedge_{x=yz}\{\nu_A(y) \vee \nu_B(z)\}, \forall x \in R$.

Definition 2.5. ([3, 4]) Let $f : R \to S$ be a homomorphism of rings and B be an IFS of S. We define an IFS $f^{-1}(B)$ of R by $f^{-1}(B)(x) := B(f(x)), \forall x \in R$.

Definition 2.6. ([3, 4]) Let $f : R \to S$ be a homomorphism of rings and A be an IFS of R. We define an IFS $f(A)$ of S by $f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y))$, where $\forall y \in S$,

$$f(A)(y) = \begin{cases} (\vee\{\mu_A(x) \mid x \in f^{-1}(y)\}, \wedge\{\nu_A(x) \mid x \in f^{-1}(y)\}), & \text{if } f^{-1}(y) \neq \phi \\ (0,1), & \text{otherwise} \end{cases}.$$

Definition 2.7. ([8]) Let R and S be any sets and let $f : R \to S$ be a function. An IFS A of R is called an f-invariant if $f(x) = f(y) \Rightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, where $x, y \in R$. If A is any f-invariant IFS of R, then $f^{-1}(f(A)) = A$.

Definition 2.8. ([9]) Let $f : R \to R'$ be a homomorphism of rings. A map $f_x : R[x] \to R'[x]$ defined by

$$f_x(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i,$$

is obviously a ring homomorphism, and we call f_x an induced homomorphism by f.

Theorem 2.9. ([13]) *Let $A = (\mu_A, \nu_A)$ be an IFI of a ring R and let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Define an IFS $A_x = (\mu_{A_x}, \nu_{A_x})$ on $R[x]$ by*

$$\mu_{A_x}(f(x)) = \min_i\{\mu_A(a_i)\} \text{ and } \nu_{A_x}(f(x)) = \max_i\{\nu_A(a_i)\}.$$

Then A_x is an IFI of $R[x]$.

The intuitionistic fuzzy ideal A_x is called the intuitionistic fuzzy polynomial ideal of $R[x]$ induced by an intuitionistic fuzzy ideal A of R.

Proposition 2.10. ([13]) *Let $f : R \to R'$ be a homomorphism of rings and let $f_x : R[x] \to R'[x]$ be an induced homomorphism of f. If A is an IFI of the ring R and A_x be its the intuitionistic fuzzy polynomial ideal of $R[x]$, then A is f -invariant if and only if A_x is f_x-invariant.*

Proposition 2.11. ([13]) *Let A be an IFI of the ring R. Then the set $S = \{f(x) \in R[x] \mid \mu_{A_x}(f(x)) = \mu_{A_x}(0)$ and $\nu_{A_x}(f(x)) = \nu_{A_x}(0)\}$ is a subring of $R[x]$.*

Remark 2.12. ([11]) Let A be an IFS of a ring R. We denote a level cut set A_* by

$$A_* = \{x \in R \mid \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}.$$

It is proved in [11] that if A is an IFI of ring R, then A_* is an ideal of ring R. Note that if A is an IFI of a ring R, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in R$.

We denote $A_*[x] = \{f(x) = \sum_{i=0}^{n} a_i x^i \in R[x] \mid a_i \in A_*, \forall i = 1, 2, \ldots, n\}$.

Theorem 2.13. ([13]) *Let A be an IFI of a ring R, then $(A_x)_* = A_*[x]$.*

Theorem 2.14. ([13]) *If A and B are two IFIs of a ring R, then*

(i) $(A \cap B)_x = A_x \cap B_x$.

(ii) $(A \cup B)_x \supseteq A_x \cup B_x$.

(iii) $A_x + B_x \subseteq (A + B)_x$.

(iv) $A_x B_x \subseteq (AB)_x$.

Theorem 2.15. ([13]) *Let $f : R \to R'$ be a homomorphism from R onto R' and let f_x be an induced homomorphism of f. If A is an f-invariant IFIs of R', then $(f(A))_x = f_x(A_x)$.*

Theorem 2.16. ([13]) *Let A be an IFI of a ring R. Then A is an intuitionistic fuzzy prime ideal of R if and only if A_x is an intuitionistic fuzzy prime ideal of $R[x]$.*

3 Radical of the intuitionistic fuzzy polynomial induced by an intuitionistic fuzzy ideal

In this section, we study some relations between the radical of the intuitionistic fuzzy polynomial ideal $R[x]$ induced by an intuitionistic fuzzy ideal of a ring R and the radical of an intuitionistic fuzzy ideal of the ring.

Definition 3.1. ([8]) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal of R. Then the intuitionistic fuzzy nil radical of A is defined to be an IFS $\sqrt{A} = (\mu_{\sqrt{A}}, \nu_{\sqrt{A}})$ defined by $\mu_A(x) = \vee\{\mu_A(x^n) \mid n > 0\}$ and $\nu_A(x) = \vee\{\nu_A(x^n) \mid n > 0\}, \forall x \in R$ and for some $n \in \mathbf{N}$.

Proposition 3.2. ([8]) *For any intuitionistic fuzzy ideals A and B of R, we have*

(i) $A \subseteq \sqrt{A}$

(ii) If $A \subseteq B$ *then* $\sqrt{A} \subseteq \sqrt{B}$

(iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Proof. Straightforward. □

Theorem 3.3. *For any intuitionistic fuzzy ideals A of R, $\sqrt{A}$ is an intuitionistic fuzzy ideal of R.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}\mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y) &= (\vee\{\mu_A(x^m) \mid m > 0\}) \wedge (\vee\{\mu_A(y^n) \mid n > 0\}) \\ &= \vee\{(\vee\{\mu_A(x^m) \wedge \mu_A(y^n) \mid n > 0\}) \mid m > 0\}.\end{aligned}$$

Thus,

$$\mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y) = \vee\{(\vee\{\mu_A(x^m) \wedge \mu_A(y^n) \mid n > 0\}) \mid m > 0\} \tag{3.1}$$

Similarly,

$$\nu_{\sqrt{A}}(x) \vee \nu_{\sqrt{A}}(y) = \wedge\{(\wedge\{\nu_A(x^m) \vee \nu_A(y^n) \mid n > 0\}) \mid m > 0\}. \tag{3.2}$$

Let m and n be any positive integers. Since R is commutative, we know that each term in the binomial expansion of $(x+y)^{m+n}$ contains either x^m or y^n as a factor. Hence there exist $r, t \in R$ such that $(x+y)^{m+n} = rx^m + ty^n$. Thus

$$\begin{aligned}\mu_A(x^m) \wedge \mu_A(y^n) &\le (\mu_A(x^m) \vee \mu_A(r)) \wedge (\mu_A(y^n) \vee \mu_A(t)) \\ &\le \mu_A(rx^m) \wedge \mu_A(ty^n) \\ &\le \mu_A(rx^m + ty^n) \\ &= \mu_A((x+y)^{m+n}) \\ &\le \vee\{\mu_A((x+y)^k) \mid k > 0\} \\ &= \mu_{\sqrt{A}}(x+y).\end{aligned}$$

Thus,

$$\mu_A(x^m) \wedge \mu_A(y^n) \le \mu_{\sqrt{A}}(x+y). \tag{3.3}$$

Similarly,

$$\nu_A(x^m) \vee \nu_A(y^n) \ge \nu_{\sqrt{A}}(x+y). \tag{3.4}$$

Notice that $\mu_A(x-y) \ge \mu_A(x) \wedge \mu_A(y) \Leftrightarrow \mu_A(x+y) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \le \nu_A(x) \vee \nu_A(y) \Leftrightarrow \nu_A(x+y) \le \nu_A(x) \vee \nu_A(y)$, respectively. Next, we have

$$\begin{aligned}\mu_{\sqrt{A}}(x) \vee \mu_{\sqrt{A}}(y) &= (\vee\{\mu_A(x^n) \mid n > 0\}) \vee (\vee\{\mu_A(y^n) \mid n > 0\}) \\ &= \vee\{(\vee\{\mu_A(x^n) \vee \mu_A(y^n) \mid n > 0\}.\end{aligned}$$

Thus,

$$\mu_{\sqrt{A}}(x) \vee \mu_{\sqrt{A}}(y) = \vee\{(\vee\{\mu_A(x^n) \vee \mu_A(y^n)\}) \mid n > 0\} \tag{3.5}$$

Similarly,

$$\nu_{\sqrt{A}}(x) \wedge \nu_{\sqrt{A}}(y) = \wedge\{(\nu_A(x^n) \wedge \nu_A(y^n)) \mid n > 0\}. \tag{3.6}$$

Since

$$\begin{aligned}\mu_A(x^n) \wedge \mu_A(y^n) &\le \mu_A(x^n y^n) \\ &= \mu_A((xy)^n) \\ &\le \vee\{\mu_A((xy)^k) \mid k > 0\} \\ &= \mu_{\sqrt{A}}(xy).\end{aligned}$$

Thus,

$$\mu_A(x^n) \wedge \mu_A(y^n) \leq \mu_{\sqrt{A}}(xy). \tag{3.7}$$

From (3.5) and (3.6) we get $\mu_{\sqrt{A}}(xy) \geq \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y)$ and $\nu_{\sqrt{A}}(xy) \leq \nu_{\sqrt{A}}(x) \vee \nu_{\sqrt{A}}(y)$. Hence $\sqrt{A}$ is an intuitionistic fuzzy ideal of R. □

Theorem 3.4. *If A and B are intuitionistic fuzzy ideals of R, then*

(i) $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$

(ii) $\sqrt{A \cup B} = \sqrt{A} \cup \sqrt{B}$

(iii) $\sqrt{A} + \sqrt{B} \subseteq \sqrt{A+B}$.

(iv) $\sqrt{A}\sqrt{B} \subseteq \sqrt{AB}$.

Proof. (i) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Therefore, by Proposition (3.2)(i) we get $\sqrt{A \cap B} \subseteq \sqrt{A}$ and $\sqrt{A \cap B} \subseteq \sqrt{B}$ and so, $\sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$.
For another inclusion, let $x \in R$ be any element. Then

$$\begin{aligned} \mu_{\sqrt{A} \cap \sqrt{B}}(x) &= \mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{B}}(x) \\ &= (\vee\{\mu_A(x^m) \mid m > 0\}) \wedge (\vee\{\mu_B(y^n) \mid n > 0\}) \\ &= \vee\{\vee\{\mu_A(x^m) \wedge \mu_B(y^n) \mid n > 0\} \mid m > 0\}. \end{aligned}$$

Similarly, we can show $\nu_{\sqrt{A} \cap \sqrt{B}}(x) = \wedge\{\wedge\{\nu_A(x^m) \vee \nu_B(y^n) \mid n > 0\} \mid m > 0\}$.
Now, let m and n be any positive integers. Then,

$$\begin{aligned} \mu_A(x^m) \wedge \mu_B(y^n) &\leq \mu_A(x^{mn}) \wedge \mu_B(y^{mn}) \\ &= \mu_{A \cap B}(x^{mn}) \\ &\leq \vee\{\mu_{A \cap B}(x^k) \mid k > 0\} \\ &= \mu_{\sqrt{A \cap B}}(x). \end{aligned}$$

Thus, $\mu_A(x^m) \wedge \mu_B(y^n) \leq \mu_{\sqrt{A \cap B}}(x)$. Therefore, $\mu_{\sqrt{A} \cap \sqrt{B}}(x) \leq \mu_{\sqrt{A \cap B}}(x)$ Similarly, we can show that $\nu_A(x^m) \vee \nu_B(y^n) \geq \nu_{\sqrt{A \cap B}}(x)$ and so, $\nu_{\sqrt{A} \cap \sqrt{B}}(x) \geq \nu_{\sqrt{A \cap B}}(x)$. Hence $\sqrt{A} \cap \sqrt{B} \subseteq \sqrt{A \cap B}$. This completes the proof of (i).
(ii) The proof follows similar to the proof of part (i).
(iii) The proof follows from the definition of sum of IFIs.
(iv) The proof follows from the definition of product of IFIs. □

If A is an intuitionistic fuzzy ideal of a ring R, then A_x is an intuitionsitic fuzzy ideal of a polynomial ring $R[x]$ by Theorem 2.9, the IFS $\sqrt{A_x}$ is the intuitionistic fuzzy nil radical of A_x. The following theorem gives that the two intuitionistic fuzzy nil radicals have the same value.

Theorem 3.5. *If A is an intuitionistic fuzzy ideal of R, then $(\sqrt{A_x})_x = (\sqrt{A})_x$.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ be any element of $R[x]$. Then by Theorem 2.9, we have $A_x(a_j^n) = (\mu_{A_x}(a_j^n), \nu_{A_x}(a_j^n))$, where

$$\mu_{A_x}(a_j^n) = \mu_{A_x(a_j^n + 0x + 0x^2 + \ldots + 0x^m)} = \min\{\mu_A(a_j^n), \mu_A(0), \ldots, \mu_A(0)\} = \mu_A(a_j^n)$$

and

$$\nu_{A_x}(a_j^n) = \nu_{A_x}(a_j^n + 0x + 0x^2 + \ldots + 0x^m) = \max\{\nu_A(a_j^n), \nu_A(0), \ldots, \nu_A(0)\} = \nu_A(a_j^n).$$

Since $\sqrt{A_x}$ is an intuitionistic fuzzy ideal of $R[x]$, we obtain

$$(\sqrt{A_x})_x(f(x)) = (\mu_{(\sqrt{A_x})_x}(f(x)), \nu_{(\sqrt{A_x})_x}(f(x))), \text{ where}$$

$$\begin{aligned}
\mu_{(\sqrt{A_x})_x}(f(x)) &= \min_{i=0}^{m}\{\mu_{\sqrt{A_x}}(a_i)\} \\
&= \min_{i=0}^{m}\{\vee\{\mu_{A_x}(a_i^n) \mid n > 0\}\} \\
&= \min_{i=0}^{m}\{\vee\{\mu_{A}(a_i^n) \mid n > 0\}\}[\because \mu_{A_x}(a_i^n) = \mu_A(a_i^n)] \\
&= \min_{i=0}^{m}\{\mu_{\sqrt{A}}(a_i)\} \\
&= \mu_{(\sqrt{A})_x}(f(x)).
\end{aligned}$$

Similarly, we can show that $\nu_{(\sqrt{A_x})_x}(f(x)) = \nu_{(\sqrt{A})_x}(f(x))$.
This proves that $(\sqrt{A_x})_x = (\sqrt{A})_x$. □

Theorem 3.6. *If A and B are intuitionistic fuzzy ideals of R, then*
(i) $(\sqrt{A \cap B})_x = (\sqrt{A})_x \cap (\sqrt{B})_x$
(ii) $(\sqrt{A})_x \cup (\sqrt{B})_x \subseteq (\sqrt{A \cup B})_x$
(iii) $(\sqrt{A})_x + (\sqrt{B})_x \subseteq (\sqrt{A + B})_x$
(iv) $(\sqrt{AB})_x \subseteq (\sqrt{A})_x(\sqrt{B})_x$.

Proof. Let A be an intuitionistic fuzzy ideal of R, then A_x and B_x are intuitionistic fuzzy polynomial ideals of $R[x]$ by Theorem 2.9.
For (i), we have

$$\begin{aligned}
(\sqrt{A \cap B})_x &= (\sqrt{(A \cap B)_x})_x \text{ [Theorem 3.5]} \\
&= (\sqrt{A_x \cap B_x})_x \text{ [Theorem 2.14 (i)]} \\
&= (\sqrt{A_x} \cap \sqrt{B_x})_x \text{ [Lemma 3.1 (i)]} \\
&= (\sqrt{A_x})_x \cap (\sqrt{B_x})_x \text{ [Theorem 2.14 (i)]} \\
&= (\sqrt{A})_x \cap (\sqrt{B})_x \text{ [Theorem 3.5].}
\end{aligned}$$

For (ii), we have

$$\begin{aligned}
(\sqrt{A})_x \cup (\sqrt{B})_x &= (\sqrt{A_x})_x \cup (\sqrt{B_x})_x \text{ [Theorem 3.5]} \\
&\subseteq (\sqrt{A_x} \cup \sqrt{B_x})_x \text{ [Theorem 2.14 (ii)]} \\
&= (\sqrt{A_x \cup B_x})_x \text{ [Lemma 3.4 (ii)]} \\
&\subseteq (\sqrt{(A \cup B)_x})_x \text{ [Theorem 2.14 (ii)]} \\
&= (\sqrt{A \cup B})_x \text{ [Theorem 3.5].}
\end{aligned}$$

For (iii), we have

$$\begin{aligned}(\sqrt{A})_x + (\sqrt{B})_x &= (\sqrt{A_x})_x + (\sqrt{B_x})_x \text{ [Theorem 3.5]}\\ &\subseteq (\sqrt{A_x} + \sqrt{B_x})_x \text{ [Theorem 2.14 (iii)]}\\ &= (\sqrt{A_x + B_x})_x \text{ [Lemma 3.4 (iii)]}\\ &\subseteq (\sqrt{(A+B)_x})_x \text{ [Theorem 2.14 (iii)]}\\ &= (\sqrt{A+B})_x \text{ [Theorem 3.5].}\end{aligned}$$

For (iv), we have

$$\begin{aligned}(\sqrt{A})_x(\sqrt{B})_x &= (\sqrt{A_x})_x(\sqrt{B_x})_x \text{ [Theorem 3.5]}\\ &\supseteq (\sqrt{A_x}\sqrt{B_x})_x \text{ [Theorem 2.14 (iv)]}\\ &= (\sqrt{A_x B_x})_x \text{ [Lemma 3.4 (iv)]}\\ &\supseteq (\sqrt{(AB)_x})_x \text{ [Theorem 2.14 (iv)]}\\ &= (\sqrt{AB})_x \text{ [Theorem 3.5].}\end{aligned}$$

This completes the proof. □

Theorem 3.7. *Let B be an intuitionistic fuzzy ideal of R' and let $f : R \to R'$ be a homomorphism of rings. If f_x is the induced homomorphism of f, i.e., $f_x(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n f(a_i)x^i$, then $f_x^{-1}[(\sqrt{B})_x] = (\sqrt{f^{-1}(B)})_x$.*

Proof. Given a polynomial $g(x) = \sum_{i=0}^m b_i x^i \in R[x]$, we have

$$(\sqrt{f^{-1}(B)})_x(g(x)) = (\mu_{(\sqrt{f^{-1}(B)})_x}(g(x)), \nu_{(\sqrt{f^{-1}(B)})_x}(g(x))), \text{ where}$$

$$\begin{aligned}\mu_{(\sqrt{f^{-1}(B)})_x}(g(x)) &= \min_i^m\{\mu_{\sqrt{f^{-1}(B)}}(b_i)\}\\ &= \min_i^m\{\vee\{\mu_{f^{-1}(B)}(b_i^n) \mid n > 0\}\}\\ &= \vee\{min_i^m\{\mu_{f^{-1}(B)}(b_i^n)\} \mid n > 0\}\\ &= \vee\{min_i^m\{\mu_B(f(b_i^n))\} \mid n > 0\}\\ &= \vee\{min_i^m\{\mu_B(f(b_i)^n)\} \mid n > 0\}\\ &= \min_i^m\{\vee\{\mu_B(f(b_i)^n) \mid n > 0\}\}\\ &= \min_i^m\{\mu_{\sqrt{B}}(f(b_i)\}\\ &= \mu_{(\sqrt{B})_x}(f_x(g(x)))\\ &= \mu_{f_x^{-1}((\sqrt{B})_x)}(g(x)).\end{aligned}$$

Similarly, we can show that $\mu_{(\sqrt{f^{-1}(B)})_x}(g(x)) = \mu_{f_x^{-1}((\sqrt{B})_x)}(g(x))$.
Hence $f_x^{-1}[(\sqrt{B})_x] = (\sqrt{f^{-1}(B)})_x$. □

Proposition 3.8. *Let $f : R \to R'$ be an epimorphism from R onto R' and let A be an intuitionistic fuzzy ideal of R, then $f(\sqrt{A}) \subseteq \sqrt{f(A)}$. Further, if A is constant on* Ker$f$*, then $f(\sqrt{A}) = \sqrt{f(A)}$.*

Proof. Clearly, $f(A)$ and $f(\sqrt{A})$ are intuitionistic fuzzy ideals of R'. If $y \in R'$ and $f(x) = y$ for some $x \in R$, then $f(x^n) = y^n$, for all $n = 1, 2, \ldots$, then

$$\begin{aligned}
\mu_{f(\sqrt{A})}(y) &= \sup\{\mu_{\sqrt{A}}(x) \mid x \in f^{-1}(y)\} \\
&= \sup\{\vee\{\mu_A(x^n) \mid n > 0\} \mid x \in f^{-1}(y)\} \\
&= \vee\{\sup\{\mu_A(x^n) \mid x \in f^{-1}(y)\} \mid n > 0\} \\
&\leq \vee\{\sup\{\mu_A(x^n) \mid x^n \in f^{-1}(y^n)\} \mid n > 0\} \\
&= \vee\{\sup\{\mu_A(z^n) \mid z \in f^{-1}(y^n)\} \mid n > 0\} \\
&= \vee\{\mu_{f(A)}(y^n) \mid n > 0\} \\
&= \mu_{\sqrt{f(A)}}(y).
\end{aligned}$$

Similarly, we can show that $\nu_{f(\sqrt{A})}(y) \geq \nu_{\sqrt{f(A)}}(y)$. Thus, we have $f(\sqrt{A}) \subseteq \sqrt{f(A)}$.

Further, if A is constant on Ker$f$ and $x_0 \in f^{-1}(y)$ is a fixed element, then by Proposition (2.3)(b) ensure that $\mu_A(x^n) = \mu_A(x_0^n)$ and $\nu_A(x^n) = \nu_A(x_0^n)$ for all $x \in f^{-1}(y)$ and $\mu_A(x) = \mu_A(x_0^n)$ and $\nu_A(x) = \nu_A(x_0^n)$ for all $x \in f^{-1}(y^n)$. Hence

$$\begin{aligned}
\mu_{f(\sqrt{A})}(y) &= \sup\{\mu_{\sqrt{A}}(x) \mid x \in f^{-1}(y)\} \\
&= \sup\{\vee\{\mu_A(x^n) \mid n > 0\} \mid x \in f^{-1}(y)\} \\
&= \vee\{\sup\{\mu_A(x^n) \mid x \in f^{-1}(y)\} \mid n > 0\} \\
&= \vee\{\sup\{\mu_A(x_0^n) \mid x \in f^{-1}(y)\} \mid n > 0\} \\
&= \vee\{\sup\{\mu_A(x) \mid x \in f^{-1}(y^n)\} \mid n > 0\} \\
&= \vee\{\mu_{f(A)}(y^n) \mid n > 0\} \\
&= \mu_{\sqrt{f(A)}}(y).
\end{aligned}$$

Similarly, we can show that $\mu_{f(\sqrt{A})}(y) = \mu_{\sqrt{f(A)}}(y)$.
Thus, we have $f(\sqrt{A}) = \sqrt{f(A)}$. □

Theorem 3.9. *Let $f : R \to R'$ be a homomorphism from R onto R' and let f_x be the induced homomorphism of f. If an intuitionistic fuzzy ideal A of R is constant on* Ker$f$*, then the intuitionistic fuzzy polynomial ideal A_x is constant on* Kerf_x*.*

Proof. Let $\mu_A(x) = \alpha_0$ and $\nu_A(x) = \beta_0, \forall x \in$ Ker$f$, where $\alpha_0, \beta_0 \in [0, 1]$ are constants such that $\alpha_0 + \beta_0 \leq 1$. Let $g(x) = \sum_{i=0}^{m} b_i x^i \in$ Kerf_x, then
$0 = f_x(g(x)) = \sum_{i=0}^{m} f(b_i)x^i \Rightarrow f(b_i) = 0, \forall i = 1, 2, \ldots, m$.
Hence $b_i \in$ Ker$f, \forall i = 1, 2, \ldots, m$, i.e., $\mu_A(b_i) = \alpha_0$ and $\nu_A(b_i) = \beta_0 \forall i = 1, 2, \ldots, m$.
$\Rightarrow \mu_{A_x}(g(x)) = \min_{i=0}^{m}\{\mu_A(b_i)\} = \alpha_0$ and $\nu_{A_x}(g(x)) = \max_{i=0}^{m}\{\nu_A(b_i)\} = \beta_0$.
Hence A_x is constant on Kerf_x. □

Corollary 3.10. *Let $f : R \to R'$ be an epimorphism from R onto R' and let f_x be the induced homomorphism of f. If an f-invariant intuitionistic fuzzy ideal A of R is constant on* Ker$f$*, then $f_x(\sqrt{A_x}) = \sqrt{(f(A))_x}$.*

Proof. It follows from Proposition 3.8 and Theorem 3.5 that
$f_x(\sqrt{A_x}) = \sqrt{f_x(A_x)} = \sqrt{(f(A))_x}$. □

Definition 3.11. Let A be an intuitionistic fuzzy ideal of a ring R. Then the intuitionistic fuzzy ideal $P(A)$ defined by

$$P(A) = \cap\{B \mid A \subseteq B, \text{ where } B \text{ is an intuitionistic fuzzy prime ideal of } R\ \}$$

is called an intuitionistic fuzzy prime radical of A.

Theorem 3.12. *Let A be an intuitionistic fuzzy ideal of a ring R and let A_x be its intuitionistic fuzzy polynomial ideal of $R[x]$. Then $P(A_x) \subseteq (P(A))_x$.*

Proof. By Theorem 2.16, B_i is an intuitionistic fuzzy ideal of R with $A \subseteq B_i$ if and only if $(B_i)_x$ is an intuitionistic fuzzy prime ideal of $R[x]$ with $A_x \subseteq (B_i)_x$. It follows from Theorem 2.14 (i) that

$$\begin{aligned}
(P(A))_x &= (\cap\{B_i \mid A \subseteq B_i, \text{ where } B_i \text{ is an intuitionistic fuzzy prime ideal of } R\})_x \\
&= (\cap\{(B_i)_x \mid A \subseteq B_i, \text{ where } B_i \text{ is an intuitionistic fuzzy prime ideal of } R\}) \\
&\subseteq \cap\{(B_i)_x \mid A_x \subseteq (B_i)_x, \text{ where } (B_i)_x \text{ is an intuitionistic fuzzy prime ideal of } R[x]\}) \\
&= \cap\{C_i \mid A_x \subseteq C_i, \text{ where } C_i \text{ is an intuitionistic fuzzy prime ideal of } R[x]\} \\
&= P(A_x).
\end{aligned}$$

This proves the theorem. □

Remark 3.13. Let A be an intuitionistic fuzzy ideal of a ring R and let A_x be its intuitionistic fuzzy polynomial ideal of $R[x]$. We denote
$IFPI(A) = \{B \mid A \subseteq B, \text{ where } B \text{ is an intuitionistic fuzzy prime ideal of } R\}$, and
$IFPI(A_x) = \{D \mid A_x \subseteq D, \text{ where } D \text{ is an intuitionistic fuzzy prime ideal of } R[x]\}$.

Theorem 3.14. *Let A be an intuitionistic fuzzy ideal of a ring R and let A_x be its intuitionistic fuzzy polynomial ideal of $R[x]$. Then the map*

$$\phi : IFPA(A) \to IFPI(A_x) \text{ defined by } \phi(B) = B_x,$$

is one-one.

Proof. Let $B, C \in IFPI(A)$ such that $\phi(B) = \phi(C)$, then $B_x = C_x$. It follows that $(B_x)(r) = (C_x)(r)$, for all $r \in R$, and hence $B(r) = C(r)$ for all $r \in R$, proving that $B = C$. Hence ϕ is one-one. □

Corollary 3.15. *Let A be an intuitionistic fuzzy ideal of a ring R and let A_x be its intuitionistic fuzzy polynomial ideal of $R[x]$. If the map ϕ defined in Theorem 3.14 is one-one map, then $(P(A))_x = P(A_x)$.*

Proof. If D is any element of $IFPI(A_x)$, then there exists $C \in IFPI(A)$ with $A \subseteq C$ such that $C_x = \phi(C) = D$. Thus $(P(A))_x = P(A_x)$. □

Example 3.16. Let **Z** be the set of all integers. Define an IFS A on **Z** by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 2\mathbf{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 2\mathbf{Z} \\ 1, & \text{otherwise} \end{cases}.$$

Then A is an intuitionistic fuzzy prime ideal of **Z**, for $A_* = 2\mathbf{Z}$ is a prime ideal of **Z**, and its induced polynomial ideal A_x is given by

$$\mu_{A_x}(f(x)) = \begin{cases} 1, & \text{if } f(x) \in 2\mathbf{Z[x]} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{A_x}(f(x)) = \begin{cases} 0, & \text{if } f(x) \in 2\mathbf{Z[x]} \\ 1, & \text{otherwise} \end{cases}.$$

By Theorem 2.16, the intuitionistic fuzzy polynomial ideal A_x induced by A is an intuitionistic fuzzy polynomial ideal of **Z[x]**. Hence $(P(A))_x = A_x = P(A_x)$.

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