Intuitionistic fuzzy based regular and normal spaces

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Abstract: In this paper, we define the notion of intuitionistic fuzzy based regular and normal spaces. We also study that classical regular and normal spaces are also intuitionistic fuzzy based regular and normal spaces but the converses are not true in general. This notion opens up a new conception of generalization of classical regular and normal spaces. The hereditary and topological properties of intuitionistic fuzzy based regular and normal spaces have been also investigated. Moreover, by setting some examples we show that every intuitionistic fuzzy based regular space as well as intuitionistic fuzzy based normal space need not be T_1 spaces. Finally, it is shown that under some conditions the images and homeomorphic images are preserved in intuitionistic fuzzy based regular and normal spaces.

Keywords: Intuitionistic fuzzy based set, Intuitionistic fuzzy based topological space, Hereditary, Regular space, Normal space.

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1 Introduction

Ever since the invention of topological spaces, many researchers have been paying remarkable contributions in this field. By investigating different properties on classical topological spaces, they also added new notions for its generalization. After the introduction of the fuzzy set by Zadeh [28] the classical topological space is directed to a new dimension namely “Fuzzy Topological Spaces” which is defined by Chang [6]. After that, Atanassov [4, 5] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set. Later, Coker et al. [7–10] defined intuitionistic fuzzy topological spaces, intuitionistic sets and intuitionistic topological spaces. From the onwards, Coker and S. Bayhan [9, 10], Singh and Srivastava [24], S. J. Lee and E. P. Lee [14], Saadati and Park [20], Estiaq Ahmed et al. [1, 2], Islam et al. [12] initiated different studies on intuitionistic fuzzy topological spaces by using intuitionistic fuzzy sets. Besides, many researchers have been working for the generalization of topological spaces and intuitionistic fuzzy based topological spaces [3, 13, 15–20, 24–28]. In 2018, S. Selvanayagi et al. [21–23] studied on normal and weak regular space in intuitionistic fuzzy based topological space. In this paper, we studied various properties of intuitionistic fuzzy based regular and normal spaces, the relation between them and their relations with intuitionistic fuzzy based T_1 space.

2 Notation and preliminaries

In this paper X is noted as a non-empty set, $\mathcal{T}$ is topology, $(X, \mathcal{T})$ is topological space, $\tilde{\mathcal{T}}$ is an intuitionistic fuzzy based topology, $(X, \tilde{\mathcal{T}})$ is intuitionistic fuzzy based topological space, $G = (G_1, G_2)$ and $H = (H_1, H_2)$ are intuitionistic fuzzy based sets, f is a function, $\tilde{\mathcal{T}}_A$ is an intuitionistic fuzzy based relative topology on A where $A \subseteq X$, $\emptyset = (\emptyset, X)$ and $X = (X, \emptyset)$ are also noted as intuitionistic fuzzy based sets.

Definition 2.1 [9]. Suppose X is a non-empty set. An intuitionistic fuzzy based set A and X is an object having the form $A = (X, A_1, A_2)$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of member of A while A_2 is called the set of non-member of A . In this paper, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic fuzzy based set.

Remark 2.1 [9]. Every subset A of a nonempty set X may obviously be regarded as an intuitionistic fuzzy based set having the form $A = (A, A^c)$ where $A^c = X/A$.

Definition 2.2 [9]. Let the intuitionistic fuzzy based sets A and B in X be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$, respectively. Furthermore, let $\{A_j, j \in J\}$ be an arbitrary family of intuitionistic fuzzy based sets in X , where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then:

- $A \subseteq B$ if only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
- $A = B$ if and if $A \subseteq B$ and $B \supseteq A$;
- $\bar{A} = (A_1, A_2)$ denotes the complement of A ;
- $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$;

- e. $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)});$
- f. $\emptyset_{\sim} = (\emptyset, X)$ and $X_{\sim} = (X, \emptyset).$

Definition 2.3 [4]. Let X be a non-empty set. A family $\mathcal{T}$ of intuitionistic fuzzy based sets in X is called an intuitionistic fuzzy based topology on X if the following conditions hold.

- a. $\emptyset_{\sim}, X_{\sim} \in \mathcal{T};$
- b. $A \cap B \in \mathcal{T}$ for all $A, B \in \mathcal{T};$
- c. $\cup A_j \in \mathcal{T}$ for any arbitrary family $\{A_j \in \mathcal{T}, j \in J\}.$

The pair $(X, \mathcal{T})$ is called an intuitionistic fuzzy based topological space (IFTS, in short), members of $\mathcal{T}$ are called intuitionistic fuzzy based open sets (IFOS, in short) in X and their complements are called intuitionistic fuzzy based closed sets (IFCS, in short) in X .

Definition 2.4 [1]. Let $(X, \mathcal{T})$ and (Y, δ) be IFTSs. A function $f: X \rightarrow Y$ is called continuous if $f^{-1}(B) \in \mathcal{T}$ for all $B \in \delta$ and f is called open if $f(A) \in \delta$ for all $A \in \mathcal{T}.$

Definition 2.5 [4]. An intuitionistic fuzzy based topological space $(X, \mathcal{T})$ is called T_1 if for all $x, y \in X$ with $x \neq y$ there exist intuitionistic fuzzy based sets $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{T}$, such that $x \in A_1, y \notin A_1$ and $y \in B_1, x \notin B_1.$

Definition 2.6 [9]. Let X and Y be two nonempty sets and $f: X \rightarrow Y$ be a function. If $A = (A_1, A_2) \subseteq X$, then $f(A)$ is defined as $f(A) = (f(A_1), f(A_2))$ and if $B = (B_1, B_2) \subseteq Y$, then $f^{-1}(B)$ is defined as $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2)).$

3 Regular and normal spaces in intuitionistic fuzzy based topological spaces

In this section, we will define the notions of intuitionistic fuzzy based subspace, intuitionistic fuzzy based regular and normal spaces and examine their properties. By setting some examples we will show that intuitionistic fuzzy based regular and normal spaces are more general than classical regular and normal spaces. We will also find the relation between them. Furthermore, it will be examined here that every intuitionistic fuzzy based regular and normal space need not be intuitionistic fuzzy based T_1 space.

Definition 3.1. Let $(X, \mathcal{T})$ be an intuitionistic fuzzy based topological space and $A \subseteq X$. If $\mathcal{T}_A$ is a topology generated by $\{G \cap (A, \emptyset): G \in \mathcal{T}\}$, then $\mathcal{T}_A$ is also an intuitionistic fuzzy based topology on A and the space $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T}).$

Definition 3.2. An intuitionistic fuzzy based space $(X, \mathcal{T})$ is regular if for any $p \in X$ and $G = (G_1, G_2) \in \mathcal{T}$ with $p \notin G_2$ there exist disjoint open sets $A = (A_1, A_2)$ and $B = (B_1, B_2)$, such that $G^c = (G_2, G_1) \subset A$ and $p \in B_1.$

Definition 3.3. An intuitionistic fuzzy based space $(X, \mathcal{T})$ is normal if $G = (G_1, G_2) \in \mathcal{T}$, $H = (H_1, H_2) \in \mathcal{T}$ with $G_2 \cap H_2 = \emptyset$ then there exist disjoint open sets $A = (A_1, A_2)$ and $B = (B_1, B_2)$ such that $G^c = (G_2, G_1) \subset A$ and $H^c = (H_2, H_1) \subset B.$

Theorem 3.1. *Every classical regular space can be represented as an intuitionistic fuzzy based regular space but the converse is not true.*

Proof. Let $(X, \mathcal{T})$ be a classical regular space. So, according to the axioms of regular space we have, if any $p \in X$ and $G \in \mathcal{T}$ with $p \notin G^c$, then there exist two disjoint open sets A and B such that $G^c \subset A$ and $p \in B$.

Set $E = (G, G^c)$, $M = (A, A^c)$, $N = (B, B^c)$, then it is clear that $p \in X$ and $E = (G, G^c) \in \check{\mathcal{T}}$, $p \notin G^c$, and we get two disjoint sets $M, N \in \check{\mathcal{T}}$ (since A, B disjoint) such that $E^c = (G^c, G) \subset M = (A, A^c)$ (as $G^c \subset A$) and $p \in B$. Hence, correspondingly $(X, \check{\mathcal{T}})$ is an intuitionistic fuzzy based regular space.

But the converse is not true. For example, Consider $X = \{a, b, c\}$, $G = (\{b, c\}, \{a\})$, $A = (\{a, c\}, \{b\})$, $B = (\{a, b\}, \{c\})$, $D = (\{a\}, \{b\})$, $E = (\{b\}, \{c\})$, $F = (\{c\}, \{a\})$. Let $\check{\mathcal{T}}$ be an intuitionistic fuzzy based topology generated by $\{G, A, D, E, F\}$. Therefore $(X, \check{\mathcal{T}})$ is an intuitionistic fuzzy based topological space.

Therefore $(X, \check{\mathcal{T}})$ is an intuitionistic fuzzy based topological space generated by $\{G, A, B, D, E, F\}$. Now, choose $b \in X$, with $b \notin \{a\}$, where $G = (\{b, c\}, \{a\}) \in \check{\mathcal{T}}$ there exist disjoint open sets A and E such that $G^c \subset A$ and $b \in E$. Similarly, we can show that this result is true for the points, a and c .

Thus $(X, \check{\mathcal{T}})$ is an intuitionistic fuzzy based regular space. But this space cannot be represented as classical regular space.

Hence the proof. $\square$

Theorem 3.2. *Every intuitionistic fuzzy based regular subspace is also regular.*

Proof. Let $A \subseteq X$ and $(A, \check{\mathcal{T}}_A)$ be a subspace of the regular space $(X, \check{\mathcal{T}})$. Consider $F = (F_2, F_1)$ is $\check{\mathcal{T}}_A$ -closed set in A then $F^c = (F_1, F_2)$ is $\check{\mathcal{T}}_A$ -open set in A . Consider $p \in A$ with $p \notin F_2$. Since $F^c = (F_1, F_2) \in \check{\mathcal{T}}_A$, then by the definition of subspace $\exists E = (E_1, E_2) \in \check{\mathcal{T}}$ such that $F^c = E \cap (A, \emptyset) \Rightarrow (F_1, F_2) = (E_1, E_2) \cap (A, \emptyset) = (E_1 \cap A, E_2)$, i.e., $F_1 = E_1 \cap A$ and $F_2 = E_2$.

Since $p \notin F_2$ then $p \notin E_2$. So we have a $\check{\mathcal{T}}$ -closed set $E^c = (E_2, E_1)$ such that $p \notin E_2$. Since $(X, \check{\mathcal{T}})$ is an intuitionistic fuzzy based regular space then there exist two disjoint open sets $G = (G_1, G_2)$ and $H = (H_1, H_2)$ such that $E^c \subset G$ and $p \in H_1$. Now, $E^c = (E_2, E_1) \subset (G_1, G_2)$ implies that $E_2 \subset G_1$ and $E_1 \supset G_2$. Furthermore, $E_2 \subset G_1 \Rightarrow F_2 \subset G_1 \Rightarrow F_2 \cap A \subset G_1 \cap A \Rightarrow F_2 = F_2 \cap A \subset G_1 \cap A$. Again, $E_1 \supset G_2 \Rightarrow E_1 \cap A \supset G_2 \cap A \Rightarrow F_1 \supset G_2 \cap A$.

Therefore, $(F_2, F_1) \subset (G_1 \cap A, G_2 \cap A) = G \cap A$, as $F_1 \cap F_2 = \emptyset$ and $G_1 \cap G_2 = \emptyset$.

Again, since $p \in A$ and $p \in H_1$ then $p \in H_1 \cap A$. Further, since G and H are disjoint then $G \cap A$ and $H \cap A$ are also disjoint.

Finally, we have, $p \in A$ with $p \notin F_2$ there exist two disjoint open sets $G \cap A \in \check{\mathcal{T}}_A$ and $H \cap A \in \check{\mathcal{T}}_A$ such that $F \subset G \cap A$ and $p \in H_1 \cap A$. Thus $(A, \check{\mathcal{T}}_A)$ is an intuitionistic fuzzy based regular space. $\square$

Theorem 3.3. *Let $(X, \check{\mathcal{T}})$ and (Y, δ) be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, onto, continuous and open map. If $(X, \check{\mathcal{T}})$ is intuitionistic fuzzy based regular space then (Y, δ) is also intuitionistic fuzzy based regular.*

Proof. Let $(X, \tilde{\tau})$ and (Y, δ) be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be a one-one, onto and open map. Let $(X, \tilde{\tau})$ is intuitionistic fuzzy based regular space, we shall show that (Y, δ) is also intuitionistic fuzzy based regular.

Let $p \in Y$ with $p \notin G_2$, where $G = (G_1, G_2) \in \delta$, then there exists an $a \in X$ such that $f(a) = p$ as f is onto. Again, p is unique for a as f is one-one. Furthermore, $G \in \delta \Rightarrow f^{-1}(G) \in \tilde{\tau}$ as f is continuous. And $p \notin G_2 \Rightarrow a = f^{-1}(p) \notin f^{-1}(G_2)$. Since $(X, \tilde{\tau})$ is intuitionistic fuzzy based regular space, so for any $a \in X$ with $a \notin f^{-1}(G_2)$ where $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2)) \in \tilde{\tau}$, then there exist two disjoint open sets $A = (A_1, A_2)$, $B = (B_1, B_2) \in \tilde{\tau}$ such that $f^{-1}(G^c) \subset A$ and $a \in B_1$.

Since $A, B \in \tilde{\tau} \Rightarrow f(A), f(B) \in \delta$ as f is open. Now we have, $f(A) = (f(A_1), f(A_2))$, $f(B) = (f(B_1), f(B_2))$. Furthermore, since $a \in B_1 \Rightarrow f(a) = p \in f(B_1)$ and $f^{-1}(G^c) \subset A \Rightarrow f f^{-1}(G^c) \subset f(A) \Rightarrow G^c \subset f(A)$. Furthermore, $f(A), f(B)$ are disjoint as A, B are disjoint.

Finally, we get, for any $p \in Y$ with $p \notin G_2$ there exist two disjoint open sets $f(A), f(B) \in \delta$ such that $f(G^c) \subset f(A)$ and $p \in f(B_1)$. Thus (Y, δ) is an intuitionistic fuzzy based regular space. $\square$

Theorem 3.4. Let $(X, \tilde{\tau})$ and (Y, δ) be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, onto, continuous and open map. If (Y, δ) is intuitionistic fuzzy based regular space then $(X, \tilde{\tau})$ is also intuitionistic fuzzy based regular.

Proof. Let $(X, \tilde{\tau})$ and (Y, δ) be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, onto and open map. Let (Y, δ) be intuitionistic fuzzy based regular space, we shall show that $(X, \tilde{\tau})$ is also intuitionistic fuzzy based regular.

Let $p \in X$ with $p \notin G_2$, where $G = (G_1, G_2) \in \tilde{\tau}$ then there exists an $a \in Y$ such that $a = f(p)$, for unique p as f is one-one. Furthermore, $G \in \tilde{\tau} \Rightarrow f(G) \in \delta$ as f is open. And $p \notin G_2 \Rightarrow a = f(p) \notin f(G_2)$. Now, since (Y, δ) is intuitionistic fuzzy based regular space, so for any $a \in Y$ with $a \notin f(G_2)$, where $G = (G_1, G_2) \in \delta$ then there exist two disjoint open sets $A = (A_1, A_2)$, $B = (B_1, B_2) \in \delta$ such that $f(G^c) \subset A$ and $a \in B_1$. Again since $A, B \in \delta \Rightarrow f^{-1}(A), f^{-1}(B) \in \tilde{\tau}$, as f is continuous. Now we have, $f^{-1}(A) = (f^{-1}(A_1), f^{-1}(A_2))$, $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.

Furthermore, $f(p) \in B_1 \Rightarrow f^{-1}f(p) \in f^{-1}(B_1) \Rightarrow p \in f^{-1}(B_1)$. And $f(G^c) \subset A \Rightarrow f^{-1}f(G^c) \subset f^{-1}(A) \Rightarrow G^c \subset f^{-1}(A)$.

Furthermore, $f^{-1}(A), f^{-1}(B)$ are disjoint as A, B are disjoint.

Finally, we get, for any $p \in X$ with $p \notin G_2$ there exist two disjoint open sets $f^{-1}(A), f^{-1}(B) \in \tilde{\tau}$ such that $G^c \subset f^{-1}(A)$ and $p \in f^{-1}(B_1)$. Thus $(X, \tilde{\tau})$ is an intuitionistic fuzzy based regular space. $\square$

Theorem 3.5. Intuitionistic fuzzy based regular space need not be intuitionistic fuzzy based normal space.

Proof. An intuitionistic fuzzy based regular space need not be an intuitionistic fuzzy based normal space. For example, consider $X = \{a, b, c\}$, $G = (\{b, c\}, \{a\})$, $A = (\{a, c\}, \{b\})$,

$B = (\{a, b\}, \{c\})$, $D = (\{a\}, \{b\})$, $E = (\{b\}, \{c\})$, $F = (\{c\}, \{a\})$. Let $\tilde{\tau}$ be an intuitionistic fuzzy based topology generated by $\{G, A, D, E, F\}$. Therefore $(X, \tilde{\tau})$ is an intuitionistic fuzzy based regular space (see Theorem 3.1). But $(X, \tilde{\tau})$ is not an intuitionistic fuzzy based normal space. Because for two disjoint closed subsets G^c and B^c there are no disjoint open sets so that G^c and B^c hold the normal space's axioms. Hence the proof. $\square$

Theorem 3.6. *Every intuitionistic fuzzy based regular space need not be intuitionistic fuzzy based T_1 space.*

Proof. An intuitionistic fuzzy based regular space need not be an intuitionistic fuzzy based T_1 space. For example, consider $X = \{a, b, c, p\}$, $G = (\{b, c\}, \{a\})$, $A = (\{a, c\}, \{b\})$, $B = (\{a, b\}, \{c\})$, $D = (\{a\}, \{b\})$, $E = (\{b\}, \{c\})$, $F = (\{c\}, \{a\})$, $M = (\{b, p\}, \{a, c\})$, $N = (\{b, p\}, \{a\})$. Let $\tilde{\tau}$ be an intuitionistic fuzzy based topology generated by $\{G, A, D, E, F, M, N\}$. Therefore, $(X, \tilde{\tau})$ is an intuitionistic fuzzy based regular space. Now, choose $b \in X$, with $b \notin \{a\}$, where $G = (\{b, c\}, \{a\}) \in \tilde{\tau}$ there exist disjoint open sets A and E such that $G^c \subset A$ and $b \in E$. Similarly we can show that this result is true for the points, a, c and p . But the space is not an intuitionistic fuzzy based T_1 space. Because for the points b, p there are no open sets such that b and p can belong to the sets separately. $\square$

Theorem 3.7. *Every classical normal space can be represented as an intuitionistic fuzzy based normal space but the converse is not true.*

Proof. Let (X, τ) be classical normal space. So according to the axioms of normal space we have if for any $G, H \in \tau$ with $G^c \cap H^c = \emptyset$, then there exist disjoint open sets A and B such that $G^c \subset A$, $H^c \subset B$.

Set $E = (G, G^c)$, $F = (H, H^c)$, $M = (A, A^c)$, $N = (B, B^c)$, then it is clear that E^c, F^c are disjoint (since G^c, H^c are disjoint), and we get two disjoint sets $M, N \in \tilde{\tau}$ (since A, B disjoint) such that $E^c = (G^c, G) \subset M = (A, A^c)$ (as $G^c \subset A$) and $F^c = (H^c, H) \subset N = (B, B^c)$ (as $H^c \subset B$). Hence, corresponding $(X, \tilde{\tau})$ is intuitionistic fuzzy based normal space. But the converse is not true. For example, consider $X = \{a, b, c, d, e, f, p, q\}$, $A = (\{c, d, p\}, \{a, b\})$, $B = (\{a, b, f\}, \{c, d\})$, $G = (\{a, b, e\}, \{c, d\})$, $H = (\{c, d, q\}, \{a\})$. Let $\tilde{\tau}$ be an intuitionistic fuzzy based topology generated by $\{A, B, G, H\}$. Therefore, $(X, \tilde{\tau})$ is an intuitionistic fuzzy based topological space. Thus $(X, \tilde{\tau})$ is also intuitionistic fuzzy based normal space because, for any two disjoint closed sets $A^c = (\{a, b\}, \{c, d, p\})$ and $B^c = (\{c, d\}, \{a, b, f\})$ there exist two disjoint open sets $G = (\{a, b, e\}, \{c, d\})$, $H = (\{c, d, q\}, \{a\}) \in \tilde{\tau}$ such that $A^c \subset G$ and $B^c \subset H$. This result is true for any other pair of disjoint closed sets. Thus $(X, \tilde{\tau})$ is an intuitionistic fuzzy based normal space. But this space cannot be represented as classical normal space. $\square$

Theorem 3.8. *Let $(X, \tilde{\tau})$ be an intuitionistic fuzzy based normal space then $(A, \tilde{\tau}_A)$ is also intuitionistic fuzzy based normal.*

Proof. Let $(A, \tilde{\tau}_A)$ be a subspace of the intuitionistic fuzzy based regular space $(X, \tilde{\tau})$. Consider $E = (E_2, E_1)$ and $F = (F_2, F_1)$ are disjoint $\tilde{\tau}_A$ -closed set in A then $E^c = (E_1, E_2)$, $F^c = (F_1, F_2)$ are $\tilde{\tau}_A$ -open sets in A .

By the definition of $\check{\tau}_A$, we have: $M = (M_1, M_2) \in \check{\tau}$ such that $E^c = M \cap (A, \emptyset)$

$$\begin{aligned} \Rightarrow (E_1, E_2) &= (M_1, M_2) \cap (A, \emptyset) \\ &= (M_1 \cap A, M_2). \end{aligned}$$

Therefore, $E_1 = M_1 \cap A$ and $E_2 = M_2$, $N = (N_1, N_2) \in \check{\tau}$ such that $F^c = N \cap (A, \emptyset)$

$$\begin{aligned} \Rightarrow (F_1, F_2) &= (N_1, N_2) \cap (A, \emptyset) \\ &= (N_1 \cap A, N_2), \end{aligned}$$

i.e., $F_1 = N_1 \cap A$ and $F_2 = N_2$.

Now $M^c \cap N^c = (M_2, M_1) \cap (N_2, N_1) = (M_2 \cap N_2, M_1 \cup N_1) = (E_2 \cap F_2, M_1 \cup N_1) = (\emptyset, M_1 \cup N_1)$ as E and F are disjoint, so M^c and N^c are disjoint.

We have M^c and N^c disjoint $\check{\tau}$ -closed sets. Since $(X, \check{\tau})$ is an intuitionistic fuzzy based normal space then there exist two disjoint $\check{\tau}$ -open sets $G = (G_1, G_2)$ and $H = (H_1, H_2)$ such that $M^c \subset G$ and $N^c \subset H$. Now, $M^c = (M_2, M_1) \subset (G_1, G_2)$ implies that $M_2 \subset G_1$ and $M_1 \supset G_2$. Furthermore, $M_2 \subset G_1$

$$\begin{aligned} \Rightarrow E_2 &\subset G_1 \\ \Rightarrow E_2 \cap A &\subset G_1 \cap A \\ \Rightarrow E_2 &= E_2 \cap A \subset G_1 \cap A. \end{aligned}$$

Again, $M_1 \supset G_2$

$$\begin{aligned} \Rightarrow M_1 \cap A &\supset G_2 \cap A \\ \Rightarrow E_1 &\supset G_2 \cap A. \end{aligned}$$

Therefore, $(E_2, E_1) = E \subset (G_1 \cap A, G_2 \cap A) = G \cap A$, as $E_1 \cap E_2 = \emptyset$ and $G_1 \cap G_2 = \emptyset$. Similarly, we can show that $(F_2, F_1) = F \subset (H_1 \cap A, H_2 \cap A) = H \cap A$, as $F_1 \cap F_2 = \emptyset$ and $H_1 \cap H_2 = \emptyset$. Further since G and H are disjoint then $G \cap A$ and $H \cap A$ are also disjoint.

Finally, we have, for any two disjoint closed sets $E = (E_2, E_1)$ and $F = (F_2, F_1)$, where $E^c = (E_1, E_2) \in \check{\tau}_A$ and $F^c = (F_1, F_2) \in \check{\tau}_A$ then there exist two disjoint open sets $G \cap A \in \check{\tau}_A$ and $H \cap A \in \check{\tau}_A$ such that $E \subset G \cap A$ and $F \subset H \cap A$. Thus $(A, \check{\tau}_A)$ is an intuitionistic fuzzy based normal space. $\square$

Theorem 3.9. Let $(X, \check{\tau})$ and $(Y, \check{\delta})$ be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, onto, continuous and open map. If $(X, \check{\tau})$ is intuitionistic fuzzy based normal space then $(Y, \check{\delta})$ is also intuitionistic fuzzy based normal.

Proof. Let $(X, \check{\tau})$ and $(Y, \check{\delta})$ be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be a one-one, onto and open map. Let $(X, \check{\tau})$ is intuitionistic fuzzy based normal space, we shall show that $(Y, \check{\delta})$ is also intuitionistic fuzzy based normal.

Let $G^c = (G_2, G_1)$, $H^c = (H_2, H_1)$ are disjoint closed set $\check{\delta}$ closed set, i.e., $G = (G_1, G_2)$, $H = (H_1, H_2) \in \check{\delta}$ then $f^{-1}(G), f^{-1}(H) \in \check{\tau}$, f is continuous. Now, $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2))$, $f^{-1}(H) = (f^{-1}(H_1), f^{-1}(H_2))$, i.e., $f^{-1}(G^c) = (f^{-1}(G))^c = (f^{-1}(G_2), f^{-1}(G_1))$ and $f^{-1}(H^c) = (f^{-1}(H))^c = (f^{-1}(H_2), f^{-1}(H_1))$ are disjoint $\check{\tau}$ closed set (as G^c and H^c are disjoint, again f is one-one and onto).

Since $(X, \tilde{\tau})$ is normal space, so for any two disjoint closed sets $f^{-1}(G^c), f^{-1}(H^c)$, where $f^{-1}(G), f^{-1}(H) \in \tilde{\tau}$ there exist two disjoint open sets $A = (A_1, A_2), B = (B_1, B_2) \in \tilde{\tau}$ such that $f^{-1}(G^c) \subset A$ and $f^{-1}(H^c) \subset B$. Since $A, B \in \tilde{\tau} \Rightarrow f(A), f(B) \in \check{\delta}$ as f is open. Now we have, $f(A) = (f(A_1), f(A_2)), f(B) = (f(B_1), f(B_2))$. Furthermore, $f^{-1}(G^c) \subset A$

$$\begin{aligned} &\Rightarrow f f^{-1}(G^c) \subset f(A) \\ &\Rightarrow G^c \subset f(A). \end{aligned}$$

And $f^{-1}(H^c) \subset B$

$$\begin{aligned} &\Rightarrow f f^{-1}(H^c) \subset f(B) \\ &\Rightarrow H^c \subset f(B). \end{aligned}$$

Again, $f(A), f(B)$ are disjoint as A, B are disjoint, f is one-one.

Finally, we get, for any two disjoint closed sets G^c, H^c where $G = (G_1, G_2), H = (H_1, H_2) \in \check{\delta}$ there exist two disjoint open sets $f(A) \in \check{\delta}$ and $f(B) \in \check{\delta}$ such that $G^c \subset f(A)$ and $H^c \subset f(B)$. Thus $(Y, \check{\delta})$ is an intuitionistic fuzzy based normal space. $\square$

Theorem 3.10. Let $(X, \tilde{\tau})$ and $(Y, \check{\delta})$ be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, onto, continuous and closed map. If $(Y, \check{\delta})$ is intuitionistic fuzzy based normal space then $(X, \tilde{\tau})$ is also intuitionistic fuzzy based normal.

Proof. Let $(X, \tilde{\tau})$ and $(Y, \check{\delta})$ be two intuitionistic fuzzy based topological spaces and $f: X \rightarrow Y$ be one-one, continuous and open map. Let $(Y, \check{\delta})$ is intuitionistic fuzzy based normal space, we shall show that $(X, \tilde{\tau})$ is also intuitionistic fuzzy based normal.

Let $p, q \in X$ with $p \neq q \Rightarrow f(p), f(q) \in Y$ with $f(p) \neq f(q)$ as f is one-one.

Let $G^c = (G_2, G_1), H^c = (H_2, H_1)$ are disjoint $\tilde{\tau}$ closed set, i.e., $G = (G_1, G_2), H = (H_1, H_2) \in \tilde{\tau}$ then $f(G^c), f(H^c)$, are disjoint $\check{\delta}$ -closed set, as f is closed, one-one and onto. Now we have, $f(G^c) = (f(G_2), f(G_1)), f(H^c) = (f(H_2), f(H_1))$. Since $(Y, \check{\delta})$ is normal space, so for any two disjoint $\check{\delta}$ closed sets $f(G^c)$ and $f(H^c)$, there exist two disjoint open sets $A = (A_1, A_2), B = (B_1, B_2) \in \check{\delta}$ such that $f(G^c) \subset A$ and $f(H^c) \subset B$.

Again since $A, B \in \check{\delta} \Rightarrow f^{-1}(A), f^{-1}(B) \in \tilde{\tau}$, as f is continuous. Now we have, $f^{-1}(A) = (f^{-1}(A_1), f^{-1}(A_2)), f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.

Furthermore, $f(G^c) \subset A$

$$\begin{aligned} &\Rightarrow f^{-1}f(G^c) \subset f^{-1}(A) \\ &\Rightarrow G^c \subset f^{-1}(A). \end{aligned}$$

And $f(H^c) \subset B$

$$\begin{aligned} &\Rightarrow f^{-1}f(H^c) \subset f^{-1}(B) \\ &\Rightarrow H^c \subset f^{-1}(B). \end{aligned}$$

Again, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint as A and B are disjoint.

Finally, we get, for any two disjoint closed sets G^c, H^c , where $G = (G_1, G_2), H = (H_1, H_2) \in \tilde{\tau}$ then there exist two disjoint open sets $f^{-1}(A) \in \check{\delta}$ and $f^{-1}(B) \in \check{\delta}$ such that $G^c \subset f^{-1}(A)$ and $H^c \subset f^{-1}(B)$.

Thus $(X, \tilde{\tau})$ is an intuitionistic fuzzy based normal space. $\square$

Theorem 3.11. *Intuitionistic fuzzy based normal space need not be intuitionistic fuzzy based regular space.*

Proof. An intuitionistic fuzzy based regular space need not be an intuitionistic fuzzy based normal space. For example, consider $X = \{a, b, c, d, e, f, p, q\}$, $A = (\{c, d, p\}, \{a, b\})$, $B = (\{a, b, f\}, \{c, d\})$, $G = (\{a, b, e\}, \{c, d\})$, $H = (\{c, d, q\}, \{a\})$. Let $\tilde{\tau}$ be an intuitionistic fuzzy based topology generated by $\{A, B, G, H\}$. Therefore $(X, \tilde{\tau})$ is an intuitionistic fuzzy based topological space. This $(X, \tilde{\tau})$ is also intuitionistic fuzzy based normal space (see Theorem 3.7). But the space is not an intuitionistic fuzzy based regular space. Because for any point $e \in X$ with $e \notin \{a, b\}$, where $A^c = (\{a, b\}, \{c, d, p\})$ then there are no two disjoint open sets such that they hold the intuitionistic fuzzy based regular space's axioms. Hence the proof. $\square$

Theorem 3.12. *Every intuitionistic fuzzy based normal space need not be intuitionistic fuzzy based T_1 space.*

Proof. An intuitionistic fuzzy based normal space need not be an intuitionistic fuzzy based T_1 space. For example, consider $X = \{a, b, c, d, e, f, p, q\}$, $A = (\{c, d, p\}, \{a, b\})$, $B = (\{a, b, f\}, \{c, d\})$, $G = (\{a, b, e\}, \{c, d\})$, $H = (\{c, d, q\}, \{a\})$. Let $\tilde{\tau}$ be an intuitionistic fuzzy based topology generated by $\{A, B, G, H\}$. Therefore $(X, \tilde{\tau})$ is an intuitionistic fuzzy based topological space. This $(X, \tilde{\tau})$ is also intuitionistic fuzzy based normal space (see Theorem 3.7). But the space is not an intuitionistic fuzzy based T_1 space. Because for the points a, b there are no open sets such that b and c can belong to the sets separately. $\square$

4 Conclusions

In this paper, we have observed that intuitionistic fuzzy based regular and normal spaces are more general than the classical regular and normal spaces. It is also showed that intuitionistic fuzzy based regular and normal spaces satisfy hereditary and topological property. Moreover, they are preserved under one-one and open mapping. Hopefully this study will open up a new approach for further investigation.

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