20th ICIFS, 2–3 September 2016, Sofia, Bulgaria
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 22, 2016, No. 3, 1–8

New Fodor's type of intuitionistic fuzzy implication and negation

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Received: 30 May 2016 **Accepted:** 2 July 2016

Abstract: A new Fodor's type of intuitionistic fuzzy implication is constructed. Its relation with some forms of Klir and Yuan's axioms, of intuitionistic logoc axioms, of Kolmogorov's axioms and of Łukasiewicz-Tarski's axioms are studied.
Keywords: Implication, Intuitionistic fuzzy logic, Intuitionistic logic, Negation.
AMS Classification: 03E72.

In memory of Prof. János Fodor (1956–2016)

1 Introduction

The concept of *"intuitionistic fuzzy propositional calculus"* has been introduced about 20 years ago (see, e.g., [1, 2]). Initially, it contained only one form of conjunction, disjunction and two forms of implication. In a series of papers, other forms of these three operations were defined.

Now, there are 4 forms of operations conjunction and disjunction, 185 forms of operation implication and 53 forms of operation negation, presented from a perspective of fulfilling some criteria. (see, e.g. [5, 4]).

In [7, 12], the authors introduced the first Fodor's type of intuitionistic fuzzy implication and studied their properties. In [13], they introduced 4 modifications of this implication.

Here, we introduce a new modification of operation implication and study some of its properties and formulate open problems, related to the operations of intuitionistic fuzzy propositional calculus. Our research is based on Janos Fodor's fuzzy implication (see, e.g., [9]), that for $a, c \in [0, 1]$ is defined by

$$a \to c = \begin{cases} 1, & \text{if } a \leq c \\ \max(1-a, c), & \text{otherwise} \end{cases} .$$

The first Fodor's type of intuitionistic fuzzy implication (see [7, 12]) has the form

$$V(x \to y) = \langle a, b\rangle \to \langle c, d\rangle = \langle \overline{\text{sg}}(a-c) + \text{sg}(a-c)\max(b, c), \text{sg}(a-c)\min(a, d)\rangle,$$

where we use functions sg and $\overline{\text{sg}}$ defined by

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases} .$$

In [5], this impication was numbered as $\to_{176}$. There, the four modifications of it were numbered as $\to_{177}$, ..., $\to_{180}$. They are obtained using the following formulas:

$$V(\langle a, b\rangle \to_{177} \langle c, d\rangle) = \Box\langle a, b\rangle \to_{176} \Box\langle c, d\rangle,$$

$$V(\langle a, b\rangle \to_{178} \langle c, d\rangle) = \Box\langle a, b\rangle \to_{176} \Diamond\langle c, d\rangle,$$

$$V(\langle a, b\rangle \to_{179} \langle c, d\rangle) = \Diamond\langle a, b\rangle \to_{176} \Diamond\langle c, d\rangle,$$

$$V(\langle a, b\rangle \to_{180} \langle c, d\rangle) = \Diamond\langle a, b\rangle \to_{176} \Box\langle c, d\rangle.$$

So, we obtained the explicit forms of these four implications as follows:

$$V(\langle a, b\rangle \to_{177} \langle c, d\rangle) = \langle \overline{\text{sg}}(a-c) + \text{sg}(a-c)\max(1-a, c), \text{sg}(a-c)\min(a, 1-c)\rangle,$$

$$V(\langle a, b\rangle \to_{178} \langle c, d\rangle) = \langle \overline{\text{sg}}(a-1+d) + \text{sg}(a-1+d)(1-\min(a, d)), \text{sg}(a-1+d)\min(a, d)\rangle,$$

$$V(\langle a, b\rangle \to_{179} \langle c, d\rangle) = \langle \overline{\text{sg}}(1-b-c) + \text{sg}(1-b-c)\max(b, c), \text{sg}(1-b-c)(1-\max(b, c))\rangle,$$

$$V(\langle a, b\rangle \to_{180} \langle c, d\rangle) = \langle \overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b, 1-d), \text{sg}(d-b)\min(1-b, d)\rangle.$$

In intuitionistic fuzzy propositional calculus, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b\rangle,$$

so that $a, b, a+b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. In [6], we called this couple an *"intuitionistic fuzzy pair" (IFP).*

Below we shall assume that for the two variables x and y the equalities: $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$ $(a, b, c, d, a+b, c+d \in [0, 1])$ hold.

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1]) by:

$$x \text{ is an IFT if and only if for } V(x) = \langle a, b\rangle \text{ holds: } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0\rangle$.

2 Main results

Now, we discuss one of the ways for generating of new implications, used for a first time in [3]. We use operation "substitution in the following form for a given propositional form $f(a, ..., b, x, c, ..., d)$:

$$g(a, ..., b, y, c, ..., d) = \frac{x}{y} f(a, ..., b, x, c, ..., d),$$

i.e., $g(a, ..., b, y, c, ..., d)$ coincides with $f(a, ..., b, x, c, ..., d)$ with exception of the participations of variable x that is changed to variable y. For example,

$$\frac{x}{y}(a + x) = a + y.$$

It is very important to note that the changes are made simultaneously, i.e., they change all instances of y to x. On the other hand, if we have to change variable x to variable y, and variable y to variable x, then we will also do it simultaneously. For example

$$\frac{x}{y}\frac{y}{x}(a + x - y) = a + y - x.$$

All above implications generate new implications by operation "substitution" with the forms:

$$V(x \to'_{176} y) = \frac{a}{d}\frac{b}{c}\frac{c}{b}\frac{d}{a}\langle a, b\rangle \to_{176} \langle c, d\rangle$$
$$= \langle \overline{\text{sg}}(d - b) + \text{sg}(d - b)\max(b, c), \text{sg}(d - b)\min(a, d)\rangle,$$
$$V(x \to'_{177} y) = \frac{a}{d}\frac{b}{c}\frac{c}{b}\frac{d}{a}\langle a, b\rangle \to_{177} \langle c, d\rangle$$
$$= \langle \overline{\text{sg}}(d - b) + \text{sg}(d - b)\max(1 - d, b), \text{sg}(d - b)\min(d, 1 - b)\rangle,$$
$$V(x \to'_{178} y) = \frac{a}{d}\frac{b}{c}\frac{c}{b}\frac{d}{a}\langle a, b\rangle \to_{178} \langle c, d\rangle$$
$$= \langle \overline{\text{sg}}(a - 1 + d) + \text{sg}(a - 1 + d)(1 - \min(a, d)), \text{sg}(a - 1 + d)\min(a, d)\rangle,$$
$$V(x \to'_{179} y) = \frac{a}{d}\frac{b}{c}\frac{c}{b}\frac{d}{a}\langle a, b\rangle \to_{179} \langle c, d\rangle$$
$$= \langle \overline{\text{sg}}(1 - b - c) + \text{sg}(1 - b - c)\max(b, c), \text{sg}(1 - b - c)(1 - \max(b, c))\rangle,$$
$$V(x \to'_{180} y) = \frac{a}{d}\frac{b}{c}\frac{c}{b}\frac{d}{a}\langle a, b\rangle \to_{180} \langle c, d\rangle$$
$$= \langle \overline{\text{sg}}(a - c) + \text{sg}(a - c)\max(c, 1 - a), \text{sg}(a - c)\min(1 - c, a)\rangle.$$

We see immediately, that

$$V(x \to'_{177} y) = V(x \to_{180} y),$$

$$V(x \to'_{178} y) = V(x \to_{178} y),$$
$$V(x \to'_{179} y) = V(x \to_{179} y),$$
$$V(x \to_{180} y) = V(x \to_{177} y)$$

and only implication $\to'_{176}$ does not coincide with the already existing implications from [5]. So, here, we discuss only it, assigning it number $\to_{186}$, i.e., implication $\to_{186}$ will coincide with implication $\to'_{176}$.

First, let

$$X \equiv \overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b,c) + \text{sg}(d-b)\min(a,d).$$

If $d \leq b$, then

$$X = 1 + 0 + 0 = 1.$$

If $d > b$, then

$$X = 0+1.\max(b,c)+1.\min(a,d) \leq \max(b,c)+\min(1-b,1-c) = \max(b,c)+1-\max(b,c) = 1.$$

Second, when $b = 1-a$ and $d = 1-c$, we obtain:

$$\langle a, 1-a\rangle \to_{186} \langle c, 1-d\rangle = \langle \overline{\text{sg}}(1-c-1+a) + \text{sg}(1-c-1+a)\max(1-a,c),$$
$$\text{sg}(1-c-1+a)\min(a,1-c)\rangle$$
$$= \langle \overline{\text{sg}}(a-c) + \text{sg}(a-c)\max(1-a,c), \text{sg}(a-c)\min(a,1-c)\rangle = \langle a,b\rangle \to_{177} \langle c,d\rangle.$$

Third, we check that

$$\langle 0,1\rangle \to_{186} \langle 0,1\rangle = \langle 1,0\rangle,$$
$$\langle 0,1\rangle \to_{186} \langle 1,0\rangle = \langle 1,0\rangle,$$
$$\langle 1,0\rangle \to_{186} \langle 0,1\rangle = \langle 0,1\rangle,$$
$$\langle 1,0\rangle \to_{186} \langle 1,0\rangle = \langle 1,0\rangle,$$

i.e., the new implication has the behaviour of the standard classical logic implication.

Fourth, we see that implication $\to_{186}$ generates the following negation using formula

$$\neg\langle a,b\rangle = \langle a,b\rangle \to \langle 0,1\rangle.$$

In a result, we obtain

$$\neg\langle a,b\rangle = \langle a,b\rangle \to_{186} \langle 0,1\rangle$$
$$\langle \overline{\text{sg}}(1-b) + \text{sg}(1-b)b, \text{sg}(1-b)a\rangle = \langle b,a\rangle,$$

because

$$\overline{\text{sg}}(1-b) + \text{sg}(1-b)b = \begin{cases} 1+0.b = 1 = b, & \text{if } b = 1 \\ 0+1.b = b, & \text{if } b < 1 \end{cases} = b$$

and

$$\text{sg}(1-b)a = \begin{cases} 0.a = 0 = a, & \text{if } b = 1, \text{ i.e., } a = 0 \\ 1.a = a, & \text{if } b < 1 \end{cases} = a.$$

Therefore, the negation, generated by implication $\rightarrow_{186}$ is the classical intuitionistic fuzzy negation $\neg_1$.

Fifth, we discuss other properties of our new intuitionistic fuzzy implication, taking into account the classic Georg Klir and Bo Yuan's book [9] that is relevant to our purposes. However, a similar, if not practically identical, analyses can be performed in the new settings and views related to fuzzy implications, notably included in the Baczynksi and Jayaram's book [8].

Some variants of fuzzy implications (marked by $I(x, y)$) are described in [9] and the following nine axioms are discussed, where

$$I(x, y) \equiv x \rightarrow y.$$

Axiom 1 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z)))$.
Axiom 2 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y)))$.
Axiom 3 $(\forall y)(I(0, y) = 1)$.
Axiom 4 $(\forall y)(I(1, y) = y)$.
Axiom 5 $(\forall x)(I(x, x) = 1)$.
Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7 $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$.
Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is one of the operations for negation.
Axiom 9 I is a continuous function.

In some research of the authors, some of these axioms are changed with the following new ones:
Axiom 3* $(\forall y)I(0, y)$ is an IFT.
Axiom 5* $(\forall x)I(x, x)$ is an IFT.
Axiom 7* $(\forall x, y)($ if $x \leq y$, then, $I(x, y) = 1)$,
Axiom 8* $(\forall x, y)(I(x, y) = N(N(I(N(y), N(x)))))$.

Theorem 1. The new implication satisfies Axioms 1, 2, 3, 3*, 5, 5*, 7* as tautologies (and therefore, as IFTs).

The proof of this and next assertions are checked in the above manner, and by this reason will be omitted.

The intuitionistic logic axioms (see, e.g., [10]) are the following

(IL1) $A \rightarrow A$,

(IL2) $A \rightarrow (B \rightarrow A)$,

(IL3) $A \rightarrow (B \rightarrow (A \wedge B))$,

(IL4) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,

(IL5) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

(IL6) $A \rightarrow \neg\neg A$,

(IL7) $\neg(A \wedge \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \wedge \neg B)$,

(IL10) $(\neg A \wedge \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \wedge B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 2. Implication $\to_{186}$ satisfies axioms (IL1), (IL2), (IL3), (IL6), (IL8), (IL9), (IL10), (IL11), (IL14), (IL15), (IL16) as tautologies.

Theorem 3. Implication $\to_{186}$ satisfies axioms (IL1), (IL2), (IL3), (IL4), (IL6), (IL7), ..., (IL16) as IFTs.

Kolmogorov's axioms are:

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 4. Implication $\to_{186}$ satisfies axioms (K1), (K2), (K3), (K4) as tautologies (and as IFTs).

Łukasiewicz–Tarski's axioms are

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 5. Implication $\rightarrow_{186}$ satisfies axioms (LT1) and (LT3) as tautologies.

Theorem 6. Implication $\rightarrow_{186}$ satisfies axioms all four axioms IFTs.

3 Conclusion

In a next author's research, other new implications will be introduced and studied. All they show that intuitonistic fuzzy sets and logics in the sense, described in [2, 4] correspond to the ideas of Brouwer's intuitionism [11].

It is important to note that the search of new implications and negations is important for constructing of rules for multicriteria and intercriteria analyses and theie evaluations.

Acknowledgements

The author is thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DFNI-I-02-5 "InterCriteria Analysis: A New Approach to Decision Making".

References

[1] Atanassov, K. (1988) Two Variants of Intuitionistic Fuzzy Propositional Calculus, *Mathematical Foundations of Artificial Intelligence Seminar*, Sofia, 1988, Preprint IM-MFAIS-5-88. Reprinted: *Int J Bioautomation*, 2016, 20(S1), S17–S26.

[2] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer, Heidelberg.

[3] Atanassov, K. (2006) On the implications and negations over intuitionistic fuzzy sets. *Proceedings of the Scientific Conference of Burgas Free University*, Burgas, 9–11 June 2006, Vol. III, 374–384.

[4] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[5] Atanassov, K. (2016) On intuitionistic fuzzy implications, *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, 12 (in press).

[6] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013) On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[7] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2015) On Fodor's type of intuitionistic fuzzy implication and negation, *Notes on Intuitionistic Fuzzy Sets*, 21(2), 25–34.

[8] Baczynski, M., & Jayaram, B. (2008) *Fuzzy Implications*, Springer, Berlin.

[9] Klir, G. & Yuan, B. (1995) *Fuzzy Sets and Fuzzy Logic*. Prentice Hall, New Jersey.

[10] Rasiovam H., & Sikorski R. (1963) *The mathematics of Metamathematics*, Polish Academy of Sciences, Warszawa.

[11] Van Stigtm W. P. (1990) *Brouwer's Intuitionism*, North-Holland, Amsterdam.

[12] Szmidt, E., Kacprzyk, J., & Atanassov, K. (2015) Properties of Fodor's intuitionistic fuzzy implication and negation. *Notes on Intuitionistic Fuzzy Sets*, 21(4), 6–12.

[13] Szmidt, E., Kacprzyk, J., & Atanassov, K. (2015) Modal forms of Fodor's type of intuitionistic fuzzy implication. *Notes on Intuitionistic Fuzzy Sets*, 21(5), 1–5.