Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 23, 2017, No. 3, 30–43

Kurzweil–Henstock integral for IF-functions

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Received: 1 June 2017 **Accepted:** 20 July 2017

Abstract: In [1] and [3] there was presented a new definition for the definite integral for real functions based on Riemann's sums with variable length of intervals in divisions. In [4] this definition was extended to functions with fuzzy values. In [2] there was introduced a notion of IF-numbers. In this contribution we are going to extend the definitions and the results for functions which has IF-numbers as their values.
Keywords: Kurzweil–Henstock integral, fuzzy numbers, fuzzy functions, IF-numbers, IF-functions, $\delta-$fine division of interval.
AMS Classification: 03E72.

1 Introduction

The notion of the definite integral is one of the most important notions of mathematical analysis and whole mathematics. Many definitions of integral inspired by various topics was introduced. One of the most general definitions is the definition by J. Kurzweil ([3]) and independently by R. Henstock ([1]). This definition is based on the Riemann-like types of integral sums. The main idea is variable length of intervals in divisions. The definition from Kurzweil and Henstock was introduced for real functions of real variable. In [4] their definition was extended to functions with real variable but with values from the set of all fuzzy numbers. Some analogical results was proved. In this contribution we take IF numbers introduced in [2] as values of integrable functions.

2 Preliminaries

In whole contribution we are going to write $\mathbb{R}_+$ instead of $(0, \infty)$.

Definition 1 ([4]). *Fuzzy set $\alpha : \mathbb{R} \to [0,1]$ is called* ***fuzzy number*** *iff:*

1. $\exists r \in \mathbb{R} \quad \alpha(r) = 1,$
2. $\forall (r_1, r_2, \lambda) \in \mathbb{R} \times \mathbb{R} \times [0,1] \quad \alpha(\lambda r_1 + (1-\lambda) r_2) \geq \min\{\alpha(r_1), \alpha(r_2)\},$
3. $\forall \lambda \in [0,1]$ *the set* $[\alpha]^\lambda = \{x \in \mathbb{R} : \alpha(x) \geq \lambda\}$ *is closed* ,
4. $cl([\alpha]^0) = cl(\{x \in \mathbb{R} : \alpha(x) > 0\})$ *is compact, where $cl(A)$ is the closure of $A \subset \mathbb{R}$ in the usual topology.*

It can be proved that for arbitrary $\lambda \in (0,1]$ and for arbitrary fuzzy number $\alpha, [\alpha]^\lambda = [\alpha_{\lambda,1}, \alpha_{\lambda,2}]$ for some real numbers $\alpha_{\lambda,1}, \alpha_{\lambda,2}$.

Definition 2 ([2]). *Let α, β be fuzzy numbers. By their* ***sum*** *we mean the fuzzy number γ, for which*

$$[\gamma]^\lambda = [\alpha_{\lambda,1} + \beta_{\lambda,1}, \alpha_{\lambda,2} + \beta_{\lambda,2}]$$

for arbitrary $\lambda \in [0,1]$. We write

$$\gamma = \alpha +_f \beta.$$

Definition 3. *Let α be a fuzzy number, let k be a real number.*
By ***k-multiplier*** *of α we mean the fuzzy number $k\alpha$ for which*

$$[k\alpha]^\lambda = \begin{cases} [k\alpha_{\lambda,1}, k\alpha_{\lambda,2}] & \text{for } k \geq 0 \\ [k\alpha_{\lambda,2}, k\alpha_{\lambda,1}] & \text{otherwise} \end{cases}$$

for arbitrary $\lambda \in [0,1]$

Definition 4 ([4]). *Let α, β be fuzzy numbers. Define*

$$\hat{\rho}(\alpha, \beta) = \sup\{\max\{|\alpha_{\lambda,1} - \beta_{\lambda,1}|, |\alpha_{\lambda,2} - \beta_{\lambda,2}|\} : \lambda \in [0,1]\}.$$

It can be proved that the mapping $\hat{\rho}$ is a metric on the set of all fuzzy numbers.

Definition 5 ([4]). *Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be real numbers. Let $\xi_i \in [x_{i-1}, x_i]$.*
Let $\delta : [a,b] \to \mathbb{R}_+$ be such that $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.
Then the ordered pair $(\{x_i : i = 0, 1, \cdots, n\}, \{\xi_i : i = 1, 2, \cdots, n\})$ is called
the $\delta -$ ***fine division of interval*** $[a,b]$.

Definition 6 ([4]). *Let $\mathbb{E}_1$ be the set of all fuzzy numbers. Let α be a fuzzy number. Let a, b be real numbers. We say that a function $f : [a,b] \to \mathbb{E}_1$* ***is Kurzweil–Henstock (we write KH) integrable*** *with KH-integral α iff for arbitrary real positive number ε there exists $\delta \in \mathbb{R}_+^{[a,b]}$ such that*

$$\hat{\rho}\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), \alpha\right) < \varepsilon$$

for arbitrary δ–fine division of $[a,b]$.

3 IF-numbers

Definition 7. *Fuzzy set $\beta : \mathbb{R} \to [0, 1]$ is called* ***fuzzy antinumber*** *iff:*

1. $\exists s \in \mathbb{R} \quad \beta(s) = 0,$
2. $\forall (r_1, r_2, \lambda) \in \mathbb{R} \times \mathbb{R} \times [0, 1] \quad \beta(\lambda r_1 + (1 - \lambda) r_2) \leq \max\{\beta(r_1), \beta(r_2)\},$
3. $\forall \lambda \in [0, 1]$ *the set* $[\beta]_\lambda = \{x \in \mathbb{R} : \beta(x) \leq \lambda\}$ *is closed* ,
4. $cl([\alpha]_0) = cl(\{x \in \mathbb{R} : \alpha(x) < 1\})$ *is compact, where $cl(A)$ is the closure of $A \subset \mathbb{R}$ in the usual topology.*

Similary as in the case of fuzzy number, we have $[\beta]_\lambda = [\beta_{\lambda,1}, \beta_{\lambda,2}]$ for arbitrary fuzzy antinumber β and arbitrary $\lambda \in [0, 1]$ for some real numbers $\beta_{\lambda,1}, \beta_{\lambda,2}$.

Theorem 1. *Let $\alpha : \mathbb{R} \to [0, 1]$ be a function. Then α is a fuzzy antinumber if and only if there exists a fuzzy number β such that*

$$\alpha(x) = 1 - \beta(x)$$

for arbitrary real number x.

Proof. Let α be a fuzzy antinumber. It suffices to prove that $\beta := 1 - \alpha$ is a fuzzy number. But if $\alpha(s) = 0,$ then $\beta(s) = 1 - \alpha(s) = 1$. Similarly $\alpha(x) \leq \gamma \Leftrightarrow \beta(x) \geq 1 - \gamma$. From this it follows, that

$$\{[\alpha]_\lambda : \lambda \in [0, 1]\} = \left\{[\beta]^\lambda : \lambda \in [0, 1]\right\}. \qquad \square$$

Definition 8. *Let α, β be fuzzy antinumbers. By their* ***sum*** *we mean the fuzzy antinumber γ, for which*

$$[\gamma]_\lambda = [\alpha_{\lambda,1} + \beta_{\lambda,1}, \alpha_{\lambda,2} + \beta_{\lambda,2}]$$

for arbitrary $\lambda \in [0, 1]$. We write $\gamma = \alpha +_a \beta$.

Definition 9. *Let α be a fuzzy antinumber, let k be a real number.*
By ***k-multiplier*** *of α we mean the fuzzy number $k\alpha$ for which*

$$[k\alpha]_\lambda = \begin{cases} [k\alpha_{\lambda,1}, k\alpha_{\lambda,2}] & \text{for } k \geq 0 \\ [k\alpha_{\lambda,2}, k\alpha_{\lambda,1}] & \text{otherwise} \end{cases}$$

for arbitrary $\lambda \in [0, 1]$.

Definition 10. *Let α, β be fuzzy antinumbers. Define*

$$\hat{\hat{\rho}}(\alpha, \beta) = \sup\{\max\{|\alpha_{\lambda,1} - \beta_{\lambda,1}|, |\alpha_{\lambda,2} - \beta_{\lambda,2}|\} : \lambda \in [0, 1]\}.$$

Theorem 2. *$\hat{\hat{\rho}}$ is a metric on the set of all fuzzy antinumbers.*

Proof. Obviously for arbitrary fuzzy antinumbers α, β it is $\hat{\hat{\rho}}(\alpha, \beta) = \hat{\hat{\rho}}(\beta, \alpha) \geq 0$ and $\rho(\alpha, \alpha) = 0$. If $\hat{\hat{\rho}}(\alpha, \beta) = 0$, then $\max\{|\alpha_{\lambda,1} - \beta_{\lambda,1}|, |\alpha_{\lambda,2} - \beta_{\lambda,2}|\} = 0$ for arbitrary $\lambda \in [0,1]$ and hence $\alpha_{\lambda,1} = \beta_{\lambda,1}$ and $\alpha_{\lambda,2} = \beta_{\lambda,2}$. From that it follows that $\alpha = \beta$. If γ is another fuzzy antinumber, then

$$|\alpha_{\lambda,1} - \gamma_{\lambda,1}| \leq |\alpha_{\lambda,1} - \beta_{\lambda,1}| + |\beta_{\lambda,1} - \gamma_{\lambda,1}|$$
$$|\alpha_{\lambda,2} - \gamma_{\lambda,2}| \leq |\alpha_{\lambda,1} - \beta_{\lambda,2}| + |\beta_{\lambda,2} - \gamma_{\lambda,2}|$$

From that we have

$$\begin{aligned}\max\{|\alpha_{\lambda,1} - \gamma_{\lambda,1}|, |\alpha_{\lambda,2} - \gamma_{\lambda,2}|\} \leq \max\{|\alpha_{\lambda,1} - \beta_{\lambda,1}|, |\alpha_{\lambda,2} - \beta_{\lambda,2}|\}+ \\ + \max\{|\beta_{\lambda,1} - \gamma_{\lambda,1}|, |\beta_{\lambda,2} - \gamma_{\lambda,2}|\}\end{aligned}$$

and finally $\hat{\hat{\rho}}(\alpha, \gamma) \leq \hat{\hat{\rho}}(\alpha, \beta) + \hat{\hat{\rho}}(\beta, \gamma)$. □

Definition 11 ([2])**.** *Let α be a fuzzy number. Let β be a fuzzy antinumber. An ordered pair (α, β) is called **IF-number** iff $\forall r \in \mathbb{R} \quad \alpha(r) + \beta(r) \leq 1$.*

Lemma 1. *Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ be IF-numbers. Let*

$$C = (\mu_A +_f \mu_B, \nu_A +_a \nu_B).$$

Then C is an IF-number.

Proof. By assumption $\mu_A + \nu_A \leq 1$ and $\mu_B + \nu_B \leq 1$. From that we get

$$\nu_A \leq 1 - \mu_A$$
$$\nu_B \leq 1 - \mu_B.$$

By Theorem 1, $1 - \mu_A$ and $1 - \mu_B$ are fuzzy antinumbers. We can see, that for arbitrary fuzzy antinumbers α, β the following relations are equivallent

$$\forall r \in \mathbb{R} \quad \alpha(r) \leq \beta(r)$$
$$\forall \lambda \in [0,1] \quad \alpha_{[\lambda]} \supset \beta_{[\lambda]}$$

From above, for arbitrary $\lambda \in [0,1]$ we have

$$(\nu_A)_{[\lambda]} \supset (1 - \mu_A)_{[\lambda]}$$
$$(\nu_B)_{[\lambda]} \supset (1 - \mu_B)_{[\lambda]}.$$

But

$$(1 - \mu_A)_{[\lambda]} = (\mu_A)^{[1-\lambda]}$$
$$(1 - \mu_A)_{[\lambda]} = (\mu_A)^{[1-\lambda]}$$

Hence

$$(\nu_A)_{[\lambda]} \supset (\mu_A)^{[1-\lambda]},$$
$$(\nu_B)_{[\lambda]} \supset (\mu_B)^{[1-\lambda]}.$$

But for arbitrary $a, b, c, d, e, f, g, h \in \mathbb{R}$ there holds

$$([a, b] \supset [c, d] \,\&\, [e, f] \supset [g, h]) \Rightarrow ([a + e, b + f] \supset [c + g, d + h]) .$$

From this it follows that

$$(\nu_C)_{[\lambda]} \supset (\mu_C)^{[1-\lambda]} = (1 - \mu_C)_{[\lambda]} .$$

Finally we get

$$\nu_C \leq 1 - \mu_C. \qquad \square$$

Definition 12. *IF-number C from above lemma is called the* ***sum*** *of IF-numbers A, B. We write*

$$C = A +_i B$$

Definition 13. *Let $A = (\mu_A, \nu_A)$ be an IF-number. By the* ***k-multiplier*** *of A we mean $kA = (k\mu_A, k\nu_a)$.*

Theorem 3. *Let A, B, C be IF-numbers. Let k be a real number. Then*

$$A +_i B = B +_i A,$$
$$(A +_i B) +_i C = A +_i (B +_i C) ,$$
$$k (A +_i B) = kA +_i kB.$$

Proof. Let $A = (\mu_A, \nu_A) , B = (\mu_B, \nu_B) , C = (\mu_C, \nu_C)$.
Let
$[\mu_A]^\lambda = [\mu_{A,\lambda,1}, \mu_{A,\lambda,2}] , [\mu_B]^\lambda = [\mu_{B,\lambda,1}, \mu_{B,\lambda,2}] , [\mu_C]^\lambda = [\mu_{C,\lambda,1}, \mu_{C,\lambda,2}] ,$
$[\nu_A]_\lambda = [\nu_{A,\lambda,1}, \nu_{A,\lambda,2}] , [\nu_B]_\lambda = [\nu_{B,\lambda,1}, \nu_{B,\lambda,2}] , [\nu_C]_\lambda = [\nu_{C,\lambda,1}, \nu_{C,\lambda,2}] .$
The statements follows from commutativity, associativity an distributivity for real numbers. $\square$

Definition 14 ([2]). *Let $A = (\mu_A, \nu_A) , B = (\mu_B, \nu_B)$ be IF-numbers. We define*

$$\rho(A, B) = \hat{\rho}(\mu_A, \mu_B) + \hat{\hat{\rho}}(\nu_A, \nu_B).$$

Theorem 4. *Let A, B, C, D be IF-numbers. Then*

$$\rho(A +_i B, C +_i D) \leq \rho(A, C) + \rho(B, D).$$

Proof.

$$\rho(A +_i B, C +_i D) = \hat{\rho}(\mu_{A+_iB}, \mu_{C+_iD}) + \hat{\hat{\rho}}(\nu_{A+_iB}, \nu_{C+_iD}) \leq$$
$$\leq \hat{\rho}(\mu_A, \mu_C) + \hat{\rho}(\mu_B, \mu_D) + \hat{\hat{\rho}}(\nu_A, \nu_C) + \hat{\hat{\rho}}(\nu_B, \mu_D) = \rho(A, C) + \rho(B, D). \qquad \square$$

Definition 15. *Let $\mathbb{D}$ be the set of all IF-numbers. Let A be an IF-number. Let a, b be real numbers. We say that a function $f : [a, b] \to \mathbb{D}$ is Kurzweil–Henstock (we write KH) integrable with the KH-integral A iff for arbitrary real positive number ε there exists $\delta \in \mathbb{R}_+^{[a,b]}$ such that*

$$\rho\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), A\right) < \varepsilon$$

for arbitrary $\delta-$fine division of $[a, b]$. In that case we write

$$A = \int_a^b f$$

Theorem 5. *Let $f, g : [a, b] \to \mathbb{D}$ be KH-integrable functions.*
Let $A = \int\limits_a^b f, B = \int\limits_a^b g$. Then

$$\int\limits_a^b (f +_i g) = A +_i B.$$

Proof. Let $\varepsilon > 0$ be arbitrary fixed positive real number, then from the assumptions there exist $\delta_1, \delta_2 : [a, b] \to \mathbb{R}_+$ such that

$$\rho\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), A\right) < \frac{\varepsilon}{2},$$
$$\rho\left(\sum_{i=1}^{n} g(\xi_i)\,(x_i - x_{i-1}), B\right) < \frac{\varepsilon}{2}$$

for arbitrary $\delta_{1,2}-$fine division of $[a, b]$. We have to prove that

$$\rho\left(\sum_{i=1}^{n} (f(\xi_i) + g(\xi_i))\,(x_i - x_{i-1}), A +_i B\right) < \varepsilon.$$

But for arbitrary IF numbers C, D, E, F, there is

$$\rho(C +_i D, E +_i F) \leq \rho(C, E) + \rho(D, F).$$

From this it follows that

$$\rho\left(\sum_{i=1}^{n} (f(\xi_i) +_i g(\xi_i))\,(x_i - x_{i-1}), A +_i B\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Definition 16. *Let $(A_n)_{n=1}^{\infty}$ be a sequence of IF-numbers.*
Let A be an IF-number.
*We say that $(A_n)_{n=1}^{\infty}$ **converges to** A iff*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \rho(A_n, A) < \varepsilon.$$

In that case we write $A_n \to A$.

Definition 17. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of IF-functions. Let f be an IF-function. We say that $(f_n)_{n=1}^{\infty}$ **uniformly converges to** f iff*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \forall x \quad \rho(f_n(x), f(x)) < \varepsilon.$$

In that case we write $f_n \rightrightarrows f$.

Theorem 6. *The set $\mathbb{D}$ of all IF-number with the metric ρ is a complete metric space, i.e. if for a sequence $(A_n)_{n=1}^{\infty}$ there holds*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall\,(m, n) \in \mathbb{N} \times \mathbb{N} \quad ((m \geq n_0 \& n \geq n_0) \Rightarrow \rho(A_m, A_n) < \varepsilon)$$

then the sequence $(A_n)_{n=1}^{\infty}$ is convergent.

Proof. Obviously the set of all fuzzy antinumbers with the metric $\hat{\hat{\rho}}$ is a complete metric space if and only if the set of all fuzzy numbers with the metric $\hat{\rho}$ has the same property. But fuzzy numbers with $\hat{\rho}$ are complete. Moreover the set of all IF-numbers with the metric ρ is a cartesian product of fuzzy numbers and fuzzy antinumbers. From that it follows that $\mathbb{D}$ is complete. $\square$

Theorem 7. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of KH-integrable IF-functions on an interval $[a,b]$. Let $f_n \rightrightarrows f$. Then*

$$\left(\int_a^b f_n\right) \to \left(\int_a^b f\right).$$

Proof. Let ε be an arbitrary fixed positive real number. From the assumptions for arbitrary positive integer k, there exists $\delta_k \in \mathbb{R}_+^{[a,b]}$ such that

$$\rho\left(\sum_{i=1}^{n} f_k(\xi_i)(x_i - x_{i-1}), \int_a^b f_k\right) < \frac{\varepsilon}{3}$$

for arbitrary δ_k-fine division of the interval $[a,b]$. From the uniform convergence we get that there exists k_0 such that for any $k \geq k_0$

$$\rho(f_k(\xi_i), f(\xi_i)) < \frac{\varepsilon}{3\,(b-a)}.$$

Then for $r, s \geq k_0$, there is

$$\begin{aligned}
\rho\left(\int_a^b f_r, \int_a^b f_s\right) &\leq \rho\left(\sum_{i=1}^{n} f_r(\xi_i)(\xi_i - \xi_{i-1}), \int_a^b f_r\right) + \\
&+ \rho\left(\sum_{i=1}^{n} f_r(\xi_i)(x_i - x_{i-1}), \sum_{i=1}^{n} f_s(\xi_i)(x_i - x_{i-1})\right) + \\
&+ \rho\left(\sum_{i=1}^{n} f_s(\xi_i)(x_i - x_{i-1}), \int_a^b f_s\right) < \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3\,(b-a)}\,(b-a) + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

From that and from the completeness of the metric space of IF-numbers it follows that there is an IF-number β such that $\left(\int_a^b f_i\right) \to \beta$. Moreover for $k \geq k_o$

$$\begin{aligned}
\rho\left(\sum_{i=1}^{n} f_k(\xi_i)(x_i - x_{i-1}), \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})\right) &\leq \sum_{i=1}^{n} (x_i - x_{i-1})\,\rho(f_k(\xi_i), f(\xi_i)) < \\
&< \frac{\varepsilon}{3\,(b-a)}\,(b-a) = \frac{\varepsilon}{3}.
\end{aligned}$$

Moreover, there exists $t > k_0$ such that

$$\rho\left(\int_a^b f_t, \beta\right) < \frac{\varepsilon}{3}.$$

And finally there exists $\delta_t \in \mathbb{R}_+^{[a,b]}$ such that for every δ_t-fine division of the interval $[a,b]$ there is

$$\rho\left(\sum_{i=1}^{n} f_t(\xi_i)(x_i - x_{i-1}), \int\limits_a^b f_t\right) < \frac{\varepsilon}{3}.$$

From that it follows that

$$\begin{aligned}\rho\left(\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \beta\right) &\leq \rho\left(\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \sum_{i=1}^{n} f_t(\xi_i)(x_i - x_{i-1})\right) + \\ &+ \rho\left(\sum_{i=1}^{n} f_t(\xi_i)(x_i - x_{i-1}), \int\limits_a^b f_t\right) + \\ &+ \rho\left(\int\limits_a^b f_t, \beta\right) < \varepsilon.\end{aligned}$$

Since ε was arbitrary, there is

$$\int\limits_a^b f = \beta. \qquad \square$$

Theorem 8. *Let $(f_k)_{k=1}^{\infty}$ be a sequence of IF-KH integrable functions. Let f be IF function such that for each $x \in [a,b]$ the sequence $(f_k(x))_{k=1}^{\infty}$ converges to $f(x)$ in the metric ρ.*
Then the following conditions are equivalent:

1. *f is IF-KH integrable and*

$$\left(\int\limits_a^b f_k\right)_{k=1}^{\infty} \to \int\limits_a^b f,$$

2. $$(\forall \varepsilon > 0)\,(\exists m \in \mathbb{N})\,(\forall k \geq m)\left(\exists \delta \in \mathbb{R}_+^{[a,b]}\right)$$

   such that for every $\delta-$fine division $\left(\Sigma^k, \xi_k\right)$ of $[a,b]$ it holds

$$\rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \sum_{i=1}^{n_k} f(\xi_{k,i})\,(x_i - x_{i-1})\right) < \varepsilon.$$

Proof. Let the condition 1 holds. Then there exists an $m \in \mathbb{N}$ such that

$$\rho\left(\int\limits_a^b f_k, \int\limits_a^b f\right) < \frac{\varepsilon}{3}$$

for arbitrary $k \geq m$. Because the functions f_k are IF-KH integrable, for each k there exists a function δ_k such that for any δ_k-fine division of $[a,b]$ it holds

$$\rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \int\limits_a^b f_k\right) < \frac{\varepsilon}{3}.$$

By our assumptions f is IF-KH integrable. From that it follows that there exists δ_0 such that

$$\rho\left(\sum_{i=1}^{n_0} f(\xi_{0,i})\,(x_i - x_{i-1}), \int_a^b f\right) < \frac{\varepsilon}{3}.$$

From that we have that for arbitrary $\min\{\delta_0, \delta_k\}-$ fine division of $[a,b]$ there holds

$$\rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \sum_{i=1}^{n_k} f(\xi_{k,i})\,(x_i - x_{i-1})\right) \leq$$
$$\leq \rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \int_a^b f_k\right) + \rho\left(\int_a^b f_k, \int_a^b f\right) +$$
$$+ \rho\left(\int_a^b f, \sum_{i=1}^{n_k} f(\xi_{k,i})\,(x_i - x_{i-1})\right) < \varepsilon.$$

Let the condition 2 holds. Then exists $m \in \mathbb{N}$ such that for arbitrary $k, l \geq m$ there are δ_k, δ_l such that

$$\rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \sum_{i=1}^{n_k} f(\xi_{k,i})\,(x_i - x_{i-1})\right) < \frac{\varepsilon}{4}$$

and

$$\rho\left(\sum_{i=1}^{n_l} f_l(\xi_{l,i})\,(x_i - x_{i-1}), \sum_{i=1}^{n_l} f(\xi_{l,i})\,(x_i - x_{i-1})\right) < \frac{\varepsilon}{4}.$$

Similarly because f_k are IF-KH integrable, there are ζ_k, ζ_l such that

$$\rho\left(\sum_{i=1}^{s_k} f(\eta_{k,i})\,(x_i - x_{i-1}), \int_a^b f_k\right) < \frac{\varepsilon}{4}$$

resp.

$$\rho\left(\sum_{i=1}^{s_l} f(\eta_{l,i})\,(x_i - x_{i-1}), \int_a^b f_l\right) < \frac{\varepsilon}{4}$$

for arbitrary ζ_k-fine, resp. ζ_l-fine division of $[a,b]$.

Then taking $\delta = \min\{\delta_k, \delta_l, \zeta_k, \zeta_l\}$ all four inequalities holds for arbitrary $\delta-$fine division of $[a,b]$. From that it follows that for arbitrary $k, l \geq m$, there holds

$$\rho\left(\int_a^b f_k, \int_a^b f_l\right) < \varepsilon.$$

We proved that $\left(\int_a^b f_k\right)_{k=1}^{\infty}$ is Cauchy sequence. But we are in complete metric space and from that it follows that there exists A such that $\left(\int_a^b f_k\right)_{k=1}^{\infty} \to A$. Then $\exists p \geq m$ such that

$$\rho\left(\int_a^b f_p, A\right) < \frac{\varepsilon}{3}.$$

From 2 there is δ_1 such that

$$\rho\left(\sum_{i=1}^{n_1} f_p(\xi_{1,i})\,(x_i - x_{i-1}), \sum_{i=1}^{n_1} f(\xi_{1,i})\,(x_i - x_{i-1})\right) < \frac{\varepsilon}{3}.$$

From the integrability of f_p there exists δ_2 such that

$$\rho\left(\sum_{i=1}^{n_2} f_p(\xi_{2,i})\,(x_i - x_{i-1}), \int_a^b f_p\right) < \frac{\varepsilon}{3}.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we get $\rho(A, \sum_{i=1}^{n} (\xi_i)(x_i - x_{i-1})) < \varepsilon$ for arbitrary $\delta-$fine division of $[a, b]$. From that it follows $A = \int_a^b f$. $\square$

Definition 18. *On the set of all IF numbers $\mathbb{D}$ we define the partial ordering $\leq_i$ by the following statement*

$$A \leq_i B \Leftrightarrow \forall\lambda \in [0,1] \left([\mu_A]^\lambda \leq [\mu_B]^\lambda \,\&\, [\nu_A]_\lambda \leq [\nu_B]_\lambda\right),$$

where

$$(\forall a, b, c, d \in \mathbb{R})\,([a, b] \leq [c, d] \Leftrightarrow (a \leq c \,\&\, b \leq d)).$$

Theorem 9. *Let $(f_k)_{k=1}^{\infty}$ be a monotone sequence of IF-KH integrable functions (i.e. $(\forall k \in \mathbb{N})$ $(\forall x \in [a, b])\,(f_k(x) \leq_i f_{k+1}(x))$ or $(\forall k \in \mathbb{N})\,(\forall x \in [a, b])\,(f_k(x) \geq_i f_{k+1}(x))$).*

Let $\left(\int_a^b f_k\right)_k^{\infty}$ be IF-bounded. Let f be the point limit of $(f_k)_{k=1}^{\infty}$ in the metric ρ. Then f is IF-KH integrable and

$$\left(\int_a^b f_k\right) \to \left(\int_a^b f\right).$$

Proof. Assume that $(f_k)_{k=1}^{\infty}$ is increasing. Then the sequence $\left(\int_a^b f_k\right)_{k=1}^{\infty}$ is IF-increasing and IF-bounded. From that it follows that it converges to some IF-number A. Then for arbitrary $\varepsilon > 0$ there exists some positive integer r such that

$$\rho\left(\int_a^b f_r, A\right) < \frac{\varepsilon}{3}$$

and $\frac{\varepsilon}{3} > \frac{1}{2^{r-2}}$ Since the functions f_k are IF-KH integrable, for arbitrary positive integer k there is $\delta_k \in \mathbb{R}_+^{[a,b]}$ such that for every δ_k-fine division of $[a, b]$ it holds

$$\rho\left(\sum_{i=1}^{n_k} f_k(\xi_{k,i})\,(x_i - x_{i-1}), \int_a^b f_k\right) < \frac{1}{2^k}.$$

f is the point limit of $(f_k)_{k=1}^{\infty}$. From that for arbitrary $x \in [a, b]$ we can find a positive integer $k_x \geq r$ such that

$$\rho(f_{k_x}, f) < \frac{\varepsilon}{3\,(b-a)}.$$

We need to show that for arbitrary positive ε there is a δ such that for arbitrary $\delta-$fine division of $[a,b]$ it holds $\rho\left(\sum\limits_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), A\right) < \varepsilon$.
But

$$\rho\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), A\right) \le \rho\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), \sum_{i=1}^{n} f_{k_{\xi_i}}(\xi_i)\,(x_i - x_{i-1})\right) +$$
$$+\,\rho\left(\sum_{i=1}^{n} f_{k_{\xi_i}}(\xi_i)\,(x_i - x_{i-1}), \sum_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i} f_{k_{\xi_i}}\right) + \rho\left(\sum_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i} f_{k_{\xi_i}}, A\right).$$

The first distance

$$\le \sum_{i=1}^{n} \rho\Big(f(x_i), f_{k_{\xi_i}}(\xi_i)\Big)\,(x_i - x_{i-1}) \le \frac{\varepsilon}{3\,(b-a)}\,(b-a) = \frac{\varepsilon}{3}.$$

Put $m = \max\{k_{\xi_i}, i = 1, 2, \cdots n\}$.
The second distance

$$\le \sum_{j=r}^{m} \sum_{i \in \{i, k_{\xi_i} = j\}} \rho\left(f_{k_{\xi_i}}(\xi_i), \int\limits_{x_i - 1}^{x_i} f_{k_{\xi_i}}\right).$$

The inner sum

$$\le \frac{1}{2^{j-1}}.$$

The outer sum

$$\le \sum_{j=r}^{m} \frac{1}{2^{t-1}} < \frac{1}{2^{r-2}} < \frac{\varepsilon}{3}.$$

Since $r \le k_{\xi_i} \le m$, there is $\int\limits_{x_{i-1}}^{x_i} f_r \le \int\limits_{x_{i-1}}^{x_i} f_{k_{\xi_i}} \le \int\limits_{x_{i-1}}^{x_i} f_m$.

From that $\int\limits_a^b f_r \le \sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i} f_{k_{\xi_i}} \le \int\limits_a^b f_m \le A$.

From that the third distance is $\le \rho\left(\int\limits_a^b f_r, A\right) < \frac{\varepsilon}{3}$. From that we have

$$\rho\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), A\right) < \varepsilon. \qquad \square$$

4 Fuzzy case as a corollary of IF case

Definition 19. *Let $\mathbb{F}$ be the set of all fuzzy numbers. Let*

$$\triangle = \left\{(f,g) \in \left(\mathbb{R}^{[0,1]}\right)^2 | f + g \le 1\right\}.$$

We define the mapping $h : \mathbb{F} \to \triangle$ by the relationship

$$h(\alpha) = (\alpha, 1 - \alpha)\,.$$

Theorem 10. *Let α be a fuzzy number. Then $h(\alpha)$ defined above is an IF-number.*

Proof. Obviously $\alpha + 1 - \alpha = 1 \leq 1$. By Theorem 1, $1 - \alpha$ is a fuzzy antinumber. From this we have that $h(\alpha) = (\alpha, 1 - \alpha)$ is an IF-number. $\square$

Theorem 11. *Let α, β be fuzzy numbers. Then*

$$h(\alpha +_f \beta) = h(\alpha) +_i h(\beta).$$

Proof.

$$\left[\mu_{h\left(\alpha +_f \beta\right)}\right]^\lambda = [\alpha]^\lambda + [\beta]^\lambda = \left[\mu_{h(\alpha) +_i h(\beta)}\right]^\lambda,$$
$$\left[\nu_{h(\alpha) +_i h(\beta)}\right]_\lambda = [\alpha]^{1-\lambda} + [\beta]^{1-\lambda} = \left[\mu_{h\left(\alpha +_f \beta\right)}\right]^{1-\lambda}.$$

$\square$

Theorem 12. *Let $(\mathbb{F}, \hat{\rho})$, resp. $(\mathbb{D}, \rho)$ be the metric space of all fuzzy numbers resp. the metric space of all IF-numbers. Then*

$$\hat{\rho}(\alpha, \beta) = \frac{\rho(h(\alpha), h(\beta))}{2}.$$

Proof.

$$\rho(h(\alpha), h(\beta)) = \hat{\rho}(\alpha, \beta) + \hat{\hat{\rho}}(1 - \alpha, 1 - \beta) = 2\hat{\rho}(\alpha, \beta).$$

$\square$

Theorem 13. *Let $(\alpha_n)_{n=1}^\infty$ be a sequence of fuzzy numbers. Then*

$$\alpha_n \to_{\hat{\rho}} \alpha \Leftrightarrow h(a_n) \to_\rho h(\alpha).$$

Proof. Let $\alpha_n \to_{\hat{\rho}} \alpha$. Then for arbitrary positive real number ε there is a positive integer n_0 such that for arbitrary integer greater than n_0, it holds

$$\frac{\varepsilon}{2} > \hat{\rho}(\alpha_n, \alpha) = \frac{\rho(h(\alpha_n), h(\alpha))}{2}.$$

Similarly, let $h(\alpha_n) \to_\rho h(\alpha)$. Then

$$2\varepsilon > \rho(h(\alpha_n), h(\alpha)) = 2\hat{\rho}(\alpha_n, \alpha).$$

$\square$

Definition 20. *Let $\mathbb{F}$ be the set of all fuzzy numbers. Let α be a fuzzy number. Let a, b be real numbers. We say that a function $f : [a, b] \to \mathbb{F}$ is Kurzweil–Henstock (we write KH) integrable with the KH-integral α iff for arbitrary real positive number ε there exists $\delta \in \mathbb{R}_+^{[a,b]}$ such that*

$$\hat{\rho}\left(\sum_{i=1}^n f(\xi_i)\,(x_i - x_{i-1}), \alpha\right) < \varepsilon$$

for arbitrary δ–fine division of $[a, b]$. In that case we write

$$\alpha = \int_a^b f$$

Theorem 14. *Let $f : [a,b] \to \mathbb{F}$ be a fuzzy function. Then f is KH-integrable in sense of definition 20 if and only if $h(f)$ if KH-integrable in sense of definition 15. In case that both are KH-integrable, it holds*

$$h\left(\int_a^b f\right) = \int_a^b h(f).$$

Proof. Let f be fuzzy KH-integrable with the KH-integral α. Let $\varepsilon > 0$ be arbitrary positive real number. Then there exists $\delta_\varepsilon \in \mathbb{R}_+^{[a,b]}$, such that

$$\hat{\rho}\left(\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}), \alpha\right) < \frac{\varepsilon}{2}$$

for arbitrary $\delta-$fine division of $[a,b]$. If we use above theorem from this section, we get

$$\rho\left(\sum_{i=1}^{n} h(f)(\xi_i)\,(x_i - x_{i-1}), h(\alpha)\right) < \varepsilon.$$

Similarly it can be proved the opposite implication. □

Corollary 1 (Theorem 2.7 (iii) in [4])**.** *Let $f, g : [a,b] \to \mathbb{F}$ be KH-integrable functions. Let $\alpha = \int_a^b f, \beta = \int_a^b g$. Then*

$$\int_a^b (f +_f g) = \alpha +_f \beta.$$

Definition 21. *Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of fuzzy numbers. Let α be a fuzzy number. We say that $(\alpha_n)_{n=1}^{\infty}$* ***converges to*** *$\alpha$ iff*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \hat{\rho}(\alpha_n, \alpha) < \varepsilon.$$

In that case we write $\alpha_n \to \alpha$.

Definition 22. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of fuzzy functions. Let f be a fuzzy function. We say that $(\alpha_n)_{n=1}^{\infty}$* ***uniformly converges to*** *α iff*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \forall x \quad \hat{\rho}(f_n(x), f(x)) < \varepsilon.$$

In that case we write $f_n \rightrightarrows f$.

Theorem 15. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of fuzzy functions. Then $f_n \rightrightarrows f \Leftrightarrow h(f_n) \rightrightarrows h(f)$.*

Proof. Similarly as in the theorem 13, we use the transformation relationship

$$\hat{\rho}(\alpha, \beta) = \frac{\rho(h(\alpha), h(\beta))}{2}.$$

□

Corollary 2 (Theorem 3.3 in [4]). *Let $(f_n)_{n=1}^{\infty}$ be a sequence of KH-integrable fuzzy functions on an interval $[a,b]$. Let $f_n \rightrightarrows f$. Then*

$$\left(\int_a^b f_n\right) \to \left(\int_a^b f\right).$$

Proof. From the fact that the transformation h is compatible with KH-integral and convergence, corollary follows from Theorem 7. □

Definition 23. *On the set of all fuzzy numbers $\mathbb{F}$ we define the partial ordering $\leq_f$ by the following statement*

$$\alpha \leq_f \beta \Leftrightarrow \forall\lambda \in [0,1] \left([\alpha]^{\lambda} \leq [\beta]^{\lambda}\right),$$

where

$$(\forall a,b,c,d \in \mathbb{R}) \left([a,b] \leq [c,d] \Leftrightarrow (a \leq c \,\&\, b \leq d)\right).$$

Theorem 16. *Let α, β be fuzzy numbers. Then $\alpha \leq_f \beta \Leftrightarrow h(\alpha) \leq_i h(\beta)$.*

Proof. Let $\alpha \leq_f \beta$ Then $[\alpha]^{\lambda} \leq [\beta]^{\lambda}$ for arbitrary $\lambda \in [0,1]$. From that it follows that $[1-\alpha]_{\lambda} = [\alpha]^{1-\lambda} \leq [\beta]^{1-\lambda} = [1-\beta]_{\lambda}$ for arbitrary $\lambda \in [0,1]$. Opposite implication is obvious. □

Corollary 3 (Theorem 3.7 in [4]). *Let $(f_k)_{k=1}^{\infty}$ be a monotone sequence of IF-KH integrable functions (i.e. $(\forall k \in \mathbb{N})\,(\forall x \in [a,b])\,(f_k(x) \leq_f f_{k+1}(x))$ or $(\forall k \in \mathbb{N})\,(\forall x \in [a,b])\,(f_k(x) \geq_i f_{k+1}(x))$). Let $\left(\int_a^b f_k\right)_k^{\infty}$ be fuzzy bounded. Let f be the point limit of $(f_k)_{k=1}^{\infty}$ in the metric ρ. Then f is IF-KH integrable and*

$$\left(\int_a^b f_k\right) \to \int_a^b f.$$

Proof. Since h is monotone and KH-integral compatible, Corollary follows from Theorem 9. □

5 Conclusion

In this contribution an extension of the notion of Kurzweil–Henstock integral to IF-functions was presented. It is reasonable to ask if it is posible to extend definition domain for integrable function to compact metric or topological space.

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