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Fuzzy Multifunctions in Intuitionistic Fuzzy Topological Spaces

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Abstract: The aim of this paper is to construct the basic concepts of fuzzy multifunctions in intuitionistic fuzzy topological spaces developed by Çoker [6,7].

Keywords: Intuitionistic fuzzy set; intuitionistic fuzzy topology; intuitionistic fuzzy topological space; fuzzy multifunction; fuzzy lower (upper) semicontinuous function; fuzzy lower (upper) almost continuous function; fuzzy lower (upper) weakly continuous function.

1. Introduction

After the introduction of the concept of fuzzy set by Zadeh [12], the idea of "intuitionistic fuzzy set" was first given by K. Atanassov:

Definition 1.1. [1,2,3,4] Let X be a nonempty set. An intuitionistic fuzzy set (IFS for short) A is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.2. [3,4,6,7,9] Let X be a nonempty set, the IFS's A, B be in the forms $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}, B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$ and let $\{A_i : i \in J\}$ be an arbitrary family of IFS's in X. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for each $x \in X$;
(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(c) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$;
(d) $\cap A_i = \{\langle x, \wedge\mu_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$; $\cup A_i = \{\langle x, \vee\mu_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$;
(e) $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$.

Definition 1.3. ([6,7,10], cf. [5]) An intuitionistic fuzzy topology (IFT for short) on a nonempty set X is a family τ of IFS's in X containing $0_\sim$, $1_\sim$, and closed under finite infima and arbitrary suprema. In this case the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS for short) in X.

Remark 1.4. [6,7] Let (X,τ) be an IFTS.

(a) $\tau_1 = \{\mu_G : G \in \tau\}$ is a fuzzy topological space on X in Chang's sense.
(b) $\tau_2^* = \{\gamma_G : G \in \tau\}$ is the family of all fuzzy closed sets of the fuzzy topological space $\tau_2 = \{1 - \gamma_G : G \in \tau\}$ on X in Chang's sense.

Definition 1.5. [6,7] The complement $\overline{A}$ of an IFOS A in an IFTS (X,τ) is called an intuitionistic fuzzy closed set (IFCS for short) in X.
Definition 1.6. [6,7] Let (X,τ) be an IFTS and $A = \langle x, \mu_A, \lambda_A \rangle$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are defined by

$cl(A) = \cap\{K : K$ is an IFCS in X and $A \subseteq K\}$,

$int(A) = \cup\{G : G$ is an IFOS in X and $G \subseteq A\}$.

Proposition 1.7. [6,7] For any IFS A in (X,τ) we have

(a) $cl(\overline{A}) = \overline{int(A)}$, (b) $int(\overline{A}) = \overline{cl(A)}$.

Definition 1.8. [10] An IFS A in an IFTS (X,τ) is said to be

(a) an intuitionistic fuzzy regularly open (closed) set (IFROS (IFRCS) for short), iff $int(cl(A)) = A$ $(cl(int(A)) = A)$,

(b) an intuitionistic fuzzy semiopen set, iff there exists an IFOS B in X such that $B \subseteq A \subseteq cl(B)$.

Definition 1.9. [8] Let X be a nonempty set, c a fixed element in X and $\alpha \in (0,1], \beta \in [0,1)$ such that $\alpha + \beta \leq 1$. The IFS $c(\alpha,\beta) = \langle x, c_\alpha, 1 - c_{1-\beta} \rangle$ is called an intuitionistic fuzzy point (IFP for short) in X. An IFP $c(\alpha,\beta)$ is said to be q-coincident with A (denoted by $c(\alpha,\beta)_q\, A$), if $\alpha > \gamma_A(c)$ or $\beta < \mu_A(c)$.
Notice that, $c(\alpha,\beta) \subseteq A$ iff $c(\alpha,\beta) \not{q} \overline{A}$ and $c(\alpha,\beta)_q\, A$ iff $c(\alpha,\beta) \not\subseteq \overline{A}$ [8].
Definition 1.10. [8] Let X be a nonempty set; A, B two IFS's in X. A is said to be q-coincident with B (denoted by $A_q B$), if there exists $x \in X$ such that $\mu_A(c) > \gamma_B(x)$ or $\gamma_A(c) < \mu_B(x)$.
Notice that, $A_q B$ iff $A \not\subseteq \overline{B}$ and $A \subseteq B$ iff $A_{\not{q}}\overline{B}$. [8].

2. Fuzzy multifunctions in IFTS's

Definition 2.1. (cf. [11]) Let (X,τ) be an ordinary topological space and (Y,Φ) be an IFTS. Then $F : X \to Y$ is said to be a fuzzy multifunction iff $F(x)$ is an IFS in Y for each $x \in X$.
In this paper, from this point on, $F : X \to Y$ will always denote a fuzzy multifunction.
Definition 2.2. (cf. [11]) For a given $F : X \to Y$ and an IFS B in Y, we define the upper inverse and lower inverse of B under F, respectively, as follows:

$$F^+(B) = \{x \in X : F(x) \subseteq B\} \text{ and } F^-(B) = \{x \in X : F(x)_q\, B\}$$

Proposition 2.3. ([cf. [11]) For a given $F : X \to Y$ and an IFS B in Y, we have

(a) $X \backslash F^+(B) = F^-(\overline{B})$, (b) $X \backslash F^-(B) = F^+(\overline{B})$.

Definition 2.4. (cf. [11]) A fuzzy multifunction $F : X \to Y$ is said to be

(a) fuzzy upper semicontinuous (f.u.s.c. for short) at a point $x_0 \in X$ iff for each IFOS V in Y satisfying $x_0 \in F^+(V)$, there exists an open nbh U of x_0 such that $U \subseteq F^+(V)$ (i.e. $F(x) \subseteq V$ for each $x \in U$),

(b) fuzzy upper semicontinuous on X iff F is fuzzy upper semicontinuous at each $x_0 \in X$,

(c) fuzzy lower semicontinuous (f.l.s.c. for short) at a point $x_0 \in X$ iff for each IFOS V in Y satisfying $x_0 \in F^-(V)$, there exists an open nbh U of x_0 such that $U \subseteq F^-(V)$ (i.e. $F(x)_q\, V$ for each $x \in U$),

(d) fuzzy lower semicontinuous on X iff F is fuzzy lower semicontinuous at each $x_0 \in X$.

Proposition 2.5. (cf. [11]) (a) $F : X \to Y$ is f.u.s.c. on X iff $F^+(V)$ is an open set in X for each IFOS V in Y.

(b) $F : X \to Y$ is f.l.s.c. on X iff $F^-(V)$ is an open set in X for each IFOS V in Y.

Now we want to express fuzzy upper (and lower) semicontinuity of F in terms of the projections of F onto the component multifunctions F_1 and F_2 defined by

$$F_1 : (X,\tau) \to (Y,\Phi_1),\ F_1(x) = \mu_{F(x)} \text{ and}$$

$$F_2 : (X,\tau) \to (Y,\Phi_2), F_2(x) = 1 - \gamma_{F(x)}.$$

Proposition 2.6. If both the multifunctions $F_1 : (X,\tau) \to (Y,\Phi_1)$ and $F_2 : (X,\tau) \to (Y,\Phi_2)$ are f.u.s.c. at $x_0 \in X$, then $F : X \to Y$ is f.u.s.c. at $x_0 \in X$.

Corollary 2.7. If both the multifunctions $F_1 : (X,\tau) \to (Y,\Phi_1)$ and $F_2 : (X,\tau) \to (Y,\Phi_2)$ are f.u.s.c. on X, then $F : X \to Y$ is f.u.s.c. on X.

Definition 2.8. (cf. [11]) A fuzzy multifunction $F : X \to Y$ said to be

(a) fuzzy upper almost continuous (f.u.a.c. for short) at a point $x_0 \in X$ iff for each IFOS V in Y satisfying $x_0 \in F^+(V)$, there exists an open nbh U of x_0 such that $U \subseteq F^+(int(cl(V)))$ (i.e. $F(x) \subseteq int(cl(V))$ for each $x \in U$),

(b) fuzzy upper almost continuous on X iff F is fuzzy upper almost continuous at each $x_0 \in X$,

(c) fuzzy lower almost continuous (f.l.a.c. for short) at a point $x_0 \in X$ iff for each IFOS V in Y satisfying $x_0 \in F^-(V)$, there exists an open nbh U of x_0 such that $U \subseteq F^-(int(cl(V)))$ (i.e. $F(x)_q\, int(cl(V))$ for each $x \in U$),

(d) fuzzy lower almost continuous on X iff F is fuzzy lower almost continuous at each $x_0 \in X$.

Proposition 2.9. (cf. [11]) The following are equivalent to each other:

(a) $F : X \to Y$ is f.u.a.c. on X.

(b) $F^+(V) \subseteq int(F^+(int(cl(V))))$ for each IFOS V in Y.

(c) $F^+(V)$ is open for each IFROS V in Y.

(d) $F^+(int(cl(V)))$ is open in X for each IFOS V in Y.

(e) $F^-(V) \subseteq cl(F^-(cl(int(V)))$ for each IFCS V in Y.

(f) $F^-(V)$ is closed for each IFRCS V in Y.

Proposition 2.10. (cf. [11]) The following are equivalent to each other:

(a) $F : X \to Y$ is f.l.a.c. on X.

(b) $F^-(V) \subseteq int(F^-(int(cl(V))))$ for each IFOS V in Y.

(c) $F^-(V)$ is open in X for each IFROS V in Y.

(d) $F^-(int(cl(V)))$ is open in X for each IFOS V in Y.

(e) $F^+(V) \subseteq cl(F^+(cl(int(V))))$ for each IFCS V in Y.

(f) $F^+(V)$ is closed for each IFRCS V in Y.

Proposition 2.11. (cf. [11]) A fuzzy multifunction $F : X \to Y$ is f.l.a.c. (f.u.a.c.) on X iff for any intuitionistic fuzzy semiopen set A in Y, $cl(F^+(A)) \subseteq F^+(cl(A))$ (respectively, $cl(F^-(A)) \subseteq F^-(cl(A))$.

Proposition 2.12. If both the multifunctions $F_1 : (X,\tau) \to (Y,\Phi_1)$ and $F_2 : (X,\tau) \to (Y,\Phi_2)$ are f.u.a.c. at $x_0 \in X$, then $F : X \to Y$ is f.u.a.c. at $x_0 \in X$.

Definition 2.13. (cf. [11]) A fuzzy multifunction $F : X \to Y$ is said to be

(a) fuzzy upper weakly continuous (f.u.w.c. for short) at a point $x_0 \in X$ iff for every IFOS V in Y satisfying $x_0 \in F^+(V)$ there exists an open nbh U of x_0 in X such that $U \subseteq F^+(cl(V))$ (i.e. $F(x) \subseteq cl(U)$ for each $x \in U$),

(b) fuzzy upper weakly continuous on X iff F is fuzzy upper weakly continuous at each $x_0 \in X$,

(c) fuzzy lower weakly continuous (f.l.w.c., for short) at a point $x_0 \in X$ iff for every IFOS V in Y satisfying $x_0 \in F^-(cl(V))$ there exists an open nbh U of x_0 in X such that $U \subseteq F^-(cl(V))$ (i.e. $F(x)_q\, cl(V)$ for each $x \in U$),

(d) fuzzy lower weakly continuous on X iff F is fuzzy lower weakly continuous at each $x_0 \in X$.

Proposition 2.14. If both the fuzzy multifunctions $F_1 : (X,\tau) \to (Y,\Phi_1)$ and $F_2 : (X,\tau) \to (Y,\Phi_2)$ are f.u.w.c. at $x_0 \in X$, then $F : X \to Y$ is f.u.w.c. at $x_0 \in X$.
Corollary 2.15. If both the multifunctions $F_1 : (X,\tau) \to (Y,\Phi_1)$ and $F_2 : (X,\tau) \to (Y,\Phi_2)$ are f.u.w.c. on X, then $F : X \to Y$ is f.u.w.c. on X.
Proposition 2.16. (cf. [11]) $F :\to Y$ is f.l.w.c. (f.u.w.c.) iff for every IFOS V in Y, $F^-(V) \subseteq int(F^-(cl(V)))$ [respectively, $F^+(V) \subseteq int(F^+(cl(V)))$].
Proposition 2.17. (cf. [11]) If a fuzzy multifunction $F : X \to Y$ is f.u.w.c. (f.l.w.c.) on X, then $cl(F^-(V)) \subseteq F^-(cl(V))$ [respectively, $cl(F^+(V)) \subseteq F^+(cl(V))$], for any IFOS V in Y.
Notice that the following implication diagram is observed. For a fuzzy multifunction $F : X \to Y$:

$$F \text{ is f.l.s.c. (f.u.s.c.)} \Rightarrow F \text{ is f.l.a.c. (f.u.a.c.)} \Rightarrow F \text{ is f.l.w.c. (f.u.w.c.)}$$

It is also possible to find examples showing that none of the above implications is reversible, in general.

3. Fuzzy δ-closed sets and fuzzy θ-closed sets in IFTS's

Definition 3.1. (cf. [11]) An intuitionistic fuzzy point $c(\alpha,\beta)$ in an IFTS X is said to be a fuzzy δ-cluster point of an IFS A in X iff each intuitionistic fuzzy regularly open q-nbh of $c(\alpha,\beta)$ is q-coincident with A. The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by $[A]_\delta$. An IFS A will be called fuzzy δ-closed iff $A = [A]_\delta$. The complement of a fuzzy δ-closed set is said to be fuzzy δ-open.
Definition 3.2. (cf. [11]) An intuitionistic fuzzy point $c(\alpha,\beta)$ in an IFTS X is said to be a fuzzy θ-cluster point of an IFS A iff for each intuitionistic fuzzy open q-nbh V of $c(\alpha,\beta)$, $cl(V)$ is q-coincident with A. The union of all θ-cluster points of A is called the fuzzy θ-closure of A and is denoted by $[A]_\theta$. An IFS A will be called fuzzy θ-closed iff $A = [A]_\theta$. The complement of a fuzzy θ-closed set is said to be fuzzy θ-open.
Proposition 3.3. (cf. [11]) An IFP $c(\alpha,\beta)$ is contained in $cl(A)$ iff each q-nbh of $c(\alpha,\beta)$ is q-coincident with A.
Proposition 3.4. Let A be an IFS in an IFTS X.
(a) If an IFP $c(\alpha,\beta)$ is contained in $cl(A)$, then $c(\alpha,\beta)$ is a fuzzy δ-cluster point of A.
(b) If an IFP $c(\alpha,\beta)$ is a δ-cluster point of A, then $c(\alpha,\beta)$ is a fuzzy θ-closure point of A.
Example 3.5. Let $X = \{a,b\}$ and

$$U_1 = \langle x, (\frac{a}{.5}, \frac{b}{.4}), (\frac{a}{.4}, \frac{b}{.4})\rangle \text{ and } U_2 = \langle x, (\frac{a}{.3}, \frac{b}{.2}), (\frac{a}{.3}, \frac{b}{.2})\rangle.$$

Then the family $\tau = \{0_\sim, 1_\sim, U_1, U_2, U_1 \cap U_2, U_1 \cup U_2\}$ is an IFTS on X, let us consider the IFS A given by $A = \langle x, (\frac{a}{.4}, \frac{b}{.4}), (\frac{a}{.5}, \frac{b}{.4})\rangle$ and consider the IFP $b(.4,.2)$. The only intuitionistic fuzzy regularly open q-nbhs of $b(.4,.2)$ are U_2 and $1_\sim$, and each of them is q-coincident with A. Thus $b(.4,.2)$ is a fuzzy δ-cluster point of A. Now the IFS U_1 is an IFO q-nbh of $b(.4,.2)$, but it is not q-coincident with A. Thus $b(0.4,0.2)$ is not contained $cl(A)$.
Example 3.6. Let $X = \{a,b\}$, $U_1 = \langle x, (\frac{a}{.4}, \frac{b}{.3}), (\frac{a}{.6}, \frac{b}{.4})\rangle$ and $\tau = \{0_\sim, 1_\sim, U_1\}$. Let us now consider the IFP $b(.5,.2)$ and the IFS $A = \langle x, (\frac{a}{.5}, \frac{b}{.3}), (\frac{a}{.3}, \frac{b}{.3})\rangle$. We have $cl(U_1) = \overline{U_1}_q\ A$ for the only intuitionistic fuzzy open q-nbh U_1 of $b(.5,.2)$. Hence $b(.5,.2)$ is a fuzzy θ-cluster point of A. Since U_1 is an IFROS such that $b(.5,.2)_q\ U_1$, then U_1 is not q-coincident with A. Thus $b(.5,.2)$ is not a fuzzy δ-cluster point of A.

Proposition 3.7. For a fuzzy lower almost continuous multifunction $F : X \to Y$ and for any IFS B in Y, if $x \in cl(F^+(B))$ and $c(\alpha,\beta) \subseteq F(x)$, then $c(\alpha,\beta)$ is a fuzzy δ-cluster point of B.

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