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ON DEFINITIONS OF INTUITIONISTIC FUZZY SUBGROUPS

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Abstract: In this paper, We give definition of intuitionistic fuzzy subgroups of a group G and discuss their properties. We also contribute normal intuitionistic fuzzy subgroups of a group G and build intuitionistic fuzzy quotient group of a group G.

Keywords: Intuitionistic fuzzy sets, group, IFP, IFSG.

1. Preliminary

Let a set E be fixed. An IFS A in E is an object having the form

$$A^* = \{<x, \mu_A(x), \upsilon_A(x)> | x \in E\}$$

where the function $\mu_A : E \to [0,1]$ and $\upsilon_A : E \to [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set A, which is subset of E, and for every $x \in E: 0 \leqslant \mu_A(x) + \upsilon_A(x) \leqslant 1$.

Let A, B be IFS, then

$$A \cap B \triangleq \{<x, \mu_A(x) \wedge \mu_B(x), \upsilon_A(x) \vee \upsilon_B(x)> | x \in E\}$$

where $\lambda \wedge \mu = \min\{\lambda, \mu\}$, $\lambda \vee \mu = \max\{\lambda, \mu\}$, for λ, $\mu \in [0,1]$.

In this paper, we denote $L = \{(\lambda, \mu) | \lambda, \mu \in [0,1], \lambda + \mu \leqslant 1\}$.

2. Intuitionistic fuzzy subgroups of a group G

Let A be a IFS over E. If IFS B satisfies:

$$\mu_B(x') = \begin{cases} \lambda, & x' = x \\ 0, & x' \neq x \end{cases} \qquad \upsilon_B(x') = \begin{cases} \mu & x' = x \\ 1 & x' \neq x \end{cases}$$

then B is called as an intuitionstic fuzzy point (IFP). It is denoted as $x_{(\lambda,\mu)}$.

If $\mu_A(x) \geqslant \lambda, \upsilon_A(x) \leqslant \mu$, then we say that $x_{(\lambda,\mu)}$ belong to A. It is denoted as $x_{(\lambda,\mu)} \in A$.

Definition 2.1 Let G be a group and A be an IFS over G, If

(1). $x_{(\lambda_1,\mu_1)}\in A, y_{(\lambda_2,\mu_2)}\in A\Rightarrow (xy)_{(\lambda_1\wedge\lambda_2,\mu_1\vee\mu_2)}\in A$,

(2). $x_{(\lambda,\mu)}\in A\Rightarrow (x^{-1})_{(\lambda,\mu)}\in A$,

then A is called as an intuitionistic fuzzy subgroup of G (IFSG).

Theorem 2.1 Let G be a group and A be a IFS over G. Then A is an IFSG of G, if and only if

(i) $\mu_A(xy)\geqslant\mu_A(x)\wedge\mu_A(y), \upsilon_A(xy)\leqslant\upsilon_A(x)\vee\upsilon_A(y)$, for any $x,y\in G$,

(ii) $\mu_A(x^{-1})\geqslant\mu_A(x), \upsilon_A(x^{-1})\leqslant\upsilon_A(x), \forall\ x\in G$.

Proof: "$\Rightarrow$", (i) Let $\lambda=\min\{\mu_A(x),\mu_A(y)\}$, $\mu=\max\{\upsilon_A(x),\upsilon_A(y)\}$, then $\lambda+\mu\leqslant 1$ and

$$\mu_A(x)\geqslant\lambda, \mu_A(y)\geqslant\lambda, \upsilon_A(x)\leqslant\mu, \upsilon_A(y)\leqslant\mu.$$

so $x_{(\lambda,\mu)}\in A$, $y_{(\lambda,\mu)}\in A$. Since A is a IFSG of G, so $(xy)_{(\lambda,\mu)}\in A$, and consequently $\mu_A(xy)\geqslant\lambda=\mu_A(x)\wedge\mu_A(y), \upsilon_A(xy)\leqslant\mu=\upsilon_A(x)\vee\upsilon_A(y)$.

(ii) is clear.

"$\Leftarrow$" (1) Let $x_{(\lambda_1,\mu_1)}\in A, y_{(\lambda_2,\mu_2)}\in A$, then $\mu_A(x)\geqslant\lambda_1, \mu_A(y)\geqslant\lambda_2, \upsilon_A(x)\leqslant\mu_1, \upsilon_A(y)\leqslant\mu_2$ and consequently $\mu_A(xy)\geqslant\mu_A(x)\wedge\mu_A(y)\geqslant\lambda_1\wedge\lambda_2, \upsilon_A(xy)\leqslant\upsilon_A(x)\vee\upsilon_A(y)\leqslant\mu_1\vee\mu_2$, then $(xy)_{(\lambda_1\wedge\lambda_2,\mu_1\vee\mu_2)}\in A$

(2) Let $x_{(\lambda,\mu)}\in A$, then $\mu_A(x^{-1})\geqslant\mu_A(x)\geqslant\lambda, \upsilon_A(x^{-1})\leqslant\upsilon_A(x)\leqslant\mu$, so $(x^{-1})_{(\lambda,\mu)}\in A$.

Hence A is an IFSG of G.

Property Let A,B be IFSGs of group G, then

(1) $\mu_A(x)\leqslant\mu_A(e)$, (2) $\upsilon_A(x)\geqslant\upsilon_A(e)$.

Notes: In this paper, we agree that $\mu_A(e)=1, \upsilon_A(e)=0$.

Lemma. Let A be an IFS over E. For $(\lambda,\mu)\in L$, Let

$$A_\lambda=\{x\mid x\in E, \mu_A(x)\geqslant\lambda\}, A^\mu=\{x\mid x\in E, \upsilon_A(x)\leqslant\mu\}.$$

then $\mu_A(x)=\vee\{\lambda\mid x\in A_\lambda\}$, $\upsilon_A(x)=\wedge\{\mu\mid x\in A^\mu\}$. where $\vee=\sup$, $\wedge=\inf$.

Theorem 2.2 Let G be a group and A be an IFS of G, then A is an IFSG of G if and only A_λ, A^μ is a subgroup of G for any $(\lambda,\mu)\in L$.

Proof: "$\Rightarrow$"Let A be an IFSG of G. We only prove that A^μ is a subgroup of G. Let $x,y\in A^\mu$, then $\upsilon_A(x)\leqslant\mu, \upsilon_A(y)\leqslant\mu$. it follows that $\upsilon_A(xy^{-1})\leqslant\upsilon_A(x)\vee\upsilon_A(y^{-1})\leqslant\upsilon_A(x)\vee\upsilon_A(y)\leqslant\mu$, so $xy^{-1}\in A^\mu$. Hence A^μ is a subgroup of G.

"$\Leftarrow$"(i) $\mu_A(xy)=\vee\{\lambda\mid xy\in A_\lambda\}\geqslant\vee\{\lambda\mid x\in A_\lambda \text{ and } y\in A_\lambda\}=\vee\{\lambda\mid\mu_A(x)\geqslant\lambda \text{ and } \mu_A(y)\geqslant\lambda\}=\vee\{\lambda\mid\mu_A(x)\wedge\mu_A(y)\geqslant\lambda\}=\mu_A(x)\wedge\mu_A(y)$.

$\upsilon_A(xy)=\wedge\{\mu\mid xy\in A^\mu\}\leqslant\wedge\{\mu\mid x\in A^\mu \text{and } y\in A^\mu\}=\wedge\{\mu\mid\upsilon_A(x)\leqslant\mu$

and $\upsilon_A(y)\leqslant\mu\} = \wedge\{\mu|\upsilon_A(x)\vee\upsilon_A(y)\leqslant\mu\} = \upsilon_A(x)\vee\upsilon_A(y)$.

(ii) $\mu_A(x^{-1}) = \vee\{\lambda|x^{-1}\in A_\lambda\} = \vee\{\lambda|x\in A_\lambda\} = \mu_A(x)$

$\upsilon_A(x^{-1}) = \wedge\{\mu|x^{-1}\in A^\mu\} = \wedge\{\mu|x\in A^\mu\} = \upsilon_A(x)$

Hence A is an IFSG of G.

Corollary Let $A_{\underset{\cdot}{\lambda}} = \{x|\mu_A(x)>\lambda\}$ $\quad A^{\underset{\cdot}{\mu}} = \{x|\upsilon_A(x)<\mu\}$, then

(1) $\mu_A(x) = \vee\{\lambda|x\in A_{\underset{\cdot}{\lambda}}\}$, $\quad \upsilon_A(x) = \wedge\{\mu|x\in A^{\underset{\cdot}{\mu}}\}$

(2) A is a IFSG of G if and only if $A_{\underset{\cdot}{\lambda}}$ and $A^{\underset{\cdot}{\mu}}$ are subgroups of G. for any $(\lambda,\mu)\in L$.

3. Normal IFSG of a group G.

Defintion 3.1 Let A be an IFSG of group G. If

$x_{(\lambda,\mu)}\in A \Rightarrow (y^{-1}xy)_{(\lambda_1\wedge\lambda,\mu_1\vee\mu)}\in A$, for any $y_{(\lambda_1,\mu_1)}\in G$,

then A is called as a normal IFSG of G(NIFSG). Clearly, we have:

Theorem 3.1 Let A be an IFSG of group G. Then the followng conditions are equivalent.

(1) A is a NIFSG of G.

(2) $\mu_A(y^{-1}xy)\geqslant\mu_A(x)$, $\upsilon_A(x^{-1}xy)\leqslant\upsilon_A(y)$, for any $x,y\in G$.

(3) $\mu_A(xy)=\mu_A(yx)$, $\upsilon_A(xy)=\upsilon_A(yx)$, for any $x,y\in G$.

(4) A_λ and A^μ are normal subgroups of G for any $(\lambda,\mu)\in L$.

(5) $A_{\underset{\cdot}{\lambda}}$ and $A^{\underset{\cdot}{\mu}}$ are normal of G for any $(\lambda,\mu)\in L$.

Definition 3.2 Let A be an IFS of G. For $a\in G$, we define:

$aA = \{<x,\mu_{aA}(x),\upsilon_{aA}(x)>|x\in G\}$, where $\mu_{aA}(x)=\mu_A(a^{-1}x)$, $\upsilon_A(x)=\upsilon_A(a^{-1}x)$.

aA is called as left coset of A over G.

Definiton 3.3 Let A, B be IFS of G. We define:

$$AB = \{<x,\mu_{AB}(x),\upsilon_{AB}(x)>|x\in G\}$$

where $\mu_{AB}(x) = \bigvee_{a\in G}(\mu_A(a)\wedge\mu_B(a^{-1}x))$, $\upsilon_{AB}(x) = \bigwedge_{a\in G}(\upsilon_A(a)\vee\upsilon_B(a^{-1}(x))$

$$A^{-1} = \{<x,\mu_A(x^{-1}),\mu_A(x^{-1})>|x\in G\}.$$

Proposition $\quad \mu_{AB}(x)+\mu_{AB}(x)\leqslant 1$, for any $x\in G$

Proof. Assume that there is a $x\in G$ such that $\mu_{AB}(x)+\upsilon_{AB}(x)>1$,

then there exists $a\in G$ such that

$$1-\upsilon_{AB}(x)<\mu_A(a)\wedge\mu_B(a^{-1}x)$$

then $1-(\mu_A(a)\vee\upsilon_B(a^{-1}x))<\mu_A(a)\wedge\mu_B(a^{-1}x)$. so

$$\mu_A(a)\wedge\mu_B(a^{-1}x)+\mu_A(a)\vee\upsilon_B(a^{-1}x)>1 \tag{1}$$

since $\mu_A(a)+\upsilon_A(a)\leqslant 1$ and $\mu_B(a^{-1}x)+\upsilon_B(a^{-1}x)\leqslant 1$, so

$\mu_A(a)\wedge\mu_B(a^{-1}x)+\upsilon_A(a)\vee\upsilon_B(a^{-1}x)\leqslant 1$, This is a contradiction with (1).

Theorem 3.2 Let A, B be IFSG of group G, If A is a NIFSG of G, then AB is a IFSG of group G.

Proof. For any $(\lambda,\mu)\in L$, then If $\mu_{AB}(x)>\lambda$, then there exists $a\in G$ such that $\mu_A(a)\wedge \mu_B(a^{-1}x)>\lambda$, then $a\in A_{\underset{\cdot}{\lambda}}$, $a^{-1}x\in B_{\underset{\cdot}{\lambda}}\Rightarrow x=a(a^{-1}x)\in A_{\underset{\cdot}{\lambda}}B_{\underset{\cdot}{\lambda}}$. So $(AB)_{\underset{\cdot}{\lambda}}\subseteq A_{\underset{\cdot}{\lambda}}B_{\underset{\cdot}{\lambda}}$. Clearly $(AB)_{\underset{\cdot}{\lambda}}\supseteq A_{\underset{\cdot}{\lambda}}B_{\underset{\cdot}{\lambda}}$ Hence $(AB)_{\underset{\cdot}{\lambda}}=A_{\underset{\cdot}{\lambda}}B_{\underset{\cdot}{\lambda}}$.

In same way, we can prove that $(AB)^{\mu}=A^{\mu}B^{\mu}$.

By corollary of theorem 2.2, we have that AB is a IFSG of G.

Theorem 3.3 Let A be a NIFSG of G, then

(1) $(aA)(bA)=(ab)A$ $\qquad$ $(aA)^{-1}=a^{-1}A$

(3) $G/A=\{aA\mid a\in G\}$ is a group (G/A is called intuitionistic fuzzy quotient group)

Proof: (1) we only prove that $\upsilon_{(aA)(bA)}(x)=\upsilon_{(ab)A}(x)$, the other is clear. In fact

$$\begin{aligned}
\upsilon_{(aA)(bA)}(x) &= \bigwedge_{g\in G}(\upsilon_{aA}(g)\vee \upsilon_{bA}(g^{-1}x)) \\
&= \bigwedge_{g\in G}\upsilon_A(a^{-1}g)\vee \upsilon_A(b^{-1}g^{-1}x) \\
&\leqslant \upsilon_A(a^{-1}a)\vee \upsilon_A(b^{-1}a^{-1}x) \\
&\leqslant \upsilon_A(e)\vee \upsilon_A((ab)^{-1}x) \\
&= \upsilon_A(ab^{-1}x) \\
&= \upsilon_{abA}(x)
\end{aligned}$$

On the other hand, $\upsilon_A(a^{-1}g)\vee \upsilon_A(b^{-1}g^{-1}x)=\upsilon_A(a^{-1}g)\vee \upsilon_A(g^{-1}xb^{-1})\geqslant \upsilon_A(a^{-1}xb^{-1})=\upsilon_A(b^{-1}a^{-1}x)=\upsilon_A((ab)^{-1}x)=\upsilon_{(ab)A}(x)$.

So $\quad \upsilon_{(aA)(bA)}(x)\geqslant \bigwedge_{g\in G}(\upsilon_A(a^{-1}g)\vee \upsilon_A(b^{-1}g^{-1}x))\geqslant \upsilon_{(ab)A}(x)$.

Hence $\quad \upsilon_{(aA)(bA)}(x)=\upsilon_{(ab)A}(x)$

Reference

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