**Notes on Intuitionistic Fuzzy Sets**
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# Intuitionistic fuzzy conjunctions and disjunctions from third type

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**Abstract:** The purpose of this article is the calculation of the intuitionistic fuzzy conjunctions and disjunctions with respect to certain IF implications and negations. With constructed operations we consider algebraic structures such monoids, lattices and study their properties.
**Keywords:** Intuitionistic fuzzy conjunctions and disjunctions.
**AMS Classification:** 03E72.

## 1 Introduction

In a series of papers, 185 intuitionistic fuzzy implications were defined. They were collected in [2]. All notations used in the present paper are from [1, 2]. Our paper is the third part of our research – the first two ones were published in [3, 4].

In the present paper, we describe the intuitionistic fuzzy conjunctions and disjunctions that are obtained by formulas:

$$\langle a, b\rangle \vee \langle c, d\rangle = \neg\langle a, b\rangle \rightarrow \langle c, d\rangle$$

and

$$\langle a, b\rangle \wedge \langle c, d\rangle = \neg(\langle a, b\rangle \rightarrow \neg\langle c, d\rangle),$$

where for formula $A$ with an intuitionistic fuzzy evaluation $V(A) = \langle a, b\rangle$,

$$V(\neg A) = \langle b, a\rangle.$$

# 2 Construction of conjunctions and disjunctions

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| $\vee_{3,1}$ | $\langle \max(a, \min(b, c)), \min(b, d)\rangle$ |
| $\wedge_{3,1}$<br>$\vee_{3,2}$ | $\langle \min(a, c), \max(b, \min(a, d))\rangle$<br>$\langle \overline{\mathrm{sg}}(b - c), d\mathrm{sg}(b - c)\rangle$ |
| $\wedge_{3,2}$ | $\langle d\mathrm{sg}(a - d), \overline{\mathrm{sg}}(a - d)\rangle$ |
| $\vee_{3,3}$ | $\langle 1 - (1 - c)\mathrm{sg}(b - c)), d\mathrm{sg}(b - c)\rangle$ |
| $\wedge_{3,3}$ | $\langle c\mathrm{sg}(a - d), 1 - (1 - d)\mathrm{sg}(a - d))\rangle$ |
| $\vee_{3,4}$ | $\langle \max(a, c), \min(b, d)\rangle$ |
| $\wedge_{3,4}$ | $\langle \min(a, c), \max(b, d)\rangle$ |
| $\vee_{3,5}$ | $\langle \min(1, a + c), \max(0, b + d - 1)\rangle$ |
| $\wedge_{3,5}$ | $\langle \max(0, a + c - 1), \min(1, d + b)\rangle$ |
| $\vee_{3,6}$ | $\langle a + b.c, b.d\rangle$ |
| $\wedge_{3,6}$ | $\langle ac, b + ad\rangle$ |
| $\vee_{3,7}$ | $\langle \min(\max(a, b), \max(a, c), \max(c, d)),$<br>$\max(\min(c, d) \min(d, b), \min(a, b))\rangle$ |
| $\wedge_{3,7}$ | $\langle \max(\min(c, d) \min(a, c), \min(a, b)),$<br>$\min(\max(a, b), \max(b, d), \max(c, d))\rangle$ |
| $\vee_{3,8}$ | $\langle 1 - (1 - \min(a, c))\mathrm{sg}(b - c), \max(b, d)\mathrm{sg}(b - c), \mathrm{sg}(d - a)\rangle$ |
| $\wedge_{3,8}$ | $\langle 1 - (1 - \min(b, d))\mathrm{sg}(a - d), \max(a, c)\mathrm{sg}(a - d), \mathrm{sg}(c - b)\rangle$ |
| $\vee_{3,9}$ | $\langle a + b^2c, ab + b^2d\rangle$ |
| $\wedge_{3,9}$ | $\langle ab + a^2c, b + a^2d\rangle$ |
| $\vee_{3,10}$ | $\langle \min(c, \overline{\mathrm{sg}}(1 - b)) + \mathrm{sg}(1 - b)(\overline{\mathrm{sg}}(1 - c) + \min(a, \mathrm{sg}(1 - c))),$<br>$\min(d, \overline{\mathrm{sg}}(1 - b)) + \min(b, \mathrm{sg}(1 - b), \mathrm{sg}(1 - c))\rangle$ |
| $\wedge_{3,10}$ | $\langle c.\overline{\mathrm{sg}}(1 - a) + a.\mathrm{sg}(1 - a).\mathrm{sg}(1 - d),$<br>$d.\overline{\mathrm{sg}}(1 - a) + \mathrm{sg}(1 - a).(\overline{\mathrm{sg}}(1 - d) + b.\mathrm{sg}(1 - d))\rangle$ |
| $\vee_{3,11}$ | $\langle 1 - (1 - c)\mathrm{sg}(b - c), d\mathrm{sg}(b - c)\mathrm{sg}(d - a)\rangle$ |
| $\wedge_{3,11}$ | $\langle c\mathrm{sg}(a - d)\mathrm{sg}(c - b), 1 - (1 - d)\mathrm{sg}(a - d)\rangle$ |
| $\vee_{3,12}$ | $\langle \max(a, c), 1 - \max(a, c)\rangle$ |
| $\wedge_{3,12}$ | $\langle 1 - \max(b, d), \max(b, d)$ |
| $\vee_{3,13}$ | $\langle a + c - ac, bd\rangle$ |
| $\wedge_{3,13}$ | $\langle ac, b + d - bd\rangle$ |
| $\vee_{3,14}$ | $\langle 1 - (1 - c)\mathrm{sg}(b - c) - d\overline{\mathrm{sg}}(b - c)\mathrm{sg}(d - a), d\mathrm{sg}(d - a)\rangle$ |
| $\wedge_{3,14}$ | $\langle c\mathrm{sg}(c - b), 1 - (1 - d)\mathrm{sg}(a - d) - c\overline{\mathrm{sg}}(a - d)\mathrm{sg}(c - b)\rangle$ |

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| $\vee_{3,15}$ | $\langle 1-(1-\min(a,c))\mathrm{sg}(\mathrm{sg}(b-c)+\mathrm{sg}(d-a))$ $-\min(a,c)\mathrm{sg}(b-c)\mathrm{sg}(d-a),\ 1-(1-\max(b,d))\mathrm{sg}(\overline{\mathrm{sg}}(b-c)$ $+\overline{\mathrm{sg}}(d-a))-\max(b,d)\overline{\mathrm{sg}}(b-c)\overline{\mathrm{sg}}(d-a)\rangle$ |
| $\wedge_{3,15}$ | $\langle\ 1-(1-\max(a,c))\mathrm{sg}(\overline{\mathrm{sg}}(a-d)$ $+\overline{\mathrm{sg}}(c-b))-\max(a,c)\overline{\mathrm{sg}}(a-d)\overline{\mathrm{sg}}(c-b),$ $1-(1-\min(b,d))\mathrm{sg}(\mathrm{sg}(a-d)+\mathrm{sg}(c-b))$ $-\min(b,d)\mathrm{sg}(a-d)\mathrm{sg}(c-b)\rangle$ |
| $\vee_{3,16}$ | $\langle\max(\overline{\mathrm{sg}}(b),c),\min(\mathrm{sg}(b),d)\rangle$ |
| $\wedge_{3,16}$ | $\langle\min(\mathrm{sg}(a),c),\max(\overline{\mathrm{sg}}(a),d)\rangle$ |
| $\vee_{3,17}$ | $\langle\max(a,c),\min(ab+b^2,d)\rangle$ |
| $\wedge_{3,17}$ | $\langle\min(ab+a^2,c),\max(b,d)\rangle$ |
| $\vee_{3,18}$ | $\langle\max(a,c),\min(1-a,d)\rangle$ |
| $\wedge_{3,18}$ | $\langle\min(1-b,c),\max(b,d)\rangle$ |
| $\vee_{3,19}$ | $\langle\max(1-\mathrm{sg}(\mathrm{sg}(b)+\mathrm{sg}(1-a)),c),\min(\mathrm{sg}(1-a),d)\rangle$ |
| $\wedge_{3,19}$ | $\langle\min(\mathrm{sg}(1-b),c),\max(1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)),d)\rangle$ |
| $\vee_{3,20}$ | $\langle\max(\overline{\mathrm{sg}}(b),\mathrm{sg}(c)),\min(\mathrm{sg}(b),\overline{\mathrm{sg}}(c))\rangle$ |
| $\wedge_{3,20}$ | $\langle\min(\mathrm{sg}(a),\overline{\mathrm{sg}}(d)),\max(\overline{\mathrm{sg}}(a),\mathrm{sg}(d))\rangle$ |
| $\vee_{3,21}$ | $\langle\max(a,c(c+d)),\min(b(a+b),d(c^2+d+cd))\rangle$ |
| $\wedge_{3,21}$ | $\langle\min(a(a+b),c(d^2+c+cd)),\max(b,d(c+d))\rangle$ |
| $\vee_{3,22}$ | $\langle\max(a,1-d),1-\max(a,1-d)\rangle$ |
| $\wedge_{3,22}$ | $\langle 1-\max(b,1-c),\max(b,1-c)\rangle$ |
| $\vee_{3,23}$ | $\langle 1-\min(\mathrm{sg}(1-a),\overline{\mathrm{sg}}(1-d)),\min(\mathrm{sg}(1-a),\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,23}$ | $\langle\min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(1-c)),1-\min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(1-c))\rangle$ |
| $\vee_{3,24}$ | $\langle\overline{\mathrm{sg}}(b-c)\overline{\mathrm{sg}}(d-a),\mathrm{sg}(b-c)\mathrm{sg}(d-a)\rangle$ |
| $\wedge_{3,24}$ | $\langle\mathrm{sg}(a-d)\mathrm{sg}(c-b),\overline{\mathrm{sg}}(a-d)\overline{\mathrm{sg}}(c-b)\rangle$ |
| $\vee_{3,25}$ | $\max(a,\overline{\mathrm{sg}}(b)\overline{\mathrm{sg}}(1-a)),c\overline{\mathrm{sg}}(d)\overline{\mathrm{sg}}(1-c)),\min(b,d)\rangle$ |
| $\wedge_{3,25}$ | $\langle\min(a,c),\max(b,\overline{\mathrm{sg}}(a)\overline{\mathrm{sg}}(1-b)),d\overline{\mathrm{sg}}(c)\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,26}$ | $\langle\max(\overline{\mathrm{sg}}(1-a),c),\min(\mathrm{sg}(b),d)\rangle$ |
| $\wedge_{3,26}$ | $\langle\min(\mathrm{sg}(a),c),\max(\overline{\mathrm{sg}}(1-b),d)\rangle$ |
| $\vee_{3,27}$ | $\langle\max(\overline{\mathrm{sg}}(1-a),\mathrm{sg}(c)),\min(\mathrm{sg}(b),\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,27}$ | $\langle\min(\mathrm{sg}(a),\overline{\mathrm{sg}}(1-c)),\max(\overline{\mathrm{sg}}(1-b),\mathrm{sg}(d))\rangle$ |
| $\vee_{3,28}$ | $\langle\max(\overline{\mathrm{sg}}(1-a),c),\min(b,d)\rangle$ |
| $\wedge_{3,28}$ | $\langle\min(a,c),\max(\overline{\mathrm{sg}}(1-b),d)\rangle$ |
| $\vee_{3,29}$ | $\langle\max(\overline{\mathrm{sg}}(1-a),\overline{\mathrm{sg}}(1-c)),\min(b,\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,29}$ | $\langle\min(a,\overline{\mathrm{sg}}(1-c)),\max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,30}$ | $\langle\max(1-b,\min(b,1-d)),\min(b,d)\rangle$ |
| $\wedge_{3,30}$ | $\langle\min(a,c),\max(1-a,\min(a,1-c)),\rangle$ |
| $\vee_{3,31}$ | $\langle\overline{\mathrm{sg}}(b+d-1),d\mathrm{sg}(b+d-1)\rangle$ |
| $\wedge_{3,31}$ | $\langle c\mathrm{sg}(a+c-1),\overline{\mathrm{sg}}(a+c-1)\rangle$ |
| $\vee_{3,32}$ | $\langle 1-d\mathrm{sg}(b+d-1),d\mathrm{sg}(b+d-1)\rangle$ |

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| $\wedge_{3,32}$ | $\langle c\mathrm{sg}(a+c-1), 1-c\mathrm{sg}(a+c-1)\rangle$ |
| $\vee_{3,33}$ | $\langle 1-\min(b,d), \min(b,d)\rangle$ |
| $\wedge_{3,33}$ | $\langle \min(a,c), 1-\min(a,c)\rangle$ |
| $\vee_{3,34}$ | $\langle \min(1, 2-b-d), \max(0, b+d-1)\rangle$ |
| $\wedge_{3,34}$ | $\langle \max(0, a+c-1), \min(1, 2-a-c)\rangle$ |
| $\vee_{3,35}$ | $\langle 1-bd, bd\rangle$ |
| $\wedge_{3,35}$ | $\langle ac, 1-ac\rangle$ |
| $\vee_{3,36}$ | $\langle \min(1-\min(b,d), \max(b,1-b), \max(1-d,d)),$ $\max(\min(b,d), \min(b,1-b), \min(1-d,d))\rangle$ |
| $\wedge_{3,36}$ | $\langle \max(\min(a,c), \min(a,1-a), \min(1-c,c)),$ $\min(1-\min(a,c), \max(a,1-a), \max(1-c,c))\rangle$ |
| $\vee_{3,37}$ | $1-\max(b,d)\mathrm{sg}(b+d-1), \max(b,d)\mathrm{sg}(b+d-1)\rangle$ |
| $\wedge_{3,37}$ | $\langle \max(a,c)\mathrm{sg}(a+c-1), 1-\max(a,c)\mathrm{sg}(a+c-1)\rangle$ |
| $\vee_{3,38}$ | $\langle 1-b+b^2(1-d), b(1-b)+b^2d\rangle$ |
| $\wedge_{3,38}$ | $\langle a(1-a)+a^2c, 1-a+a^2(1-c)\rangle$ |
| $\vee_{3,39}$ | $\langle (1-d)\overline{\mathrm{sg}}(1-b)+\mathrm{sg}(1-b)(\overline{\mathrm{sg}}(d)+(1-b)\mathrm{sg}(d)),$ $d\overline{\mathrm{sg}}(1-b)+b\mathrm{sg}(1-b)\mathrm{sg}(d)\rangle$ |
| $\wedge_{3,39}$ | $\langle c\overline{\mathrm{sg}}(1-a)+a\mathrm{sg}(1-a)\mathrm{sg}(c),$ $(1-c)\overline{\mathrm{sg}}(1-a)+\mathrm{sg}(1-a)(\overline{\mathrm{sg}}(c)+(1-a)\mathrm{sg}(c))\rangle$ |
| $\vee_{3,40}$ | $\langle 1-\mathrm{sg}(b+d-1), 1-\overline{\mathrm{sg}}(b+d-1)\rangle$ |
| $\wedge_{3,40}$ | $\langle 1-\overline{\mathrm{sg}}(a+c-1), 1-\mathrm{sg}(a+c-1), \rangle$ |
| $\vee_{3,41}$ | $\langle \max(\overline{\mathrm{sg}}(b), 1-d), \min(\mathrm{sg}(b), d)\rangle$ |
| $\wedge_{3,41}$ | $\langle \min(\mathrm{sg}(a), c), \max(\overline{\mathrm{sg}}(a), 1-c)\rangle$ |
| $\vee_{3,42}$ | $\langle \max(\overline{\mathrm{sg}}(b), \mathrm{sg}(1-d)), \min(\mathrm{sg}(b), \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,42}$ | $\langle \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(1-c)), \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(1-c))\rangle$ |
| $\vee_{3,43}$ | $\langle \max(\overline{\mathrm{sg}}(b), 1-d), \min(\mathrm{sg}(b), d)\rangle$ |
| $\wedge_{3,43}$ | $\langle \min(\mathrm{sg}(a), c), \max(\overline{\mathrm{sg}}(a), 1-c)\rangle$ |
| $\vee_{3,44}$ | $\langle \max(\overline{\mathrm{sg}}(b), 1-d), \min(b, d)\rangle$ |
| $\wedge_{3,44}$ | $\langle \min(a, c), \max(\overline{\mathrm{sg}}(a), 1-c)\rangle$ |
| $\vee_{3,45}$ | $\langle \max(\overline{\mathrm{sg}}(b), \overline{\mathrm{sg}}(d)), \min(b, \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,45}$ | $\langle \min(a, \overline{\mathrm{sg}}(1-c)), \max(\overline{\mathrm{sg}}(a), \overline{\mathrm{sg}}(c))\rangle$ |
| $\vee_{3,46}$ | $\langle \max(a, \min(1-a, c)), 1-\max(a, c)\rangle$ |
| $\wedge_{3,46}$ | $\langle 1-\max(b, d), \max(b, \min(1-b, d))\rangle$ |
| $\vee_{3,47}$ | $\langle \overline{\mathrm{sg}}(1-a-c), (1-c)\mathrm{sg}(1-a-c)\rangle$ |
| $\wedge_{3,47}$ | $\langle (1-d)\mathrm{sg}(1-b-d), \overline{\mathrm{sg}}(1-b-d)\rangle$ |
| $\vee_{3,48}$ | $\langle 1-(1-c)\mathrm{sg}(1-a-c), (1-c)\mathrm{sg}(1-a-c)\rangle$ |
| $\wedge_{3,48}$ | $\langle (1-d)\mathrm{sg}(1-b-d), 1-(1-d)\mathrm{sg}(1-b-d)\rangle$ |
| $\vee_{3,49}$ | $\langle \min(1, a+c), \max(0, 1-a-c)\rangle$ |
| $\wedge_{3,49}$ | $\langle \max(0, 1-b-d), \min(1, b+d)\rangle$ |
| $\vee_{3,50}$ | $\langle a+c-ac, 1-a-c+ac\rangle$ |

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| $\wedge_{3,50}$ | $\langle 1-b-d+bd, b+d-bd\rangle$ |
| $\vee_{3,51}$ | $\langle \min(\max(a,c),\max(1-a,a),\max(c,1-c)),$ $\max(1-\max(a,c),\min(1-a,a),\min(c,1-c))\rangle$ |
| $\wedge_{3,51}$ | $\langle \max(1-\max(b,d),\min(1-b,b),\min(d,1-d)),$ $\min(\max(b,d),\max(1-b,b),\max(d,1-d))\rangle$ |
| $\vee_{3,52}$ | $\langle 1-(1-\min(a,c))\mathrm{sg}(1-a-c), 1-\min(a,c)\mathrm{sg}(1-a-c)\rangle$ |
| $\wedge_{3,52}$ | $\langle 1-\min(b,d)\mathrm{sg}(1-b-d), 1-(1-\min(b,d))\mathrm{sg}(1-b-d)\rangle$ |
| $\vee_{3,53}$ | $\langle a+(1-a)^2c, (1-a)a+(1-a)^2(1-c)\rangle$ |
| $\wedge_{3,53}$ | $\langle (1-b)b+(1-b)^2(1-d), b+(1-b)^2d\rangle$ |
| $\vee_{3,54}$ | $\langle c\overline{\mathrm{sg}}(a)+\mathrm{sg}(a)(\overline{\mathrm{sg}}(1-c)+a\mathrm{sg}(1-c)),$ $(1-c)\overline{\mathrm{sg}}(a)+(1-a)\mathrm{sg}(a)\mathrm{sg}(1-c)\rangle$ |
| $\wedge_{3,54}$ | $\langle (1-d)\overline{\mathrm{sg}}(b)+(1-b)\mathrm{sg}(b)\mathrm{sg}(1-d),$ $(d\overline{\mathrm{sg}}(b)+\mathrm{sg}(b)(\overline{\mathrm{sg}}(1-d)+b\mathrm{sg}(1-d))\rangle$ |
| $\vee_{3,55}$ | $\langle \overline{\mathrm{sg}}(1-a-c), \mathrm{sg}(1-a-c)\rangle$ |
| $\wedge_{3,55}$ | $\langle \mathrm{sg}(1-b-d), \overline{\mathrm{sg}}(1-b-d)\rangle$ |
| $\vee_{3,56}$ | $\langle \max(\overline{\mathrm{sg}}(1-a),c), \min(\mathrm{sg}(1-a),1-c)\rangle$ |
| $\wedge_{3,56}$ | $\langle \min(\mathrm{sg}(1-b),1-d), \max(\overline{\mathrm{sg}}(1-b),d)\rangle$ |
| $\vee_{3,57}$ | $\langle \max(\overline{\mathrm{sg}}(1-a),\mathrm{sg}(c)), \min(\mathrm{sg}(1-a),\overline{\mathrm{sg}}(c))\rangle$ |
| $\wedge_{3,57}$ | $\langle \min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(d)), \max(\overline{\mathrm{sg}}(1-b),\mathrm{sg}(d))\rangle$ |
| $\vee_{3,58}$ | $\langle \max(\overline{\mathrm{sg}}(1-a),\overline{\mathrm{sg}}(1-c)), 1-\max(a,c)\rangle$ |
| $\wedge_{3,58}$ | $\langle 1-\max(b,d), \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,59}$ | $\langle \max(\overline{\mathrm{sg}}(1-a),c), 1-\max(a,c)\rangle$ |
| $\wedge_{3,59}$ | $\langle 1-\max(b,d), \max(\overline{\mathrm{sg}}(1-b),d)\rangle$ |
| $\vee_{3,60}$ | $\langle \max(\overline{\mathrm{sg}}(1-a),\overline{\mathrm{sg}}(1-c)), \min(1-a,\overline{\mathrm{sg}}(c))\rangle$ |
| $\wedge_{3,60}$ | $\langle \min(1-b,\overline{\mathrm{sg}}(d)), \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,61}$ | $\langle \max(c,\min(a,d)), \min(b,d)\rangle$ |
| $\wedge_{3,61}$ | $\langle \min(a,c), \max(d,\min(b,c))\rangle$ |
| $\vee_{3,62}$ | $\langle \overline{\mathrm{sg}}(d-a), b\mathrm{sg}(d-a)\rangle$ |
| $\wedge_{3,62}$ | $\langle a\mathrm{sg}(c-b), \overline{\mathrm{sg}}(c-b)\rangle$ |
| $\vee_{3,63}$ | $\langle 1-(1-a)\mathrm{sg}(d-a), b\mathrm{sg}(d-a)\rangle$ |
| $\wedge_{3,63}$ | $\langle a\mathrm{sg}(c-b), 1-(1-b)\mathrm{sg}(c-b)\rangle$ |
| $\vee_{3,64}$ | $\langle c+ad, bd\rangle$ |
| $\wedge_{3,64}$ | $\langle ac, d+bc\rangle$ |
| $\vee_{3,65}$ | $\langle 1-(1-\min(a,c))\mathrm{sg}(d-a), \max(b,d)\mathrm{sg}(d-a)\mathrm{sg}(b-c)\rangle$ |
| $\wedge_{3,65}$ | $\langle \max(a,c)\mathrm{sg}(c-b)\mathrm{sg}(a-d), 1-(1-\min(b,d))\mathrm{sg}(c-b)\rangle$ |
| $\vee_{3,66}$ | $\langle c+d^2a, ad+d^2b\rangle$ |
| $\wedge_{3,66}$ | $\langle bc+c^2a, d+c^2b\rangle$ |
| $\vee_{3,67}$ | $\langle a\overline{\mathrm{sg}}(1-d)+\mathrm{sg}(1-d)(\overline{\mathrm{sg}}(1-a)+c\mathrm{sg}(1-a)),$ $b\overline{\mathrm{sg}}(1-d)+d\mathrm{sg}(1-d)\mathrm{sg}(1-a)\rangle$ |
| $\wedge_{3,67}$ | $\langle a\overline{\mathrm{sg}}(1-c)+d\mathrm{sg}(1-c)\mathrm{sg}(1-b),$ |

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| | $b\overline{\mathrm{sg}}(1-c)+\mathrm{sg}(1-c)(\overline{\mathrm{sg}}(1-b)+d\mathrm{sg}(1-b))\rangle$ |
| $\vee_{3,68}$ | $\langle 1-(1-a)\mathrm{sg}(d-a), b\mathrm{sg}(d-a)\mathrm{sg}(b-c)\rangle$ |
| $\wedge_{3,68}$ | $\langle a\mathrm{sg}(c-b)\mathrm{sg}(a-d), 1-(1-b)\mathrm{sg}(c-b)\rangle$ |
| $\vee_{3,69}$ | $\langle 1-(1-a)\mathrm{sg}(d-a)-b\overline{\mathrm{sg}}(d-a)\mathrm{sg}(b-c), b\mathrm{sg}(b-c)\rangle$ |
| $\wedge_{3,69}$ | $\langle a\mathrm{sg}(a-d), 1-(1-b)\mathrm{sg}(c-b)-a\overline{\mathrm{sg}}(c-b)\mathrm{sg}(a-d)\rangle$ |
| $\vee_{3,70}$ | $\langle \max(\overline{\mathrm{sg}}(d), a), \min(\mathrm{sg}(d), b)\rangle$ |
| $\wedge_{3,70}$ | $\langle \min(\mathrm{sg}(c), a), \max(\overline{\mathrm{sg}}(c), b)\rangle$ |
| $\vee_{3,71}$ | $\langle \max(c, a), \min(dc+d^2, b)\rangle$ |
| $\wedge_{3,71}$ | $\langle \min(cd+c^2, a), \max(b, d)\rangle$ |
| $\vee_{3,72}$ | $\langle \max(a, c), \min(1-c, b)\rangle$ |
| $\wedge_{3,72}$ | $\langle \min(1-d, a), \max(b, d)\rangle$ |
| $\vee_{3,73}$ | $\langle \max(1-\max(\mathrm{sg}(d), \mathrm{sg}(1-c)), a), \min(\mathrm{sg}(1-c), b)\rangle$ |
| $\wedge_{3,73}$ | $\langle \min(\mathrm{sg}(1-d), a), \max(1-\max(\mathrm{sg}(c), \mathrm{sg}(1-d)), b)\rangle$ |
| $\vee_{3,74}$ | $\langle \max(\mathrm{sg}(a), \overline{\mathrm{sg}}(d)), \min(\overline{\mathrm{sg}}(a), \mathrm{sg}(d))\rangle$ |
| $\wedge_{3,74}$ | $\langle \min(\overline{\mathrm{sg}}(b), \mathrm{sg}(c)), \max(\mathrm{sg}(b), \overline{\mathrm{sg}}(c))\rangle$ |
| $\vee_{3,75}$ | $\langle \max(c, a(a+b)), \min(d(c+d), b(a^2+b)+ab)\rangle$ |
| $\wedge_{3,75}$ | $\langle \min(c(c+d), a(b^2+a)+ab), \max(d, b(a+b))\rangle$ |
| $\vee_{3,76}$ | $\langle \max(c, 1-b), \min(1-c, b)\rangle$ |
| $\wedge_{3,76}$ | $\langle \min(1-d, a), \max(d, 1-a)\rangle$ |
| $\vee_{3,77}$ | $\langle (1-\min(\overline{\mathrm{sg}}(1-b), \mathrm{sg}(1-c))), \min(\overline{\mathrm{sg}}(1-b), \mathrm{sg}(1-c))\rangle$ |
| $\wedge_{3,77}$ | $\langle \min(\overline{\mathrm{sg}}(1-a), \mathrm{sg}(1-d)), (1-\min(\overline{\mathrm{sg}}(1-a), \mathrm{sg}(1-d)))\rangle$ |
| $\vee_{3,78}$ | $\langle \max(\overline{\mathrm{sg}}(1-c), a), \min(\mathrm{sg}(d), b)\rangle$ |
| $\wedge_{3,78}$ | $\langle \min(\mathrm{sg}(c), a), \max(\overline{\mathrm{sg}}(1-d), b)\rangle$ |
| $\vee_{3,79}$ | $\langle \max(\overline{\mathrm{sg}}(1-c), \mathrm{sg}(a)), \min(\mathrm{sg}(d), \overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,79}$ | $\langle \min(\mathrm{sg}(c), \overline{\mathrm{sg}}(1-a)), \max(\overline{\mathrm{sg}}(1-d), \mathrm{sg}(b))\rangle$ |
| $\vee_{3,80}$ | $\langle \max(\overline{\mathrm{sg}}(1-c), a), \min(d, b)\rangle$ |
| $\wedge_{3,80}$ | $\langle \min(c, a), \max(\overline{\mathrm{sg}}(1-d), b)\rangle$ |
| $\vee_{3,81}$ | $\langle \max(\overline{\mathrm{sg}}(1-a), \overline{\mathrm{sg}}(1-c)), \min(d, \overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,81}$ | $\langle \min(c, \overline{\mathrm{sg}}(1-a)), \max(\overline{\mathrm{sg}}(1-b), \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,82}$ | $\langle \max(1-d, \min(d, 1-b)), \min(d, b)\rangle$ |
| $\wedge_{3,82}$ | $\langle \min(c, a), \max(1-c, \min(c, 1-a))\rangle$ |
| $\vee_{3,83}$ | $\langle \overline{\mathrm{sg}}(b+d-1), b\mathrm{sg}(b+d-1)\rangle$ |
| $\wedge_{3,83}$ | $\langle a\mathrm{sg}(a+c-1), \overline{\mathrm{sg}}(a+c-1), \rangle$ |
| $\vee_{3,84}$ | $\langle 1-b\mathrm{sg}(b+d-1), b\mathrm{sg}(b+d-1)\rangle$ |
| $\wedge_{3,84}$ | $\langle a\mathrm{sg}(c+a-1), 1-a\mathrm{sg}(c+a-1)\rangle$ |
| $\vee_{3,85}$ | $\langle 1-d+d^2(1-b), d(1-d)+d^2\rangle$ |
| $\wedge_{3,85}$ | $\langle c^2+c(1-c), 1-c+c^2(1-a)\rangle$ |
| $\vee_{3,86}$ | $\langle (1-b)\overline{\mathrm{sg}}(1-d)+\mathrm{sg}(1-d)(\overline{\mathrm{sg}}(b)+(1-d)\mathrm{sg}(d)),$ $b\overline{\mathrm{sg}}(1-d)+d\mathrm{sg}(1-d)\mathrm{sg}(b)\rangle$ |
| $\wedge_{3,86}$ | $\langle a\overline{\mathrm{sg}}(1-c)+c\mathrm{sg}(1-c)\mathrm{sg}(a),$ |

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| | $(1-a)\overline{\mathrm{sg}}(1-c)+\mathrm{sg}(1-c)(\overline{\mathrm{sg}}(a)+(1-c)\mathrm{sg}(c))\rangle$ |
| $\vee_{3,87}$ | $\langle\max(\overline{\mathrm{sg}}(d),1-b),\min(\mathrm{sg}(d),b)\rangle$ |
| $\wedge_{3,87}$ | $\langle\min(\mathrm{sg}(c),a),\max(\overline{\mathrm{sg}}(c),1-a)\rangle$ |
| $\vee_{3,88}$ | $\langle\max(\overline{\mathrm{sg}}(d),\mathrm{sg}(1-b)),\min(\mathrm{sg}(d),\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,88}$ | $\langle\min(\mathrm{sg}(c),\overline{\mathrm{sg}}(1-a)),\max(\overline{\mathrm{sg}}(c),\mathrm{sg}(1-a))\rangle$ |
| $\vee_{3,89}$ | $\langle\max(\overline{\mathrm{sg}}(d),1-b),\min(d,b)\rangle$ |
| $\wedge_{3,89}$ | $\langle\min(c,a),\max(\overline{\mathrm{sg}}(c),1-a)\rangle$ |
| $\vee_{3,90}$ | $\langle\max(\overline{\mathrm{sg}}(b),\overline{\mathrm{sg}}(d)),\min(d,\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,90}$ | $\langle\min(c,\overline{\mathrm{sg}}(1-a)),\max(\overline{\mathrm{sg}}(a),\overline{\mathrm{sg}}(c))\rangle$ |
| $\vee_{3,91}$ | $\langle\max(c,\min(1-c,a)),1-\max(a,c)\rangle$ |
| $\wedge_{3,91}$ | $\langle 1-\max(b,d),\max(d,\min(1-d,b))\rangle$ |
| $\vee_{3,92}$ | $\langle\overline{\mathrm{sg}}(1-a-c),\min(1-a,\mathrm{sg}(1-a-c))\rangle$ |
| $\wedge_{3,92}$ | $\langle\min(1-b,\mathrm{sg}(1-b-d)),\overline{\mathrm{sg}}(1-b-d)\rangle$ |
| $\vee_{3,93}$ | $\langle(1-\min(1-a,\mathrm{sg}(1-a-c)),\min(1-a,\mathrm{sg}(1-a-c))\rangle$ |
| $\wedge_{3,93}$ | $\langle\min(1-b,\mathrm{sg}(1-b-d)),1-\min(1-b,\mathrm{sg}(1-b-d))\rangle$ |
| $\vee_{3,94}$ | $\langle c+(1-c)^2a,(1-c)c+(1-c)^2(1-a)\rangle$ |
| $\wedge_{3,94}$ | $\langle(1-d)d+(1-d)^2(1-b),d+(1-d)^2b\rangle$ |
| $\vee_{3,95}$ | $\langle\min(a,\overline{\mathrm{sg}}(c))+\mathrm{sg}(c)(\overline{\mathrm{sg}}(1-a)+\min(c,\mathrm{sg}(1-a))),$ $\min(1-a,\overline{\mathrm{sg}}(c))+\min(1-c,\mathrm{sg}(c),\mathrm{sg}(1-a))\rangle$ |
| $\wedge_{3,95}$ | $\langle\min(1-b,\overline{\mathrm{sg}}(d))+\min(1-d,\mathrm{sg}(d),\mathrm{sg}(1-b)),$ $\min(b,\overline{\mathrm{sg}}(d))+\mathrm{sg}(d)(\overline{\mathrm{sg}}(1-b)+\min(d,\mathrm{sg}(1-b)))\rangle$ |
| $\vee_{3,96}$ | $\langle\max(\overline{\mathrm{sg}}(1-c),a),\min(\mathrm{sg}(1-a),1-c)\rangle$ |
| $\wedge_{3,96}$ | $\langle\min(\mathrm{sg}(1-b),1-d),\max(\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\vee_{3,97}$ | $\langle\max(\overline{\mathrm{sg}}(1-c),\mathrm{sg}(a)),\min(\mathrm{sg}(1-c),\overline{\mathrm{sg}}(a))\rangle$ |
| $\wedge_{3,97}$ | $\langle\min(\mathrm{sg}(1-d),\overline{\mathrm{sg}}(b)),\max(\overline{\mathrm{sg}}(1-d),\mathrm{sg}(b))\rangle$ |
| $\vee_{3,98}$ | $\langle\max(\overline{\mathrm{sg}}(1-c),a),1-\max(a,c)\rangle$ |
| $\wedge_{3,98}$ | $\langle 1-\max(b,d),\max(\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\vee_{3,99}$ | $\langle\max(\overline{\mathrm{sg}}(1-c),\overline{\mathrm{sg}}(1-a)),\min(1-c,\overline{\mathrm{sg}}(a))\rangle$ |
| $\wedge_{3,99}$ | $\langle\min(1-d,\overline{\mathrm{sg}}(b)),\max(\overline{\mathrm{sg}}(1-d),\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\vee_{3,100}$ | $\langle\max(\min(a,\mathrm{sg}(b)),c),\min(b,\mathrm{sg}(a),d)\rangle$ |
| $\wedge_{3,100}$ | $\langle\min(a,\mathrm{sg}(b),c),\max(\min(b,\mathrm{sg}(a)),d)\rangle$ |
| $\vee_{3,101}$ | $\langle\max(\min(a,\mathrm{sg}(b)),\min(c,\mathrm{sg}(d))),\min(b,\mathrm{sg}(a),\mathrm{sg}(c),d)\rangle$ |
| $\wedge_{3,101}$ | $\langle\min(a,\mathrm{sg}(b),\mathrm{sg}(d),c),\max(\min(b,\mathrm{sg}(a)),\min(d,\mathrm{sg}(c)))\rangle$ |
| $\vee_{3,102}$ | $\langle\max(a,\min(c,\mathrm{sg}(d))),\min(b,\mathrm{sg}(c),d)\rangle$ |
| $\wedge_{3,102}$ | $\langle\min(a,\mathrm{sg}(d),c),\max(b,\min(d,\mathrm{sg}(c))),\rangle$ |
| $\vee_{3,103}$ | $\langle\max(\min(1-b,\mathrm{sg}(b)),1-d),\min(b,\mathrm{sg}(1-b),d)\rangle$ |
| $\wedge_{3,103}$ | $\langle\min(a,\mathrm{sg}(1-a),c),\max(\min(1-a,\mathrm{sg}(a)),1-c)\rangle$ |
| $\vee_{3,104}$ | $\langle\max(\min(1-b,\mathrm{sg}(b)),\min(1-d,\mathrm{sg}(d))),$ $\min(b,\mathrm{sg}(1-b),d,\mathrm{sg}(1-d))\rangle$ |

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| $\wedge_{3,104}$ | $\langle \min(a, \mathrm{sg}(1-a), c, \mathrm{sg}(1-c)),$ $\max(\min(1-a, \mathrm{sg}(a)), \min(1-c, \mathrm{sg}(c)))\rangle$ |
| $\vee_{3,105}$ | $\langle \max(1-b, \min(1-d, \mathrm{sg}(d))), \min(b, d, \mathrm{sg}(1-d))\rangle$ |
| $\wedge_{3,105}$ | $\langle \min(a, c, \mathrm{sg}(1-c)), \max(1-a, \min(1-c, \mathrm{sg}(c)))\rangle$ |
| $\vee_{3,106}$ | $\langle \max(\min(a, \mathrm{sg}(1-a)), c), \min(1-a, \mathrm{sg}(a), 1-c)\rangle$ |
| $\wedge_{3,106}$ | $\langle \min(1-b, \mathrm{sg}(b), 1-d), \max(\min(b, \mathrm{sg}(1-b)), d)\rangle$ |
| $\vee_{3,107}$ | $\langle \max(\min(a, \mathrm{sg}(1-a)), \min(c, \mathrm{sg}(1-c))),$ $\min(1-a, \mathrm{sg}(a), 1-c, \mathrm{sg}(c))\rangle$ |
| $\wedge_{3,107}$ | $\langle \min(1-b, \mathrm{sg}(b), 1-d, \mathrm{sg}(d)),$ $\max(\min(b, \mathrm{sg}(1-b)), \min(d, \mathrm{sg}(1-d)))\rangle$ |
| $\vee_{3,108}$ | $\langle \max(a, \min(c, \mathrm{sg}(1-c))), \min(1-a, 1-c, \mathrm{sg}(c))\rangle$ |
| $\wedge_{3,108}$ | $\langle \min(1-b, 1-d, \mathrm{sg}(d)), \max(b, \min(d, \mathrm{sg}(1-d)))\rangle$ |
| $\vee_{3,109}$ | $b + \min(\overline{\mathrm{sg}}(1-b), c), ab + \min(\overline{\mathrm{sg}}(1-b), d))\rangle$ |
| $\wedge_{3,109}$ | $\langle ab + \min(\overline{\mathrm{sg}}(1-a), c)), b + \min(\overline{\mathrm{sg}}(1-a), d)\rangle$ |
| $\vee_{3,110}$ | $\langle \max(a, c), \min(ab + \overline{\mathrm{sg}}(1-b), d)\rangle$ |
| $\wedge_{3,110}$ | $\langle \min(ab + \overline{\mathrm{sg}}(1-a), c), \max(b, d)\rangle$ |
| $\vee_{3,111}$ | $\langle \max(a, cd + \overline{\mathrm{sg}}(1-c)), \min(ab + \overline{\mathrm{sg}}(1-b),$ $d(cd + \overline{\mathrm{sg}}(1-c)) + \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,111}$ | $\langle c(cd + \overline{\mathrm{sg}}(1-d)) + \overline{\mathrm{sg}}(1-c)),$ $\max(b, cd + \overline{\mathrm{sg}}(1-d)), \min(ab + \overline{\mathrm{sg}}(1-a)\rangle$ |
| $\vee_{3,112}$ | $\langle a + c - ac, ab + \overline{\mathrm{sg}}(1-b)d\rangle$ |
| $\wedge_{3,112}$ | $\langle ab + \overline{\mathrm{sg}}(1-a)c, b + d - bd\rangle$ |
| $\vee_{3,113}$ | $\langle a + cd - a(cd + \overline{\mathrm{sg}}(1-c)),$ $(ab + \overline{\mathrm{sg}}(1-b))(d(cd + \overline{\mathrm{sg}}(1-c)) + \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,113}$ | $\langle (ab + \overline{\mathrm{sg}}(1-a))(c(cd + \overline{\mathrm{sg}}(1-d)) + \overline{\mathrm{sg}}(1-c)),$ $b + cd - b(cd + \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,114}$ | $\langle 1 - b + \min(\overline{\mathrm{sg}}(1-b), 1-d), b(1-b) + \min(\overline{\mathrm{sg}}(1-b), d)\rangle$ |
| $\wedge_{3,114}$ | $\langle a(1-a) + \min(\overline{\mathrm{sg}}(1-a), c), 1 - a + \min(\overline{\mathrm{sg}}(1-a), 1-c)\rangle$ |
| $\vee_{3,115}$ | $\langle 1 - \min(b, d), \min(b(1-b) + \overline{\mathrm{sg}}(1-b), d)\rangle$ |
| $\wedge_{3,115}$ | $\langle \min(a(1-a) + \overline{\mathrm{sg}}(1-a), c), 1 - \min(a, c)\rangle$ |
| $\vee_{3,116}$ | $\langle \max(1-b, (1-d)d + \overline{\mathrm{sg}}(d)),$ $\min(b(1-b) + \overline{\mathrm{sg}}(1-b), d((1-d)d + \overline{\mathrm{sg}}(d)) + \overline{\mathrm{sg}}(1-d))\rangle$ |
| $\wedge_{3,116}$ | $\langle \min(a(1-a) + \overline{\mathrm{sg}}(1-a), c((1-c)c + \overline{\mathrm{sg}}(c)) + \overline{\mathrm{sg}}(1-c)),$ $\max(1-a, (1-c)c + \overline{\mathrm{sg}}(c))\rangle$ |
| $\vee_{3,117}$ | $\langle 1 - b - d + bd, (b(1-b) + \overline{\mathrm{sg}}(1-b))d\rangle$ |
| $\wedge_{3,117}$ | $\langle (a(1-a) + \overline{\mathrm{sg}}(1-a))c, 1 - a - c + ac, \rangle$ |
| $\vee_{3,118}$ | $\langle 1 - b + (1-d)d - (1-b)((1-d)d + \overline{\mathrm{sg}}(d)),$ $(b(1-b) + \overline{\mathrm{sg}}(1-b))d((1-d)d + \overline{\mathrm{sg}}(d)) + \overline{\mathrm{sg}}(1-d)\rangle$ |
| $\wedge_{3,118}$ | $\langle (a(1-a) + \overline{\mathrm{sg}}(1-a))c((1-c)c + \overline{\mathrm{sg}}(c)) + \overline{\mathrm{sg}}(1-c),$ $1 - a + (1-c)c - (1-a)((1-c)c + \overline{\mathrm{sg}}(c))\rangle$ |

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| $\vee_{3,119}$ | $\langle a+\min(\overline{\mathrm{sg}}(a),c),(1-a)a+\min(\overline{\mathrm{sg}}(a),1-c)\rangle$ |
| $\wedge_{3,119}$ | $\langle(1-b)b+\min(\overline{\mathrm{sg}}(b),1-d),b+\min(\overline{\mathrm{sg}}(b),d)\rangle$ |
| $\vee_{3,120}$ | $\langle\max(a,c),\min((1-a)a+\overline{\mathrm{sg}}(a),1-c)\rangle$ |
| $\wedge_{3,120}$ | $\langle\min((1-b)b+\overline{\mathrm{sg}}(b),1-d),\max(b,d)\rangle$ |
| $\vee_{3,121}$ | $\langle\max(a,c(1-c)+\overline{\mathrm{sg}}(1-c)),$ <br> $\min((1-a)a+\overline{\mathrm{sg}}(a),(1-c)(c(1-c)+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c)\rangle$ |
| $\wedge_{3,121}$ | $\langle\min((1-b)b+\overline{\mathrm{sg}}(b),(1-d)(d(1-d)+\overline{\mathrm{sg}}(1-d)))+\overline{\mathrm{sg}}(d),$ <br> $\max(b,d(1-d)+\overline{\mathrm{sg}}(1-d))\rangle$ |
| $\vee_{3,122}$ | $\langle a+c-ac,((1-c)a+\overline{\mathrm{sg}}(a))(1-c)\rangle$ |
| $\wedge_{3,122}$ | $\langle((1-d)b+\overline{\mathrm{sg}}(b))(1-d),b+d-bd\rangle$ |
| $\vee_{3,123}$ | $\langle a+c(1-c)-(a(c(1-c)+\overline{\mathrm{sg}}(1-c))),$ <br> $((1-a)a+\overline{\mathrm{sg}}(a))(((1-c)(c(1-c)+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c))\rangle$ |
| $\wedge_{3,123}$ | $\langle((1-b)b+\overline{\mathrm{sg}}(b))(((1-d)(d(1-d)+\overline{\mathrm{sg}}(1-d)))+\overline{\mathrm{sg}}(d)),$ <br> $b+d(1-d)-(b(d(1-d)+\overline{\mathrm{sg}}(1-d)))\rangle$ |
| $\vee_{3,124}$ | $\langle c+\min(\overline{\mathrm{sg}}(1-d),a),cd+\min(\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\wedge_{3,124}$ | $\langle cd+\min(\overline{\mathrm{sg}}(1-c),a),d+\min(\overline{\mathrm{sg}}(1-c),b)\rangle$ |
| $\vee_{3,125}$ | $\langle\max(a,c),\min(cd+\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\wedge_{3,125}$ | $\langle\min(cd+\overline{\mathrm{sg}}(1-c),a),\max(b,d)\rangle$ |
| $\vee_{3,126}$ | $\langle\max(c,ab+\overline{\mathrm{sg}}(1-a)),$ <br> $\min(cd+\overline{\mathrm{sg}}(1-d),b(ab+\overline{\mathrm{sg}}(1-a))+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,126}$ | $\langle\min(cd+\overline{\mathrm{sg}}(1-c),a(ab+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(1-a)),$ <br> $\max(d,ab+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\vee_{3,127}$ | $\langle a+c-ac,(cd+\overline{\mathrm{sg}}(1-d))b\rangle$ |
| $\wedge_{3,127}$ | $\langle(cd+\overline{\mathrm{sg}}(1-c))a,b+d-bd\rangle$ |
| $\vee_{3,128}$ | $\langle c+ab-c(ab+\overline{\mathrm{sg}}(1-a)),$ <br> $(cd+\overline{\mathrm{sg}}(1-d))(b(ab+\overline{\mathrm{sg}}(1-a))+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,128}$ | $\langle(cd+\overline{\mathrm{sg}}(1-c))(b(ab+\overline{\mathrm{sg}}(1-a))+\overline{\mathrm{sg}}(1-b)),$ <br> $d+ab-d(ab+\overline{\mathrm{sg}}(1-a))\rangle$ |
| $\vee_{3,129}$ | $\langle 1-d+\min(\overline{\mathrm{sg}}(1-d),1-b),d(1-d)+\min(\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\wedge_{3,129}$ | $\langle c(1-c)+\min(\overline{\mathrm{sg}}(1-c),a),1-c+\min(\overline{\mathrm{sg}}(1-c),1-a)\rangle$ |
| $\vee_{3,130}$ | $\langle 1-\min(d,b),\min(d(1-d)+\overline{\mathrm{sg}}(1-d),b)\rangle$ |
| $\wedge_{3,130}$ | $\langle\min(c(1-c)+\overline{\mathrm{sg}}(1-c),a),1-\min(c,a)\rangle$ |
| $\vee_{3,131}$ | $\langle\max(1-d,(1-b)b+\overline{\mathrm{sg}}(b)),$ <br> $\min(d(1-d)+\overline{\mathrm{sg}}(1-d),b((1-b)b+\overline{\mathrm{sg}}(b))+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\wedge_{3,131}$ | $\langle\min(c(1-c)+\overline{\mathrm{sg}}(1-c),a((1-a)a+\overline{\mathrm{sg}}(a))+\overline{\mathrm{sg}}(1-a)),$ <br> $\max(1-c,(1-a)a+\overline{\mathrm{sg}}(a))\rangle$ |
| $\vee_{3,132}$ | $\langle 1-bd,(d(1-d)+\overline{\mathrm{sg}}(1-d))b\rangle$ |
| $\wedge_{3,132}$ | $\langle(c(1-c)+\overline{\mathrm{sg}}(1-c))a,1-ac\rangle$ |
| $\vee_{3,133}$ | $\langle 1-d+(1-b)b-(1-d)((1-b)b+\overline{\mathrm{sg}}(b)),$ <br> $(d(1-d)+\overline{\mathrm{sg}}(1-d))(b((1-b)b+\overline{\mathrm{sg}}(b))+\overline{\mathrm{sg}}(1-b))\rangle$ |

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| $\wedge_{3,133}$ | $\langle (c(1-c)+\overline{\mathrm{sg}}(1-c))(a((1-a)a+\overline{\mathrm{sg}}(a))+\overline{\mathrm{sg}}(1-a)),$ $1-c+(1-a)a-(1-c)((1-a)a+\overline{\mathrm{sg}}(a))\rangle$ |
| $\vee_{3,134}$ | $\langle c+\min(\overline{\mathrm{sg}}(c),a),(1-c)c+\min(\overline{\mathrm{sg}}(c),(1-a))\rangle$ |
| $\wedge_{3,134}$ | $(1-d)d+\min(\overline{\mathrm{sg}}(d),(1-b)),d+\min(\overline{\mathrm{sg}}(d),b)\rangle$ |
| $\vee_{3,135}$ | $\langle \max(a,c),\min((1-c)c+\overline{\mathrm{sg}}(c),1-a)\rangle$ |
| $\wedge_{3,135}$ | $\langle \min((1-d)d+\overline{\mathrm{sg}}(d),1-b),\max(b,d)\rangle$ |
| $\vee_{3,136}$ | $\langle \max(c,a(1-a)+\overline{\mathrm{sg}}(1-a)),$ $\min((1-c)c+\overline{\mathrm{sg}}(c),(1-a)(a(1-a)+\overline{\mathrm{sg}}(1-a))+\overline{\mathrm{sg}}(a))\rangle$ |
| $\wedge_{3,136}$ | $\langle \min((1-d)d+\overline{\mathrm{sg}}(d),(1-b)(b(1-b)+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(b)),$ $\max(d,b(1-b)+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\vee_{3,137}$ | $\langle a+c-ac,((1-c)c+\overline{\mathrm{sg}}(c))(1-a)\rangle$ |
| $\wedge_{3,137}$ | $\langle ((1-d)d+\overline{\mathrm{sg}}(d))(1-b),b+d-bd\rangle$ |
| $\vee_{3,138}$ | $\langle c+a(1-a)-c(a(1-a)+\overline{\mathrm{sg}}(1-a)),$ $((1-c)c+\overline{\mathrm{sg}}(c))((1-a)(a(1-a)+\overline{\mathrm{sg}}(1-a))+\overline{\mathrm{sg}}(a))\rangle$ |
| $\wedge_{3,138}$ | $\langle ((1-d)d+\overline{\mathrm{sg}}(d))((1-b)(b(1-b)+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(b)),$ $d+b(1-b)-d(b(1-b)+\overline{\mathrm{sg}}(1-b))\rangle$ |
| $\vee_{3,139}$ | $\langle \frac{a+c}{2},\frac{b+d}{2}\rangle$ |
| $\wedge_{3,139}$ | $\langle \frac{a+c}{2},\frac{b+d}{2}\rangle$ |
| $\vee_{3,140}$ | $\langle \frac{a+c+\min(a,c)}{3},\frac{b+d+\max(b,d)}{3}\rangle$ |
| $\wedge_{3,140}$ | $\langle \frac{a+c+\max(a,c)}{3},\frac{b+d+\min(b,d)}{3}\rangle$ |
| $\vee_{3,141}$ | $\langle \frac{a+c+\max(a,c)}{3},\frac{b+d+\min(b,d)}{3}\rangle$ |
| $\wedge_{3,141}$ | $\langle \frac{a+c+\min(a,c)}{3},\frac{b+d+\max(b,d)}{3},\rangle$ |
| $\vee_{3,142}$ | $\langle \frac{3-b-d-\max(b,d)}{3},\frac{b+d+\max(b,d)}{3}\rangle$ |
| $\wedge_{3,142}$ | $\langle \frac{a+c+\max(a,c)}{3},\frac{3-a-c-\max(a,c)}{3}\rangle$ |
| $\vee_{3,143}$ | $\langle \frac{1-b+c+\min(1-b,c)}{3},\frac{2+b-c-\min(1-b,c)}{3}\rangle$ |
| $\wedge_{3,143}$ | $\langle \frac{2+a-d-\min(1-a,d)}{3},\frac{1-a+d+\min(1-a,d)}{3}\rangle$ |
| $\vee_{3,144}$ | $\langle \frac{1+a-d+\min(a,1-d)}{3},\frac{2-a+d-\min(a,1-d)}{3}\rangle$ |
| $\wedge_{3,144}$ | $\langle \frac{2-b+c-\min(b,1-c)}{3},\frac{1+b-c+\min(b,1-c)}{3}\rangle$ |
| $\vee_{3,145}$ | $\langle \frac{a+c+\min(a,c)}{3},\frac{3-a-c-\min(a,c)}{3}\rangle$ |
| $\wedge_{3,145}$ | $\langle \frac{3-b-d-\min(b,d)}{3},\frac{b+d+\min(b,d)}{3}\rangle$ |
| $\vee_{3,146}$ | $\langle \frac{3-b-d-\min(b,d)}{3},\frac{b+d+\min(b,d)}{3}\rangle$ |
| $\wedge_{3,146}$ | $\langle \frac{a+c+\min(a,c)}{3},\frac{3-a-c-\min(a,c)}{3}\rangle$ |
| $\vee_{3,147}$ | $\langle \frac{1-b+c+\max(1-b,c)}{3},\frac{2+b-c-\max(1-b,c)}{3}\rangle$ |
| $\wedge_{3,147}$ | $\langle \frac{2+a-d-\max(1-a,d)}{3},\frac{1-a+d+\max(1-a,d)}{3}\rangle$ |
| $\vee_{3,148}$ | $\langle \frac{1+a-d+\max(a,1-d)}{3},\frac{2-a+d-\max(a,1-d)}{3}\rangle$ |
| $\wedge_{3,148}$ | $\langle \frac{2-b+c-\max(b,1-c)}{3},\frac{1+b-c+\max(b,1-c)}{3}\rangle$ |
| $\vee_{3,149}$ | $\langle \frac{a+c+\max(a,c)}{3},\frac{3-a-c-\max(a,c)}{3}\rangle$ |
| $\wedge_{3,149}$ | $\langle \frac{3-b-d-\max(b,d)}{3},\frac{b+d+\max(b,d)}{3}\rangle$ |
| $\vee_{3,150}$ | $\langle \frac{a+c+\lambda-1}{2\lambda},\frac{b+d+\lambda-1}{2\lambda}\rangle$ |
| $\wedge_{3,150}$ | $\langle \frac{a+c+\lambda-1}{2\lambda},\frac{b+d+\lambda-1}{2\lambda}\rangle$ |

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| $\vee_{3,151}$ | $\langle \frac{a+c+\gamma}{2\gamma+1}, \frac{b+d+\gamma-1}{2\gamma+1} \rangle$ |
| $\wedge_{3,151}$ | $\langle \frac{b+d+\gamma}{2\gamma+1}, \frac{a+c+\gamma-1}{2\gamma+1} \rangle$ |
| $\vee_{3,152}$ | $\langle \frac{a+c+\alpha-1}{\alpha+\beta}, \frac{b+d+\beta-1}{\alpha+\beta} \rangle$ |
| $\wedge_{3,152}$ | $\langle \frac{a+c+\beta-1}{\alpha+\beta}, \frac{b+d+\alpha-1}{\alpha+\beta} \rangle$ |
| $\vee_{3,153}$ | $\langle \min(1, \max(c, a+\varepsilon)), \max(0, \min(d, b-\eta)) \rangle$ |
| $\wedge_{3,153}$ | $\langle \max(0, \min(c, a-\eta)), \min(1, \max(d, b+\varepsilon)) \rangle$ |
| $\vee_{3,154}$ | $\langle \frac{-b+c+\lambda}{2\lambda}, \frac{b-c+\lambda}{2\lambda} \rangle$ |
| $\wedge_{3,154}$ | $\langle \frac{a-d+\lambda}{2\lambda}, \frac{-a+d+\lambda}{2\lambda} \rangle$ |
| $\vee_{3,155}$ | $\langle \frac{1-b-d+\lambda}{2\lambda}, \frac{b+d+\lambda-1}{2\lambda} \rangle$ |
| $\wedge_{3,155}$ | $\langle \frac{1-a-c+\lambda}{2\lambda}, \frac{a+c+\lambda-1}{2\lambda} \rangle$ |
| $\vee_{3,156}$ | $\langle \frac{a+c+\lambda-1}{2\lambda}, \frac{1-a-c+\lambda}{2\lambda} \rangle$ |
| $\wedge_{3,156}$ | $\langle \frac{1-b-d+\lambda}{2\lambda}, \frac{b+d+\lambda-1}{2\lambda} \rangle$ |
| $\vee_{3,157}$ | $\langle \frac{a-d+\lambda}{2\lambda}, \frac{-a+d+\lambda}{2\lambda} \rangle$ |
| $\wedge_{3,157}$ | $\langle \frac{-b+c+\lambda}{2\lambda}, \frac{b-c+\lambda}{2\lambda} \rangle$ |
| $\vee_{3,158}$ | $\langle \frac{1-b+c+\gamma}{2\gamma+1}, \frac{b-c+\gamma}{2\gamma+1} \rangle$ |
| $\wedge_{3,158}$ | $\langle \frac{a-d+\gamma}{2\gamma+1}, \frac{1-a+d+\gamma}{2\gamma+1} \rangle$ |
| $\vee_{3,159}$ | $\langle \frac{2-b-d+\gamma}{2\gamma+1}, \frac{b+d+\gamma-1}{2\gamma+1} \rangle$ |
| $\wedge_{3,159}$ | $\langle \frac{a+c+\gamma-1}{2\gamma+1}, \frac{2-a-c+\gamma}{2\gamma+1} \rangle$ |
| $\vee_{3,160}$ | $\langle \frac{a-d+\gamma+1}{2\gamma+1}, \frac{-a+d+\gamma}{2\gamma+1} \rangle$ |
| $\wedge_{3,160}$ | $\langle \frac{-b+c+\gamma}{2\gamma+1}, \frac{b-c+\gamma+1}{2\gamma+1} \rangle$ |
| $\vee_{3,161}$ | $\langle \frac{a+c+\gamma}{2\gamma+1}, \frac{1-a-c+\gamma}{2\gamma+1} \rangle$ |
| $\wedge_{3,161}$ | $\langle \frac{1-b-d+\gamma}{2\gamma+1}, \frac{b+d+\gamma}{2\gamma+1} \rangle$ |
| $\vee_{3,162}$ | $\langle \frac{-b+c+\alpha}{\alpha+\beta}, \frac{b-c+\beta}{\alpha+\beta} \rangle$ |
| $\wedge_{3,162}$ | $\langle \frac{a-d+\beta}{\alpha+\beta}, \frac{-a+d+\alpha}{\alpha+\beta} \rangle$ |
| $\vee_{3,163}$ | $\langle \frac{1-b-d+\alpha}{\alpha+\beta}, \frac{b+d+\beta-1}{\alpha+\beta} \rangle$ |
| $\wedge_{3,163}$ | $\langle \frac{a+c+\beta-1}{\alpha+\beta}, \frac{1-a-c+\alpha}{\alpha+\beta} \rangle$ |
| $\vee_{3,164}$ | $\langle \frac{a-d+\alpha}{\alpha+\beta}, \frac{-a+d+\beta}{\alpha+\beta} \rangle$ |
| $\wedge_{3,164}$ | $\langle \frac{-b+c+\beta}{\alpha+\beta}, \frac{b-c+\alpha}{\alpha+\beta}, \rangle$ |
| $\vee_{3,165}$ | $\langle \frac{a+c+\alpha-1}{\alpha+\beta}, \frac{1-a-c+\beta}{\alpha+\beta} \rangle$ |
| $\wedge_{3,165}$ | $\langle \frac{1-b-d+\beta}{\alpha+\beta}, \frac{b+d+\alpha-1}{\alpha+\beta}, \rangle$ |
| $\vee_{3,166}$ | $\langle \max(a, \min(b, c)), \min(b, \max(a, d)) \rangle$ |
| $\wedge_{3,166}$ | $\langle \min(a, \max(b, c)), \max(b, \min(a, d)) \rangle$ |
| $\vee_{3,167}$ | $\langle \max(1-b, \min(b, c)), \min(b, 1-\min(b, c)) \rangle$ |
| $\wedge_{3,167}$ | $\langle \min(a, 1-\min(a, d)), \max(1-a, \min(a, d)) \rangle$ |
| $\vee_{3,168}$ | $\langle \max(1-b, \min(b, 1-d)), 1-\max(1-b, \min(b, 1-d)) \rangle$ |
| $\wedge_{3,168}$ | $\langle 1-\max(1-a, \min(a, c)), \max(1-a, \min(a, 1-c)) \rangle$ |
| $\vee_{3,169}$ | $\langle \max(a, \min(1-a, c)), 1-\max(a, \min(1-a, c)) \rangle$ |
| $\wedge_{3,169}$ | $\langle 1-\max(b, \min(1-b, d)), \max(b, \min(1-b, d)) \rangle$ |
| $\vee_{3,170}$ | $\langle \max(a, \min(1-a, 1-d)), 1-\max(a, \min(1-a, 1-d)) \rangle$ |

| | |
|---|---|
| $\wedge_{3,170}$ | $\langle 1-\max(b,\min(1-b,1-c)),\max(b,\min(1-b,1-c))\rangle$ |
| $\vee_{3,171}$ | $\langle \overline{\mathrm{sg}}(\max(b,d)-\max(a,c)),\mathrm{sg}(\max(b,d)-\max(a,c))\rangle$ |
| $\wedge_{3,171}$ | $\langle \mathrm{sg}(\max(a,c)-\max(b,d)),\overline{\mathrm{sg}}(\max(a,c)-\max(b,d)),\rangle$ |
| $\vee_{3,172}$ | $\langle \overline{\mathrm{sg}}(b-c),\mathrm{sg}(b-c)\rangle$ |
| $\wedge_{3,172}$ | $\langle \mathrm{sg}(a-d),\overline{\mathrm{sg}}(a-d)\rangle$ |
| $\vee_{3,173}$ | $\langle \overline{\mathrm{sg}}(b+d-1),\mathrm{sg}(b+d-1)\rangle$ |
| $\wedge_{3,173}$ | $\langle \mathrm{sg}(a+c-1),\overline{\mathrm{sg}}(a+c-1),\rangle$ |
| $\vee_{3,174}$ | $\langle \overline{\mathrm{sg}}(1-a-c),\mathrm{sg}(1-a-c)\rangle$ |
| $\wedge_{3,174}$ | $\langle \mathrm{sg}(1-b-d),\overline{\mathrm{sg}}(1-b-d)\rangle$ |
| $\vee_{3,175}$ | $\langle \overline{\mathrm{sg}}(d-a),\mathrm{sg}(d-a)\rangle$ |
| $\wedge_{3,175}$ | $\langle \mathrm{sg}(c-b),\overline{\mathrm{sg}}(c-b)\rangle$ |
| $\vee_{3,176}$ | $\langle \overline{\mathrm{sg}}(b-c)+\mathrm{sg}(b-c)\max(a,c),\mathrm{sg}(b-c)\min(b,d)\rangle$ |
| $\wedge_{3,176}$ | $\langle \mathrm{sg}(a-d)\min(a,c),\overline{\mathrm{sg}}(a-d)+\mathrm{sg}(a-d)\max(b,d)\rangle$ |
| $\vee_{3,177}$ | $\langle \overline{\mathrm{sg}}(b-c)+\mathrm{sg}(b-c)\max(1-b,c),\mathrm{sg}(b-c)\min(b,1-c)\rangle$ |
| $\wedge_{3,177}$ | $\langle \mathrm{sg}(a-d)\min(a,1-d),\overline{\mathrm{sg}}(a-d)+\mathrm{sg}(a-d)\max(1-a,d)\rangle$ |
| $\vee_{3,178}$ | $\langle \overline{\mathrm{sg}}(b-1+d)+\mathrm{sg}(b-1+d)(1-\min(b,d)),$ <br> $\mathrm{sg}(b-1+d)\min(b,d)\rangle$ |
| $\wedge_{3,178}$ | $\langle \mathrm{sg}(a-1+c)\min(a,c),$ <br> $\overline{\mathrm{sg}}(a-1+c)+\mathrm{sg}(a-1+c)(1-\min(a,c))\rangle$ |
| $\vee_{3,179}$ | $\langle \overline{\mathrm{sg}}(1-a-c)+\mathrm{sg}(1-a-c)\max(a,c),$ <br> $\mathrm{sg}(1-a-c)(1-\max(a,c))\rangle$ |
| $\wedge_{3,179}$ | $\langle \mathrm{sg}(1-b-d)(1-\max(b,d)),$ <br> $\overline{\mathrm{sg}}(1-b-d)+\mathrm{sg}(1-b-d)\max(b,d)\rangle$ |
| $\vee_{3,180}$ | $\langle \overline{\mathrm{sg}}(d-1+a)+\mathrm{sg}(d-1+a)\max(1-a,1-d),$ <br> $\mathrm{sg}(d-1+a)\min(a,d)\rangle$ |
| $\wedge_{3,180}$ | $\langle \mathrm{sg}(c-b)\min(1-b,c),\overline{\mathrm{sg}}(c-b)+\mathrm{sg}(c-b)\max(b,1-c)\rangle$ |
| $\vee_{3,181}$ | $\langle 1-\mathrm{sg}(b).(1-c),d.\mathrm{sg}(b)\rangle$ |
| $\wedge_{3,181}$ | $\langle c\mathrm{sg}(a),1-\mathrm{sg}(a).(1-d)\rangle$ |
| $\vee_{3,182}$ | $\langle 1-\mathrm{sg}(b)(1-c),(1-c).\mathrm{sg}(b)\rangle$ |
| $\wedge_{3,182}$ | $\langle (1-d).\mathrm{sg}(a),1-\mathrm{sg}(a).(1-d)\rangle$ |
| $\vee_{3,183}$ | $\langle 1-d\mathrm{sg}(b),d\mathrm{sg}(b)\rangle$ |
| $\wedge_{3,183}$ | $\langle c.\mathrm{sg}(a),1-c.\mathrm{sg}(a)\rangle$ |
| $\vee_{3,184}$ | $\langle 1-d\mathrm{sg}(1-a),d\mathrm{sg}(1-a)\rangle$ |
| $\wedge_{3,184}$ | $\langle c\mathrm{sg}(1-b),1-c\mathrm{sg}(1-b)\rangle$ |
| $\vee_{3,185}$ | $\langle 1-(1-c)\mathrm{sg}(1-a),(1-c)\mathrm{sg}(1-a)\rangle$ |
| $\wedge_{3,185}$ | $\langle (1-d)\mathrm{sg}(1-b),1-(1-d)\mathrm{sg}(1-b)\rangle$ |

# 3 Conclusion

With constructed conjunctions and disjunctions we will consider algebraic structures such as monoids, groups and lattices and study their properties.


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