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New equalities on the intuitionistic fuzzy operators and operations

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Abstract: In this study, properties of the modal operators $\Box$, $\Diamond$ introduced on intuitionistic fuzzy sets and some intuitionistic fuzzy operations ($\longrightarrow$, @, $\cup$,$\cap$, \$, $\#$) have been investigated. New equalities have been obtained and proved.
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1 Introduction

Definition 1 (cf. [1]). *Let X be a nonempty set. A fuzzy set A drawn from X is defined as*

$$A = \{\langle x, \mu_A(x)\rangle |\, x \in X\},$$

where $\mu_A(x) : X \to [0,1]$ is the membership function of the fuzzy set A.

Definition 2 ([2, 3]). *Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\, x \in X\},$$

where the functions

$$\mu_A(x), \nu_A(x) : X \to [0,1]$$

define respectively, the degree of membership and degree of nonmembership of the element $x \in X$, to the set A, which is a subset of X, and for every element $x \in X$,

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

Furthermore, we have

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

called the intuitionistic fuzzy set index or hesitation on margin of x in A. $\pi_A(x)$ is degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ i.e.,

$$\pi_A : X \to [0,1]$$

for every $x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 3 ([2, 3]). *Let A and $B \in X$ be IFS. For every two IFS's A and B the following operations and relations are valid.*

$$\begin{aligned}
&A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\; x \in X\} \\
&B = \{\langle x, \mu_B(x), \nu_B(x)\rangle |\; x \in X\} \\
&A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle |\; x \in X\} \\
&A@B = \{\langle x, \tfrac{\mu_A(x)+\mu_B(x)}{2}, \tfrac{\nu_A(x)+\nu_B(x)}{2}\rangle\; x \in X\} \\
&A \to B = \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x)\rangle |\; x \in X\} \\
&A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle |\; x \in X\} \\
&A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle |\; x \in X\} \\
&A \oplus B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle |\; x \in X\} \\
&A \otimes B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle |\; x \in X\} \\
&A \,\$\, B = \{\langle x, \sqrt{\mu_A(x).\mu_B(x)}, \sqrt{\nu_A(x).\nu_B(x)}\rangle |\; x \in X\} \\
&A \,\#\, B = \{\langle x, \frac{2\mu_A(x).\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x).\nu_B(x)}{\nu_A(x)+\nu_B(x)}\rangle |\; x \in X\}
\end{aligned}$$

Definition 4 ([2,3]). *Let X be nonempty. If A is an IFS drawn from X, then;*

(i) $\Box A = \{\langle x, \mu_A(x)\rangle |\; x \in X\} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle |\; x \in X\}$
(ii) $\Diamond A = \{\langle x, 1 - \nu_A(x)\rangle |\; x \in X\} = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle |\; x \in X\}$

Theorem 1 ([2,3]). *Let X be nonempty. For every IFS A in X;*

(a) $\Box\Box A = \Box A$
(b) $\Box\Diamond A = \Diamond A$
(c) $\Diamond\Box A = \Box A$
(d) $\Diamond\Diamond A = \Diamond A$

Theorem 2 ([2,3]). *Let X be nonempty. For every IFS A and B in X:*

$$\Box A \;@\; \Box B = \Box(A@B)$$

Theorem 3 ([2,3]). *Let X be nonempty. For every IFS A and B in X:*

$$\Diamond A \;@\; \Diamond B = \Diamond(A@B)$$

Theorem 4. *Let X be nonempty. For every IFS A and B in X:*

$$(\Box A \oplus \Box B) \;@\; (\Box A \otimes \Box B) = \Box A \;@\; \Box B$$

Proof.

$$(\Box A \oplus \Box B) = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), (1-\mu_A(x)).(1-\mu_B(x))\rangle | \ x \in X\}$$
$$(\Box A \otimes \Box B) = \{\langle x, \mu_A(x).\mu_B(x), 1-\mu_A(x) + 1 - \mu_B(x) - (1-\mu_A(x)).(1-\mu_B(x))\rangle | \ x \in X\}$$
$$(\Box A \oplus \Box B) \ @ \ (\Box A \otimes \Box B) = \left\{\langle \ x, \frac{\mu_A(x)+\mu_B(x)}{2}, 1 - \frac{\mu_A(x)+\mu_B(x)}{2}\rangle | \ x \in X\right\}$$
$$(\Box A \oplus \Box B) \ @ \ (\Box A \otimes \Box B) = \Box A \ @ \ \Box B$$
□

Theorem 5. *Let X be nonempty. For every IFS A and B in X;*

$$(\Diamond A \oplus \Diamond B) \ @ \ (\Diamond A \otimes \Diamond B) = \Diamond A \ @ \ \Diamond B$$

Proof.

$$(\Diamond A \oplus \Diamond B) = \{\langle x, 1-\nu_A(x) + 1 - \nu_B(x) - (1-\nu_A(x)).(1-\nu_B(x)), \nu_A(x).\nu_B(x)\rangle | \ x \in X\}$$
$$(\Diamond A \otimes \Diamond B) = \{\langle x, (1-\nu_A(x)).(1-\nu_B(x)), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle | \ x \in X\}$$
$$(\Diamond A \oplus \Diamond B) \ @ \ (\Diamond A \otimes \Diamond B) = \left\{\langle \ x, \frac{2-\nu_B(x)-\nu_A(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle | \ x \in X\right\}$$
$$(\Diamond A \oplus \Diamond B) \ @ \ (\Diamond A \otimes \Diamond B) = \Diamond A \ @ \ \Diamond B$$
□

Theorem 6. *Let X be nonempty. For every IFS A and B in X;*

$$[(\Box A \ @ \ \Box B) \ \$ \ (\Box A \ \# \ \Box B)] = \Box A \ \$ \ \Box B$$

Proof.

$$(\Box A \ @ \ \Box B) = \left\{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{1-\mu_A(x)+1-\mu_B(x)}{2}\rangle | \ x \in X\right\}$$
$$(\Box A \ \# \ \Box B) = \left\{\langle x, \frac{2\mu_A(x).\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2.(1-\mu_A(x)).(1-\mu_B(x))}{1-\mu_A(x)+1-\mu_B(x)}\rangle | \ x \in X\right\}$$
$$(\Box A \ @ \ \Box B) \ \$ \ (\Box A \ \# \ \Box B) = \left\{\langle x, \sqrt{\mu_A(x).\mu_B(x)}, \sqrt{(1-\mu_A(x)).(1-\mu_B(x))}\rangle | \ x \in X\right\}$$
$$[(\Box A \ @ \ \Box B) \ \$ \ (\Box A \ \# \ \Box B)] = \Box A \ \$ \ \Box B$$

□

Theorem 7. *Let X be nonempty. For every IFS A and B in X:*

$$[(\Diamond A \ @ \ \Diamond B) \ \$ \ (\Diamond A \ \# \ \Diamond B)] = \Diamond A \ \$ \ \Diamond B$$

Proof.

$$(\Diamond A \ @ \ \Diamond B) = \left\{\langle x, \frac{1-\nu_A(x)+1-\nu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle | \ x \in X\right\}$$
$$(\Diamond A \ \# \ \Diamond B) = \left\{\langle x, \frac{2.(1-\nu_A(x)).(1-\nu_B(x))}{1-\nu_A(x)+1-\nu_B(x)}, \frac{2.\nu_A(x).\nu_B(x)}{\nu_A(x)+\nu_B(x)}\rangle | \ x \in X\right\}$$
$$(\Diamond A \ @ \ \Diamond B) \ \$ \ (\Diamond A \ \# \ \Diamond B) = \left\{\langle x, \sqrt{(1-\nu_A(x)).(1-\nu_B(x))}, \sqrt{\nu_A(x).\nu_B(x)}\rangle | \ x \in X\right\}$$
$$[(\Diamond A \ @ \ \Diamond B) \ \$ \ (\Diamond A \ \# \ \Diamond B)] = \Diamond A \ \$ \ \Diamond B$$
□

Theorem 8. *Let X be nonempty. For every IFS A and B in X:*

$$\Box[(\Diamond A \,@\, \Diamond B)^c] = [\Diamond(A \,@\, B)]^c$$

Proof.

$$(\Diamond A \,@\, \Diamond B) = \left\{\langle x, 1 - \tfrac{\nu_A(x)+\nu_B(x)}{2}, \tfrac{\nu_A(x)+\nu_B(x)}{2}\rangle | \; x \in X\right\}$$
$$(\Diamond A \,@\, \Diamond B)^c = \left\{\langle x, \tfrac{\nu_A(x)+\nu_B(x)}{2}, 1 - \tfrac{\nu_A(x)+\nu_B(x)}{2}\rangle | \; x \in X\right\}$$
$$\Box[(\Diamond A \,@\, \Diamond B)^c] = \left\{\langle x, \tfrac{\nu_A(x)+\nu_B(x)}{2}, 1 - \tfrac{\nu_A(x)+\nu_B(x)}{2}\rangle | \; x \in X\right\}$$
$$\Box[(\Diamond A \,@\, \Diamond B)^c] = [\Diamond(A \,@\, B)]^c \quad \Box$$

Theorem 9. *Let X be nonempty. For every IFS A and B in X:*

$$[(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)] \cup (\Box A)^c = (\Box A)^c$$

Proof.

$$(\Box A \oplus \Diamond B)$$
$$= \{\langle x, \mu_A(x) + 1 - \nu_B(x) - \mu_A(x).(1 - \nu_B(x)), (1 - \mu_A(x)).\nu_B(x)\rangle \mid x \in X\}$$

$$(\Box A \oplus \Diamond B)$$
$$= \{\langle x, \mu_A(x) + 1 - \nu_B(x) - \mu_A(x) + \mu_A(x).\nu_B(x), \nu_B(x) - \mu_A(x).\nu_B(x)\rangle \mid x \in X\}$$

$$(\Box A \oplus \Diamond B)^c$$
$$= \{\langle x, \nu_B(x) - \mu_A(x).\nu_B(x), \mu_A(x) + 1 - \nu_B(x) - \mu_A(x) + \mu_A(x).\nu_B(x)\rangle \mid x \in X\}$$

$$(\Box A)^c = \{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle \mid x \in X\}$$

$$((\Box A)^c \otimes \Diamond B)$$
$$= \{\langle x, (1 - \mu_A(x)).(1 - \nu_B(x)), \mu_A(x) + \nu_B(x) - \mu_A(x).\nu_B(x)\rangle \mid x \in X\}$$

$$((\Box A)^c \otimes \Diamond B)$$
$$= \{\langle x, 1 - \mu_A(x) - \nu_B(x) + \mu_A(x).\nu_B(x), \mu_A(x) + \nu_B(x) - \mu_A(x).\nu_B(x)\rangle \mid x \in X\}$$

$$(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)$$
$$= \{\langle \tfrac{\nu_B(x)-\mu_A(x).\nu_B(x)+1-\mu_A(x)-\nu_B(x)+\mu_A(x).\nu_B(x)}{2}$$
$$, \tfrac{\mu_A(x)+1-\nu_B(x)-\mu_A(x)+\mu_A(x).\nu_B(x)+\mu_A(x)+\nu_B(x)-\mu_A(x).\nu_B(x)}{2}\rangle | \; x \in X\}$$

$$(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)$$
$$= \left\{\left\langle x, \tfrac{1-\mu_A(x)}{2}, \tfrac{1+\mu_A(x)}{2}\right\rangle \mid x \in X\right\}$$

$$(\Box A)^c = \{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle \mid x \in X\}$$

$$[(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)] \cup (\Box A)^c$$
$$= \left\{\left\langle x, \max(\tfrac{1-\mu_A(x)}{2}, 1 - \mu_A(x)), \min(\tfrac{\mu_A(x)+1}{2}, \mu_A(x)\right\rangle \mid x \in X\right\}$$

$$[(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)] \cup (\Box A)^c$$
$$= \{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle \mid x \in X\}$$

$$[(\Box A \oplus \Diamond B)^c \,@\, ((\Box A)^c \otimes \Diamond B)] \cup (\Box A)^c = (\Box A)^c \quad \Box$$

References

[1] Zadeh, L. A. (1965). Fuzzy Sets. *Information and Control*, 8, 338–353.

[2] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[3] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets*, Springer, Heidelberg.