Notes on Intuitionistic Fuzzy Sets
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Norms over intuitionistic fuzzy subrings and ideals of a ring

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Abstract: In this paper, we apply norms over intuitionistic fuzzy subrings and ideals of a ring. We introduce the notions of intuitionistic fuzzy subrings and ideals of a ring with respect a t-norm T and a t-conorm C and investigate some related properties under homomorphism.
Keywords: Ring theory, Norms, Fuzzy set theory, Intuitionistic fuzzy subrings, Intuitionistic fuzzy ideals, Homomorphisms, Direct products.
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1 Introduction

The concept of a fuzzy set was introduced by Zadeh [16], and it is now a rigorous area of research with manifold applications raging from engineering and computer science to medical diagnosis and social behavior studies. In particular, some researchers [2, 14, 17] applied the notion of fuzzy sets to ideals of a ring. As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by K. T. Atanassov [3, 4]. Recently, Coker [8], Coker and Es [9], Gurcay, Coker and Es [10] and S. J. Lee and E. P. Lee [13] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigated some of their properties. In 1989, Biswas [6] introdused the concept of intuitionistic fuzzy subgroups and studied some of it's properties. In 2003, Banejee and Basnet [5] investigated intuitionistic fuzzy subrings and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. Also Hur, Jang and Kang [11] and Hur, Kang and Song [12] studied various properties of intuitionistic fuzzy subgroupoids, intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings. In this work, We introduce the notions of intuitionistic fuzzy subrings and ideals of a ring with respect a t-norm T and a t-conorm C and establish necessary and sufficient conditions for them. We also investigate the algebraic nature of such type of intuitionistic fuzzy subrings and ideals under homomorphism and direct product.

2 Preliminaries

In this section, we list some basic concepts and well known results needed in the later sections. Throughout this paper, R will be a commutative ring with unity.

Definition 2.1. (See [3]) For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.2. (See [3]) Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, *IFS*) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$, respectively.
We will denote the set of all *IFSs* in X as *IFS*(X).

Definition 2.3. (See [3]) Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be *IFSs* in X. Then
(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 2.4. (See [1]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element)
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)
(T3) $T(x,y) = T(y,x)$ (commutativity)
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

Recall that T is idempotent if for all $x \in [0,1]$, $T(x,x) = x$.

Example 2.5. The basic t-norms are $T_m(x,y) = \min\{x,y\}$,$T_b(x,y) = \max\{0, x+y-1\}$ and $T_p(x,y) = xy$, with $x, y \in [0,1]$, are called standard intersection, bounded sum and algebraic product respectively.

Lemma 2.6. *(See [1]) Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.7. (See [7]) A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(C1) $C(x,0) = x$
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$
(C3) $C(x,y) = C(y,x)$
(C4) $C(x, C(y,z)) = C(C(x,y), z)$,
for all $x, y, z \in [0,1]$.

Example 2.8. The basic t-conorms are $C_m(x,y) = \max\{x,y\}$,$C_b(x,y) = \min\{1, x+y\}$ and $C_p(x,y) = x + y - xy$ for all $x, y \in [0,1]$.
C_m is standard union, C_b is bounded sum, C_p is algebraic sum.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, $C(x,x) = x$.

Theorem 2.9. *(See [15]) Let R be a ring. A nonempty subset S of R is a subring of R if and only if $x - y \in S$ and $xy \in S$ for all $x, y \in S$.*

Definition 2.10. (See [15]) Let R be a ring and I be a nonempty subset of R. We say that I is a left(right) ideal of R if for all $x, y \in I$ and for all $r \in R$, $x - y \in I$, $rx \in I$ ($x - y \in I$, $xr \in I$).

3 Intuitionistic fuzzy subrings with respect to a t-norm T and a t-conorm C

Definition 3.1. Let R be a ring. An $A = (\mu_A, \nu_A)$ is said to be intuitionistic fuzzy subring with respect to a t-norm T and a t-conorm C (in short, *IFSTC*(R)) of R if
(1) $\mu_A(x - y) \geq T(\mu_A(x), \mu_A(y))$
(2) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y))$
(3) $\nu_A(x - y) \leq C(\nu_A(x), \nu_A(y))$
(4) $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y))$,
for all $x, y \in R$.

Example 3.2. Let $R = (Z, +, .)$ be a ring of integer. For all $x \in R$ we define a fuzzy subset μ_A and ν_A of R as

$$\mu_A(x) = \begin{cases} 0.6 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.5 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0.2 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.4 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

let $T(x,y) = T_p(x,y) = xy$ and $C(x,y) = C_p(x,y) = x + y - xy$ for all $x, y \in R$, then $A = (\mu_A, \nu_A) \in$ *IFSTC*(R).

Proposition 3.3. *If $A = (\mu_A, \nu_A) \in$ IFSTC(R) and T, C be idempotent, then*
(1) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
(2) $A(-x) = A(x)$, for all $x \in R$.

Proof. (1) If $x \in R$, then $\mu_A(0) = \mu_A(x - x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$.
Also $\nu_A(0) = \nu_A(x - x) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x)$.
(2) Let $x \in R$. Then

$$\mu_A(-x) = \mu_A(0 - x) \geq T(\mu_A(0), \mu_A(x)) \geq T(\mu_A(x), \mu_A(x))$$

$$= \mu_A(x) = \mu_A(0 - (-x)) \geq T(\mu_A(0), \mu_A(-x)) \geq T(\mu_A(-x), \mu_A(-x))$$

$$= \mu_A(-x)$$

and so $\mu_A(-x) = \mu_A(x)$.
Also

$$\nu_A(-x) = \nu_A(0 - x) \leq C(\nu_A(0), \nu_A(x)) \leq C(\nu_A(x), \nu_A(x))$$

$$= \nu_A(x) = \nu_A(0 - (-x)) \leq C(\nu_A(0), \nu_A(-x)) \leq C(\nu_A(-x), \nu_A(-x))$$

$$= \nu_A(-x)$$

and so $\nu_A(-x) = \nu_A(x)$.
Thus $A(-x) = (\mu_A(-x), \nu_A(-x)) = (\mu_A(x), \nu_A(x)) = A(x)$. □

Proposition 3.4. *Let* $A = (\mu_A, \nu_A) \in$ *IFSTC*(R) *and* $x, y \in R$.
(1) If $\mu_A(x - y) = 1$, *then* $\mu_A(x) \geq \mu_A(y)$
(2) If $\nu_A(x - y) = 0$, *then* $\nu_A(x) \leq \nu_A(y)$.

Proof. Let $x, y \in R$. Then
(1) $\mu_A(x) = \mu_A(x - y + y) \geq T(\mu_A(x - y), \mu_A(y)) = T(1, \mu_A(y)) = \mu_A(y)$.
(2) $\nu_A(x) = \nu_A(x - y + y) \leq C(\nu_A(x - y), \nu_A(y)) = C(0, \nu_A(y)) = \nu_A(y)$. □

Proposition 3.5. *Let* $A = (\mu_A, \nu_A) \in$ *IFSTC*(R) *and* T, C *be idempotent. Then* $A(x - y) = A(y)$ *if and only if* $A(x) = A(0)$ *for all* $x, y \in R$.

Proof. Let $A(x - y) = A(y)$ then by letting $y = 0$ we get $A(x) = A(0)$.
Conversely, assume that $A(x) = A(0)$. Then
(1) $\mu_A(x) = \mu_A(0)$ and from Proposition 3.3 we get $\mu_A(x) \geq \mu_A(x - y), \mu_A(y)$. Now

$$\mu_A(x - y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\mu_A(y), \mu_A(y))$$

$$= \mu_A(y) = \mu_A(-y) = \mu_A(x - y - x) \geq T(\mu_A(x - y), \mu_A(x))$$

$$\geq T(\mu_A(x - y), \mu_A(x - y)) = \mu_A(x - y),$$

so $\mu_A(x - y) = \mu_A(y)$.
(2) $\nu_A(x) = \nu_A(0)$ and by Proposition 3.3 we have $\nu_A(x) \leq \nu_A(x - y), \nu_A(y)$. Now

$$\nu_A(x - y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\nu_A(y), \nu_A(y))$$

$$= \nu_A(y) = \nu_A(-y) = \nu_A(x - y - x)$$

$$\leq C(\nu_A(x - y), \nu_A(x)) \leq C(\nu_A(x - y), \nu_A(x - y)) = \nu_A(x - y),$$

hence $\nu_A(x - y) = \nu_A(y)$.
Therefore from (1) and (2) we obtain that $A(x - y) = A(y)$. □

Proposition 3.6. *Let $A = (\mu_A, \nu_A) \in$ IFSTC(R) and T, C be idempotent.*
(1) $S = \{x \in R \mid \mu_A(x) = 1, \nu_A(x) = 0\}$ is a subring of R.
(2) Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, then $R_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a subring of R.

Proof. (1) Let $x, y \in S$. Then from $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y)) = T(1,1) = 1$, we get $\mu_A(x-y) = 1$. Since $\nu_A(x-y) \leq C(\nu_A(x), \nu_A(y)) = C(0,0) = 0$ so $\nu_A(x-y) = 0$. Hence $x - y \in S$.
Also from $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y)) = T(1,1) = 1$ and $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y)) = C(0,0) = 0$, we get $\mu_A(xy) = 1$ and $\nu_A(xy) = 0$ respectively. Hence $xy \in S$.
Thus $S = \{x \in R \mid \mu_A(x) = 1, \nu_A(x) = 0\}$ is a subring of R.
(2) Let $x, y \in R_{\alpha,\beta}$. Then by $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$ and $\nu_A(x-y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$, we get $\mu_A(x-y) \geq \alpha$ and $\nu_A(x-y) \leq \beta$ respectively. Hence $x - y \in R_{\alpha,\beta}$.
Also from $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$ and $\nu_A(xy) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$, we obtain that $\mu_A(x-y) \geq \alpha$ and $\nu_A(x-y) \leq \beta$ respectively. Thus $xy \in R_{\alpha,\beta}$.
Therefore $R_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a subring of R. □

Proposition 3.7. *Let $A = (\mu_A, \nu_A) \in$ IFS(R) and T, C be idempotent such that for all $x, y \in R$ we have $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y)), \nu_A(x-y) \leq C(\nu_A(x), \nu_A(y))$ and $\mu_A(rx) \geq \mu_A(x), \nu_A(rx) \leq \nu_A(x)$. Then*
(1) $R_0 = \{x \in R \mid A(x) = A(0)\}$ is a left ideal of R.
(2) $R_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a left ideal of R for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.

Proof. (1) Suppose that $x, y \in R_0$ then $\mu_A(x) = \mu_A(y) = \mu_A(0)$ and $\nu_A(x) = \nu_A(y) = \nu_A(0)$. From $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y)) = T(\mu_A(0), \mu_A(0)) = \mu_A(0) \geq \mu_A(x-y)$, we get $\mu_A(x-y) = \mu_A(0)$. By $\nu_A(x-y) \leq C(\nu_A(x), \nu_A(y)) = C(\nu_A(0), \nu_A(0)) = \nu_A(0) \leq \nu_A(x-y)$, we obtain $\nu_A(x-y) = \nu_A(0)$. Hence $A(x-y) = A(0)$ and so $x - y \in R_0$.
Also if $x \in R_0$ and $r \in R$, then $\mu_A(rx) \geq \mu_A(x) = \mu_A(0) \geq \mu_A(rx)$ and $\nu_A(rx) \leq \nu_A(x) = \nu_A(0) \leq \nu_A(rx)$ and we get $\mu_A(rx) = \mu_A(0)$ and $\nu_A(rx) = \nu_A(0)$ respectively. Thus $A(rx) = A(0)$ and $rx \in R_0$.
Therefore R_0 is a left ideal of R.
(2) Assume that $x, y \in R_{\alpha,\beta}$. Then by $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$ and $\nu_A(x-y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$ we have that $x - y \in R_{\alpha,\beta}$.
Also if $x \in R_{\alpha,\beta}$ and $r \in R$, then by $\mu_A(rx) \geq \mu_A(x) \geq \alpha$ and $\nu_A(rx) \leq \nu_A(x) \leq \beta$ we get $rx \in R_{\alpha,\beta}$.
Hence $R_{\alpha,\beta}$ is a left ideal of R. □

4 Intuitionistic fuzzy ideals with respect to a t-norm T and a t-conorm C

Definition 4.1. Let $A = (\mu_A, \nu_A) \in$ *IFS*(R). Then A is called an intuitionistic fuzzy ideal with respect to a t-norm T and a t-conorm C (in short, *IFITC*(R)) of R if
(1) $\mu_A(x - y) \geq T(\mu_A(x), \mu_A(y))$
(2) $\mu_A(xy) \geq \mu(x), \mu(y)$
(3) $\nu_A(x - y) \leq C(\nu_A(x), \nu_A(y))$
(4) $\nu_A(xy) \leq \nu(x), \nu(y)$,
for all $x, y \in R$.

Proposition 4.2. *Let $A = (\mu_A, \nu_A) \in$ IFITC(R) and $x, y \in R$. Then $A(x - y) = A(0)$ if and only if $A(x) = A(0)$.*

Proof. Let $x, y \in R$. If $A(x - y) = A(0)$ and $y = 0$, then $A(x) = A(0)$.
Conversely, let $A(x) = A(0)$. Now $\mu_A(x - y) \geq T(\mu_A(x), \mu_A(y)) = T(\mu_A(0), \mu_A(0)) = \mu_A(0) \geq \mu_A(x - y)$ and so $\mu_A(x - y) = \mu_A(0)$.
Also $\nu(x - y) \leq C(\nu(x), \nu(y)) = C(\nu(0), \nu(0)) = \nu(0) \leq \nu(x - y)$ and then $\nu(x - y) = \nu(0)$.
Thus $A(x - y) = (\mu_A(x - y), \nu_A(x - y)) = (\mu_A(0), \nu_A(0)) = A(0)$. □

Proposition 4.3. *Let $A = (\mu_A, \nu_A) \in$ IFS(R) and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in$ IFITC(R) if and only if $R_{\alpha,\beta} = \{x \in R \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is an ideal of R, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, where $\mu_A(0) \geq \alpha$ and $\nu_A(0) \leq \beta$.*

Proof. Let $A = (\mu_A, \nu_A) \in$ *IFITC*(R). If $x, y \in R_{\alpha,\beta}$, then by $\mu_A(x - y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$ and $\nu_A(x - y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\beta, \beta) = \beta$, we obtain that $x - y \in R_{\alpha,\beta}$.
Now let $x \in R_{\alpha,\beta}$ and $r \in R$. Then from $\mu_A(rx) \geq \mu_A(x) \geq \alpha$ and $\nu_A(rx) \leq \nu_A(x) \leq \beta$, we get $rx \in R_{\alpha,\beta}$. Similary we have $xr \in R_{\alpha,\beta}$. Thus $R_{\alpha,\beta}$ is an ideal of R.
Conversely, let $R_{\alpha,\beta}$ be an ideal of R and $x, y \in R_{\alpha,\beta}$ such that $\mu_A(x) = \mu_A(y) = \alpha$ and $\nu_A(x) = \nu_A(y) = \beta$. Since $x - y \in R_{\alpha,\beta}$ so $\mu_A(x - y) \geq \alpha = T(\alpha, \alpha) = T(\mu_A(x), \mu_A(y))$ and $\nu_A(x - y) \leq \beta = C(\beta, \beta) = C(\nu_A(x), \nu_A(y))$.
Also since $xy \in R_{\alpha,\beta}$ then $\mu_A(xy) \geq \alpha = \mu_A(x)$ and $\nu_A(xy) \leq \beta = \nu_A(x)$.
Therefore $A = (\mu_A, \nu_A) \in$ *IFITC*(R). □

Corollary 4.4. *$A = (\mu_A, \nu_A) \in$ IFS(R) and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in$ IFITC(R) if and only if $U = \{x \in R \mid \mu_A(x) \geq \alpha\}$ and $L = \{x \in R \mid \nu_A(x) \leq \beta\}$ are ideals of R for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.*

Proof. Let $A = (\mu_A, \nu_A) \in$ *IFITC*(R). Then $U = R_{\alpha,1}$ and $L = R_{0,\beta}$ are ideals of R.
Conversely, $U \cap L = R_{\alpha,\beta}$ is also an ideal of R and so $A = (\mu_A, \nu_A) \in$ *IFITC*(R). □

Definition 4.5. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of R. Define $A \cap B = (\mu_{A\cap B}, \nu_{A\cup B})$ as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cup B}(x) = C(\nu_A(x), \nu_B(y))$ for all $x \in R$.

Proposition 4.6. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in R. If $A, B \in$ IFITC(R), then $(A \cap B) \in$ IFITC(R).*

Proof. Let $x, y \in R$. Then
(1)

$$\mu_{A\cap B}(x-y) = T(\mu_A(x-y), \mu_B(x-y)) \geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y)))$$
$$= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) = T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)).$$

(2) $\mu_{A\cap B}(xy) = T(\mu_A(xy), \mu_B(xy)) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$.
(3)

$$\nu_{A\cup B}(x-y) = C(\nu_A(x-y), \nu_B(x-y)) \leq C(C(\nu_A(x), \nu_A(y)), C(\nu_B(x), \nu_B(y)))$$
$$= C(C(\nu_A(x), \nu_B(x)), C(\nu_A(y), \nu_B(y))) = C(\nu_{A\cup B}(x), \nu_{A\cup B}(y)).$$

(4) $\nu_{A\cup B}(xy) = C(\nu_A(xy), \nu_B(xy)) \leq C(\nu_A(x), \nu_B(x)) = \nu_{A\cup B}(x)$.
Hence $(A \cap B) \in$ *IFITC*(R). □

Corollary 4.7. *Let $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i = 1, 2, 3, ..., n\} \subseteq$ IFITC(R). Then so does $\cap_{A_i} = (\mu_{\cap A_i}, \nu_{\cup A_i})$.*

Definition 4.8. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in R. Then the intuitionistic fuzzy product of A and B with respect to a t-norm T and a t-conorm C, $A \circ B$ is defined as follows:
for all $x \in R$,

$$A \circ B(x) = (\mu_{A\circ B}(x), \nu_{A\circ B}(x))$$
$$= \begin{cases} (\sup_{x=yz}\{T(\mu_A(y), \mu_B(z)\}, \inf_{x=yz}\{C(\nu_A(y), \nu_B(z)\}) & \text{if } x = yz \\ (0, 1) & \text{if } x \neq yz \end{cases}$$

Recall that a ring R is said to be regular if for each $a \in R$ there exists an $x \in R$ such that $a = axa$.

Proposition 4.9. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in R. If R is regular and $A, B \in$ IFITC(R), then $A \circ B = A \cap B$.*

Proof. Let $x \in R$ and suppose $A \circ B(x) = (0, 1)$. Then there is nothing to show. Assume $A \circ B(x) \neq (0, 1)$. Then $A \circ B(x) = (\sup_{x=yz}\{T(\mu_A(y), \mu_B(z)\}, \inf_{x=yz}\{C(\nu_A(y), \nu_B(z)\})$.
Since $A, B \in$ *IFITC*(R),

$$\mu_A(y) \leq \mu_A(yz) = \mu_A(x), \nu_A(y) \geq \nu_A(yz) = \nu_A(x)$$

and

$$\mu_B(z) \leq \mu_B(yz) = \mu_B(x), \nu_B(z) \geq \nu_B(yz) = \nu_B(x).$$

Hence

$$\mu_{A\circ B}(x) = \sup_{x=yz}\{T(\mu_A(y), \mu_B(z)\} \leq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$$

and

$$\nu_{A\circ B}(x) = \inf_{x=yz}\{C(\nu_A(y), \nu_B(z)\} \geq C(\nu_A(x), \nu_B(x)) = \nu_{A\cap B}(x).$$

Therefore $A \circ B \subset A \cap B$.

Now we show $A \cap B \subset A \circ B$. Let $a \in R$ and since R is regular so there exists an $x \in R$ such that $a = axa$. Hence

$$\mu_A(a) = \mu_A(axa) \geq \mu_A(ax) \geq \mu_A(a)$$

and

$$\nu_A(a) = \nu_A(axa) \leq \nu_A(ax) \leq \nu_A(a)$$

so $\mu_A(ax) = \mu_A(a)$ and $\nu_A(ax) = \nu_A(a)$. Thus $A(ax) = A(a)$. Now

$$\mu_{A\circ B}(a) = \sup_{a=yz}\{T(\mu_A(y), \mu_B(z)\} \geq T(\mu_A(ax), \mu_B(a)) \qquad \text{(Since } a = axa\text{)}$$

$$= T(\mu_A(a), \mu_B(a)) = \mu_{A\cap B}(a)$$

and

$$\nu_{A\circ B}(a) = \inf_{a=yz}\{C(\nu_A(y), \nu_B(z)\} \leq C(\nu_A(ax), \nu_B(a)) \qquad \text{(Since } a = axa\text{)}$$

$$C(\nu_A(a), \nu_B(a)) = \nu_{A\cap B}(a).$$

Therefore $A \cap B \subset A \circ .B$

Hence $A \cap B = A \circ B$. This completes the proof. □

Definition 4.10. Let Let φ be a morphism from ring R into ring S such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in R and S respectively.

For all $x \in R, y \in S$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$$
$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in R, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in R, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0, 1) & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Proposition 4.11. *Let φ be an epimorphism from ring R into ring S. If $A = (\mu_A, \nu_A) \in$ IFITC(R), then $\varphi(A) \in$ IFITC(S).*

Proof. Let $y_1, y_2 \in S$. Then

(1)

$$\varphi(\mu_A)(y_1 - y_2) = \sup\{\mu_A(x_1 - x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\geq \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$= T(\sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\})$$
$$= T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2)).$$

(2)
$$\varphi(\mu_A)(y_1y_2) = \sup\{\mu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\geq \sup\{\mu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\} = \varphi(\mu_A)(y_1).$$

(3)
$$\varphi(\nu_A)(y_1 - y_2) = \inf\{\nu_A(x_1 - x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\leq \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$= C(\inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in R, \varphi(x_2) = y_2\})$$
$$= C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2)).$$

(4)
$$\varphi(\nu_A)(y_1y_2) = \inf\{\nu_A(x_1x_2) \mid x_1, x_2 \in R, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\leq \inf\{\nu_A(x_1) \mid x_1 \in R, \varphi(x_1) = y_1\} = \varphi(\nu_A)(y_1).$$

Hence $\varphi(A) \in$ *IFITC*(S). □

Proposition 4.12. *Let Let φ be a morphism from ring R into ring S. If $B = (\mu_B, \nu_B) \in$ IFITC(S), then $\varphi^{-1}(B) \in$ IFITC(R).*

Proof. Let $x_1, x_2 \in R$.
(1)
$$\varphi^{-1}(\mu_B)(x_1 - x_2) = \mu_B(\varphi(x_1 - x_2)) = \mu_B(\varphi(x_1) - \varphi(x_2))$$
$$\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) = T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2)).$$

(2)
$$\varphi^{-1}(\mu_B)(x_1x_2) = \mu_B(\varphi(x_1x_2)) = \mu_B(\varphi(x_1)\varphi(x_2))$$
$$\geq \mu_B(\varphi(x_1)) = \varphi^{-1}(\mu_B)(x_1).$$

(3)
$$\varphi^{-1}(\nu_B)(x_1 - x_2) = \nu_B(\varphi(x_1 - x_2)) = \nu_B(\varphi(x_1) - \varphi(x_2))$$
$$\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) = C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2)).$$

(4) $\varphi^{-1}(\nu_B)(x_1x_2) = \nu_B(\varphi(x_1x_2)) = \nu_B(\varphi(x_1)\varphi(x_2)) \leq \nu_B(\varphi(x_1)) = \varphi^{-1}(\nu_B)(x_1)$.
Thus $\varphi^{-1}(B) \in$ *IFITC*(R). □

Definition 4.13. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in R and S, respectively. The direct product of A and B, denoted by $A \times B = (\mu_A \times \mu_B, \nu_A \times \nu_B)$, is an intuitionistic fuzzy set in $R \times S$ such that for all x in R and y in S,$(\mu_A \times \mu_B)(x, y) = T(\mu_A(x), \mu_B(y))$ and $(\nu_A \times \nu_B)(x, y) = C(\nu_A(x), \nu_B(y))$

Proposition 4.14. *If $A_i = (\mu_{A_i}, \nu_{A_i}) \in$ IFITC(R_i) for $i = 1, 2$, then $A_1 \times A_2 \in$ IFITC$(R_1 \times R_2)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in R_1 \times R_2$. Then

(1)

$$(\mu_{A_1} \times \mu_{A_2})((x_1, y_1) - (x_2, y_2)) = (\mu_{A_1} \times \mu_{A_2})(x_1 - x_2, y_1 - y_2)$$

$$= T(\mu_{A_1}(x_1 - x_2), \mu_{A_2}(y_1 - y_2)) \geq T(T(\mu_{A_1}(x_1), \mu_{A_1}(x_2)), T(\mu_{A_2}(y_1), \mu_{A_2}(y_2)))$$

$$= T(T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)), T(\mu_{A_1}(x_2), \mu_{A_2}(y_2))) = T((\mu_{A_1} \times \mu_{A_2})(x_1, y_1), (\mu_{A_1} \times \mu_{A_2})(x_2, y_2)).$$

(2)

$$(\mu_{A_1} \times \mu_{A_2})((x_1, y_1)(x_2, y_2)) = (\mu_{A_1} \times \mu_{A_2})(x_1x_2, y_1y_2)$$

$$= T(\mu_{A_1}(x_1x_2), \mu_{A_2}(y_1y_2)) \geq T(\mu_{A_1}(x_1), \mu_{A_2}(y_1)) = (\mu_{A_1} \times \mu_{A_2})(x_1, y_1).$$

(3)

$$(\nu_{A_1} \times \mu_{A_2})((x_1, y_1) - (x_2, y_2)) = (\nu_{A_1} \times \nu_{A_2})(x_1 - x_2, y_1 - y_2)$$

$$= C(\nu_{A_1}(x_1 - x_2), \nu_{A_2}(y_1 - y_2)) \leq C(C(\nu_{A_1}(x_1), \nu_{A_1}(x_2)), C(\nu_{A_2}(y_1), \nu_{A_2}(y_2)))$$

$$= C(C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)), C(\nu_{A_1}(x_2), \nu_{A_2}(y_2))) = C((\nu_{A_1} \times \nu_{A_2})(x_1, y_1), (\nu_{A_1} \times \nu_{A_2})(x_2, y_2)).$$

(4)

$$(\nu_{A_1} \times \nu_{A_2})((x_1, y_1)(x_2, y_2)) = (\nu_{A_1} \times \nu_{A_2})(x_1x_2, y_1y_2)$$

$$= C(\nu_{A_1}(x_1x_2), \nu_{A_2}(y_1y_2)) \leq C(\nu_{A_1}(x_1), \nu_{A_2}(y_1)) = (\nu_{A_1} \times \nu_{A_2})(x_1, y_1).$$

Then $A_1 \times A_2 \in$ ***IFITC***$(R_1 \times R_2)$. □

Corollary 4.15. *Let $A_i = (\mu_{A_i}, \nu_{A_i}) \in$ **IFITC**(R_i) for $i = 1, 2, ..., n$. Then*

$$A_1 \times A_2 \times ... \times A_n \in \textbf{\textit{IFITC}}(R_1 \times R_2 \times ... \times R_n).$$

Next we will introduce the concept of intuitionistic fuzzy set in R/I.

Definition 4.16. $A = (\mu_A, \nu_A) \in$ ***IFS***(R) and I be an ideal of R.
Define $A^* = (\mu_{A^*}, \nu_{A^*}) \in$ ***IFS***(R/I) by

$$\mu_{A^*}(x + I) = \begin{cases} T(\mu_A(x), \mu_A(i)) & \text{if } x \neq i \\ (1, 0) & \text{if } x = i \end{cases}$$

and

$$\nu_{A^*}(x + I) = \begin{cases} C(\nu_A(x), \nu_A(i)) & \text{if } x \neq i \\ (0, 1) & \text{if } x = i \end{cases}$$

for all $x \in R$ and $i \in I$.

Proposition 4.17. *Let $A = (\mu_A, \nu_A) \in$ **IFITC**(R). If T and C be idempotent, then $A^* = (\mu_{A^*}, \nu_{A^*}) \in$ **IFITC**(R/I).*

Proof. Let $x + I, y + I \in R/I$ and $i \in I$ such that $x \neq i \neq y$.
(1)

$$\mu_{A^*}((x + I) - (y + I)) = \mu_{A^*}((x - y) + I) = T(\mu_A(x - y), \mu_A(i))$$
$$\geq T(T(\mu_A(x), \mu_A(y)), \mu_A(i)) = T(T(\mu_A(x), \mu_A(y)), T(\mu_A(i), \mu_A(i)))$$
$$= T(T(\mu_A(x), \mu_A(i)), T(\mu_A(y), \mu_A(i))) = T(\mu_{A^*}(x + I), \mu_{A^*}(y + I)).$$

(2) $\mu_{A^*}((x+I)(y+I)) = \mu_{A^*}((xy)+I) = T(\mu_A(xy), \mu_A(i)) \geq T(\mu_A(x), \mu_A(i)) = \mu_{A^*}(x+I)$.
(3)

$$\nu_{A^*}((x + I) - (y + I)) = \nu_{A^*}((x - y) + I) = C(\nu_A(x - y), \nu_A(i))$$
$$\leq C(C(\nu_A(x), \nu_A(y)), \nu_A(i)) = C(C(\nu_A(x), \nu_A(y)), C(\nu_A(i), \nu_A(i)))$$
$$= C(C(\nu_A(x), \nu_A(i)), C(\nu_A(y), \nu_A(i))) = C(\nu_{A^*}(x + I), \nu_{A^*}(y + I)).$$

(4) $\nu_{A^*}((x + I)(y + I)) = \nu_{A^*}((xy) + I) = C(\nu_A(xy), \nu_A(i)) \leq C(\nu_A(x), \nu_A(i)) = \nu_{A^*}(x + I)$. □

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