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std-Statistical convergence in intuitionistic fuzzy normed space

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Abstract: In this paper, we introduce the notion of *std*-statistical convergence and *std*-statistical Cauchy with respect to the intuitionistic fuzzy norm, study their relationship, and obtain some important results.
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1 Introduction

Fuzzy theory was introduced by Zadeh [17] in 1965 and applied by researchers to the well-known results. Afterwards, fuzzy theory was generalized by Atanassov [1] as intuitionistic fuzzy theory and by Saadati and Park [11] as intuitionistic fuzzy normed space. Some important works for intuitionistic fuzzy normed space can be found in the literature [2, 6, 8, 12, 13, 15].

Recently, a powerful notion than Cauchy sequence, called standard Cauchy (shortly, *std*-Cauchy) have been obtained by Ricarte and Romaguera [10]. By using *std*-Cauchy, they established relationships between the theory of complete fuzzy metric spaces and domain theory. Quite recently, Gregori and Minana [5] answered two questions posed by Morillas and Sapera [7] concerned to standard convergence (*std*-convergence) in fuzzy metric spaces in the sense of George and Veeramani [4].

Now we recall some definitions and notations.

Definition 1. ([14]) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm provided that following conditions are satisfied:

(i) $*$ is associate and commutative,
(ii) $*$ is continuous,
(iii) $a*1=a$ for all $a\in[0,1]$,
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for every $a,b,c,d\in[0,1]$.

Definition 2. ([14]) A binary operation $\lozenge:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-conorm provided that following conditions are satisfied:
(i) $\lozenge$ is associate and commutative,
(ii) $\lozenge$ is continuous,
(iii) $a\lozenge 0=a$ for all $a\in[0,1]$,
(iv) $a\lozenge b\le c\lozenge d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

In [11], Saadati and Park introduced the notion of intuitionistic fuzzy normed space using the continuous t-norm and t-conorm as follows.

Definition 3. ([11]) The 5-tuple $(X,\mu,v,*,\lozenge)$ is said to be an intuitionistic fuzzy normed space (shortly, IFNS) provided that X is a vector space, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm, and μ, v are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions for every $x,y\in X$, and $s,t>0$:
(a) $\mu(x,t)+v(x,t)\le 1$,
(b) $\mu(x,t)>0$,
(c) $\mu(x,t)=1$ if and only if $x=0$,
(d) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$,
(e) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,
(f) $\mu(x,.):(0,\infty)\to[0,1]$ is continuous,
(g) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
(h) $v(x,t)<1$,
(i) $v(x,t)=0$ if and only if $x=0$,
(j) $v(\alpha x,t)=v\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$,
(k) $v(x,t)\lozenge v(y,s)\ge v(x+y,t+s)$,
(l) $v(x,.):(0,\infty)\to[0,1]$ is continuous,
(m) $\lim_{t\to\infty}v(x,t)=0$ and $\lim_{t\to 0}v(x,t)=1$.

In this case (μ,v) is called an intuitionistic fuzzy norm (IFN). An IFNS $(X,\mu,v,*,\lozenge)$ will be denoted simply by X.

As a standard example, we can give the following example.

Let $(X,\|.\|)$ be a normed space, and let $a*b=ab$ and $a\lozenge b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in X$ and every $t>0$, take into consider

$$\mu(x,t)=\frac{t}{t+\|x\|} \text{ and } v(x,t)=\frac{\|x\|}{t+\|x\|}.$$

Then X is an intuitionistic fuzzy normed space (IFNS).

The following definitions are due to Saadati and Park [11].

Definition 4. Let X be an IFNS. Then a sequence $x=\{x_k\}$ in X is said to be convergent to $\alpha\in X$ with respect to the intuitionistic fuzzy norm (μ,v) provided that for every $\varepsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-\alpha,t)>1-\varepsilon$ and $v(x_k-\alpha,t)<\varepsilon$ for all $k\geq k_0$. It is denoted by $(\mu,v)-\lim x=\alpha$ or $x_k\overset{(\mu,v)}{\rightarrow}\alpha$ as $k\to\infty$.

Definition 5. Let X be an IFNS. Then a sequence $x=\{x_k\}$ in X is said to be *Cauchy* to $\alpha\in X$ with respect to the intuitionistic fuzzy norm (μ,v) if, for every $\varepsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-x_m,t)>1-\varepsilon$ and $v(x_k-x_m,t)<\varepsilon$ for all $k,m\geq k_0$..

2 *std*-Statistical convergence on IFNS

Let $K\subset\mathbb{N}$ and $\delta(K):=\lim_{n\to\infty}\frac{1}{n}|\{k\in K\ :\ k\leq n\}|$ denote the natural density of set $K=\{k\in K\ :\ k\leq n\}$, where the vertical bars denote number of elements of K not exceeding $n\in\mathbb{N}$. A sequence $x=(x_k)_{k\in\mathbb{N}}$ of real (or complex) numbers is said to be statistically convergent to α provided that for every $\varepsilon>0$, natural density of the set $\{k\in\mathbb{N}\ :\ |x_k-\alpha|\geq\varepsilon\}$ is zero. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to α we write $st\text{-}\lim x_k=\alpha$ [3, 16].

Definition 6. Let X be an IFNS. Then a sequence $x=\{x_k\}_{k\in\mathbb{N}}$ is said to be *std-statistically convergent* to $\alpha\in X$ with respect to the intuitionistic fuzzy norm (μ,v) provided that for each $\varepsilon>0$ and $t>0$,

$$\delta\left\{k\in\mathbb{N}\ :\ \mu(x_k-\alpha,t)\leq\frac{t}{t+\varepsilon} \text{ or } v(x_k-\alpha,t)\geq\frac{\varepsilon}{t+\varepsilon}\right\}=0 \tag{1}$$

or equivalently

$$\frac{1}{n}\left|\left\{k\in\mathbb{N}\ :\ \mu(x_k-\alpha,t)\leq\frac{t}{t+\varepsilon} \text{ or } v(x_k-\alpha,t)\geq\frac{\varepsilon}{t+\varepsilon}\right\}\right|=0$$

In this case we abbreviate $\overset{std}{st_{(\mu,v)}}-\lim x=\alpha$.

Definition 7. Let X be an IFNS. Then a sequence $x=\{x_k\}_{k\in\mathbb{N}}$ is said to be *std*-statistically Cauchy with respect to the intuitionistic fuzzy norm (μ,v) provided that for each $\varepsilon>0$ and $t>0$,

$$\delta\left\{k \in \mathbb{N} \ : \ \mu(x_k - x_m, t) \leq \frac{t}{t+\varepsilon} \text{ or } v(x_k - x_m, t) \geq \frac{\varepsilon}{t+\varepsilon}\right\} = 0.$$

From (1) and property of density, we can easily following result.

Lemma 1. Let X be an IFNS. Then, for every $\varepsilon > 0$ and $t > 0$, the following conditions are equivalent:

(i) $st_{(\mu,v)} \overset{std}{-} \lim x = \alpha,$

(ii) $\delta\left\{k \in \mathbb{N} \ : \ \mu(x_k - \alpha, t) \leq \frac{t}{t+\varepsilon}\right\} = \delta\left\{k \in \mathbb{N} \ : \ v(x_k - \alpha, t) \geq \frac{\varepsilon}{t+\varepsilon}\right\} = 0,$

(iii) $\delta\left\{k \in \mathbb{N} \ : \ \mu(x_k - \alpha, t) > \frac{t}{t+\varepsilon} \text{ or } v(x_k - \alpha, t) < \frac{\varepsilon}{t+\varepsilon}\right\} = 1,$

(iv) $st \overset{std}{-} \lim \mu(x_k - \alpha, t) = 1$ and $st \overset{std}{-} \lim v(x_k - \alpha, t) = 0.$

Theorem 1. Let X be an IFNS. If a sequence $x = \{x_k\}_{k \in \mathbb{N}}$ is *std*-statisticaly convergent with respect to the IFN (μ, v), then the $st_{(\mu,v)} \overset{std}{-}$ limit is unique.

Proof. Suppose that $st_{(\mu,v)} \overset{std}{-} \lim x = \alpha_1$ and $st_{(\mu,v)} \overset{std}{-} \lim x = \alpha_2$. Given $\varepsilon > 0$ and $t > 0$ choose $\eta > 0$ such that

$$\left(\frac{t}{t+\eta}\right) * \left(\frac{t}{t+\eta}\right) > \frac{t}{t+\varepsilon} \text{ and } \left(\frac{\eta}{t+\eta}\right) \lozenge \left(\frac{\eta}{t+\eta}\right) < \frac{\varepsilon}{t+\varepsilon}.$$

Then, for any $t > 0$, define the following sets:

$$M_{\mu,1}(\eta, t) = \left\{k \in \mathbb{N} \ : \ \mu(x_k - \alpha_1, t) \leq \frac{t}{t+\eta}\right\},$$
$$M_{\mu,2}(\eta, t) = \left\{k \in \mathbb{N} \ : \ \mu(x_k - \alpha_2, t) \leq \frac{t}{t+\eta}\right\},$$
$$M_{v,1}(\eta, t) = \left\{k \in \mathbb{N} \ : \ v(x_k - \alpha_1, t) \geq \frac{\eta}{t+\eta}\right\},$$
$$M_{v,2}(\eta, t) = \left\{k \in \mathbb{N} \ : \ v(x_k - \alpha_2, t) \geq \frac{\eta}{t+\eta}\right\}.$$

As $st_{(\mu,v)} \overset{std}{-} \lim x = \alpha_1$, we can obtain

$$\delta(M_{\mu,1}(\varepsilon, t)) = \delta(M_{v,1}(\varepsilon, t)) = 0$$

for all $t > 0$. Moreover, $st_{(\mu,v)} \overset{std}{-} \lim x = \alpha_2$, we have

$$\delta(M_{\mu,2}(\varepsilon, t)) = \delta(M_{v,2}(\varepsilon, t)) = 0$$

for all $t > 0$. Let

$$M_{\mu,v}(\varepsilon,t)=\{M_{\mu,1}(\varepsilon,t)\cup M_{\mu,2}(\varepsilon,t)\}\cap\{M_{v,1}(\varepsilon,t)\cup M_{v,2}(\varepsilon,t)\}.$$

Then see that $\delta(M_{\mu,v}(\varepsilon,t))=0$ which implies $\delta(\mathsf{N}/M_{\mu,v}(\varepsilon,t))=1$. If $k\in\mathsf{N}/M_{\mu,v}(\varepsilon,t)$, there are two possible cases. The first is the case of $k\in\mathsf{N}/M_{\mu,1}(\varepsilon,t)\cup M_{\mu,2}(\varepsilon,t)$, and the second is the case of $k\in\mathsf{N}/M_{v,1}(\varepsilon,t)\cup M_{v,2}(\varepsilon,t)$. We first take into consideration that $k\in\mathsf{N}/M_{\mu,1}(\varepsilon,t)\cup M_{\mu,2}(\varepsilon,t)$. Then we obtain

$$\begin{aligned}\mu(\alpha_1-\alpha_2,t)&\geq \mu\left(x_k-\alpha_1,\frac{t}{2}\right)*\mu\left(x_k-\alpha_2,\frac{t}{2}\right)\\&>\left(\frac{t}{t+\eta}\right)*\left(\frac{t}{t+\eta}\right)\\&>\frac{t}{t+\varepsilon}.\end{aligned}$$

As $\varepsilon>0$ is arbitrary, we have $\mu(\alpha_1-\alpha_2,t)=1$ for all $t>0$. Therefore, we get $\alpha_1-\alpha_2=0$, that is, $\alpha_1=\alpha_2$.

On the other hand, if $k\in\mathsf{N}/M_{v,1}(\varepsilon,t)\cup M_{v,2}(\varepsilon,t)$, the we get

$$\begin{aligned}v(\alpha_1-\alpha_2,t)&\leq v\left(x_k-\alpha_1,\frac{t}{2}\right)\Diamond v\left(x_k-\alpha_2,\frac{t}{2}\right)\\&<\left(\frac{\eta}{t+\eta}\right)\Diamond\left(\frac{\eta}{t+\eta}\right)\\&<\frac{\varepsilon}{t+\varepsilon}.\end{aligned}$$

Again, since $\varepsilon>0$ is arbitrary, we have $v(\alpha_1-\alpha_2,t)=0$ for all $t>0$, which implies $\alpha_1=\alpha_2$. As a consequence, in all cases, we conclude that the the $st_{(\mu,v)}\overset{std}{-}$ limit is unique. □

Theorem 2. Let X be an IFNS. If $(\mu,v)\overset{std}{-}\lim x=\alpha$, then $st_{(\mu,v)}\overset{std}{-}\lim x=\alpha$.

Proof. Since $(\mu,v)\overset{std}{-}\lim x=\alpha$, for every $\varepsilon>0$ and $t>0$, there exits $k_\varepsilon\in\mathsf{N}$ such that

$$\mu(x_k-\alpha,t)>\frac{t}{t+\varepsilon}\text{ and }v(x_k-\alpha,t)<\frac{\varepsilon}{t+\varepsilon}$$

for all $k\geq k_\varepsilon$. This guarantees that the set

$$\left\{k\in\mathsf{N}\ :\ \mu(x_k-\alpha,t)\leq\frac{t}{t+\varepsilon}\text{ or }v(x_k-\alpha,t)\geq\frac{\varepsilon}{t+\varepsilon}\right\}$$

has at most finitely many terms. As every finite subset of the natural numbers has density zero, we get that

$$\delta\left\{k\in\mathsf{N}\ :\ \mu(x_k-\alpha,t)\leq\frac{t}{t+\varepsilon}\text{ or }v(x_k-\alpha,t)\geq\frac{\varepsilon}{t+\varepsilon}\right\}=0,$$

as a desired. □

The following two results show that the notions of *std*-statistically convergence and *std*-statistically Cauchy are both stronger than usual statistically convergence and statistically Cauchy respectively in an IFNS.

Theorem 3. Let X be an IFNS and the sequence $x = \{x_k\}_{k\in\mathrm{N}}$ in X be *std*-statistically convergent to $\alpha \in X$. Then $\{x_k\}$ is statistically convergent to $\alpha \in X$ with respect to the IFN (μ, v).

Proof. Suppose $st_{(\mu,v)}\overset{std}{-}\lim x = \alpha.$ Then we have

$$\delta\left\{k \in \mathrm{N} \ : \ \mu(x_k - \alpha, t) \le \frac{t}{t+\varepsilon} \text{ or } v(x_k - \alpha, t) \ge \frac{\varepsilon}{t+\varepsilon}\right\} = 0$$

for $\varepsilon > 0$ and $t > 0$. Since $\frac{\varepsilon}{t+\varepsilon} < \varepsilon$ for all $t > 0$, we obtain that

$$\frac{t}{t+\varepsilon} = 1 - \frac{\varepsilon}{t+\varepsilon} > 1 - \varepsilon.$$

As a result,

$$\delta\{k \in \mathrm{N} \ : \ \mu(x_k - \alpha, t) \le 1 - \varepsilon \text{ or } v(x_k - \alpha, t) \ge \varepsilon\} = 0,$$

as desired. □

By same arguments in Theorem 3, the following can be easily proved.

Theorem 4. Let X be an IFNS and the sequence $x = \{x_k\}_{k\in\mathrm{N}}$ in X be *std*-statistically Cauchy. Then $\{x_k\}$ is statistically Cauchy with respect to the IFN (μ, v).

Definition 8. A sequence $x = \{x_k\}_{k\in\mathrm{N}}$ in IFNS is said to be strong *std*-statistically convergent if it is both statistically convergent and *std*-statistically Cauchy with respect to the IFN (μ, v).

Theorem 5. Let X be an IFNS and $x = \{x_k\}_{k\in\mathrm{N}}$ be a strong *std*-statistically convergent sequence in X. Then $x = \{x_k\}_{k\in\mathrm{N}}$ is *std*-statistically convergent with respect to the IFN (μ, v).

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