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The inclusion-exclusion principle for product operations on IF-events

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Abstract: The paper contains an extension of inclusion-exclusion principle published in [3] for a larger set of probabilities. A new method of the proof is presented as well.
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1 Introduction

Let us have a finite number of sets $A_i,\ i \in N$. The probability of their union is given by a formula

$$\mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i) - \sum_{i<j}^{n} \mathcal{P}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathcal{P}\left(\bigcap_{i=1}^{n} A_i\right),$$

which is also called 'the inclusion-exclusion principle'. As it was shown in [3], after defining the union and intersection, respectively other operations and a probability, we can simply replace the word 'set' by an 'IF-set'. Aim of this paper is to simplify the proof of Grzegorzewski and to generalize the inclusion-exslusion principle for larger set of probabilities.

May (Ω, S) be a measurable space, where Ω is a nonempty set and S is a σ-algebra of subsets of Ω. By IF-set (see [1]) we mean each pair

$$A = (\mu_A, \nu_A),$$

where $\mu_A, \nu_A : \Omega \to [0, 1]$ and following condition is satisfied:

$$\mu_A + \nu_A \leq 1.$$

μ_A is called a membership function, ν_A is a non-membership function. In addition, if μ_A and ν_A are Borel measurable, i. e.

$$I \subset R \text{ is an interval} \Rightarrow \mu_A^{-1}(I) \in S,\ \nu_A^{-1}(I) \in S,$$

then A is an IF-event. The union and intersection of IF-sets are defined by this way:

$$A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),\ A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B).$$

Let us denote the family of all IF-events by $\mathcal{F}$. We will use Łukasiewicz connectives for $A, B \in \mathcal{F}$:

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$
$$A \odot B = ((\nu_A + \nu_B - 1) \vee 0, (\mu_A + \mu_B) \wedge 1).$$

A partial ordering on $\mathcal{F}$ is defined by the formula

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B,\ \nu_A \geq \nu_B.$$

Evidently

$(0_\Omega, 1_\Omega)$ is the least element of $(\mathcal{F}, \leq)$,
$(1_\Omega, 0_\Omega)$ is the greatest element of $(\mathcal{F}, \leq)$.

The following definition comes from quantum theory ([2]).

Definition 1.1. *A mapping $m : \mathcal{F} \to [0, 1]$ is called a state of the following properties are satisfied:*

1. $m((1_\Omega, 0_\Omega)) = 1,\ m((0_\Omega, 1_\Omega)) = 0$,
2. $A \odot B = (0_\Omega, 1_\Omega) \Rightarrow m((A \oplus B)) = m(A) + m(B)$,
3. $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$,

$\forall A, B, A_i \in \mathcal{F}\ (i = 1, \ldots, n)$.

Grzegorzewski ([3]) defined a probability of an IF-event A as an interval

$$\mathcal{P}(A) = \left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP \right], \tag{1}$$

where P is a probability measure over Ω. More general, axiomatic approach to probability on IF-events was created by Riečan ([5]):

Definition 1.2. *A mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, where $\mathcal{J} = \{[a, b]; a, b \in R, a \leq b\}$ is called a probability if the following conditions hold:*

1. $\mathcal{P}((1_\Omega, 0_\Omega)) = [1, 1],\ \mathcal{P}((0_\Omega, 1_\Omega)) = [0, 0]$,
2. $A \odot B = (0_\Omega, 1_\Omega) \Rightarrow \mathcal{P}((A \oplus B)) = \mathcal{P}(A) + \mathcal{P}(B)$,
3. $A_n \nearrow A \Rightarrow \mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

Of course, $\mathcal{P}(A)$ is an interval on R, let us denote it by

$$\mathcal{P}(A) = \left[\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)\right].$$

It is easy to see, that the following proposition holds:

Proposition 1.3. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be defined by $\mathcal{P}(A) = \left[\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)\right]$. Then $\mathcal{P}$ is a probability if and only if $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ are states.*

Hence, we will be interested in states in this paper. In [2], Ciungu and Riečan have proved the following theorem (see also [6]):

Theorem 1.4. *For any state $m : \mathcal{F} \to [0,1]$ there exist probability measures $P,\ Q : S \to [0,1]$ and $\alpha \in [0,1]$ such that*

$$m((\mu_A, \nu_A)) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right).$$

This Theorem will be used as a main idea of our proof.

Corollary 1.5. *If*

$$\mathcal{P}^\flat(A) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right),$$

$$\mathcal{P}^\sharp(A) = \int_\Omega \mu_A dR + \beta \left(1 - \int_\Omega (\mu_A + \nu_A) dS\right)$$

and we put $Q = R = S$, $\alpha = 0$ and $\beta = 1$, we obtain the Grzegorzewski definition of probability.

2 The inclusion-exclusion principle

It is natural to start with a probability of an union and intersection of IF-events, but this was already done in [3] and [4]. We will only summarize the main results without proofs here.

Theorem 2.1. *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \dots, n$. Let m be a state and $m((\mu_A, \nu_A)) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right)$ $\forall A \in \mathcal{F}$. Then m satisfies the inclusion-exclusion principle, i. e.*

$$m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i) - \sum_{i<j}^n m(A_i \cap A_j) + \ldots + (-1)^{n+1} m\left(\bigcap_{i=1}^n A_i\right).$$

The proof can be found in [4].

Corollary 2.2. *From (1.3) and (2.1) we get the assertion*

$$\mathcal{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathcal{P}(A_i) - \sum_{i<j}^n \mathcal{P}(A_i \cap A_j) + \ldots + (-1)^{n+1} \mathcal{P}\left(\bigcap_{i=1}^n A_i\right).$$

Further, we will work with so called product operations between IF-events:

$$A + B = (\mu_A + \mu_B - \mu_A \mu_B, \nu_A \nu_B),$$

$$A \cdot B = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B).$$

We will use a notation:

$$\sum_{i=1}^n A_i = A_1 + A_2 + \ldots + A_n,$$

$$\prod_{i=1}^{n} A_i = A_1 \cdot A_2 \cdot \ldots \cdot A_n.$$

Then the following theorem holds:

Theorem 2.3. *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \ldots, n$. Let m be a state and $m((\mu_A, \nu_A)) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right)$ $\forall A \in \mathcal{F}$. Then m satisfies the inclusion-exclusion principle, i. e.*

$$m\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i \cdot A_j) + \ldots + (-1)^{n+1} m\left(\prod_{i=1}^{n} A_i\right).$$

Proof. To make the notation more simple, let us denote

$$a \boxplus b = a + b - ab.$$

$$a \boxdot b = ab, \ \forall a, b \in \mathbb{R}.$$

Hence, we can write

$$A + B = (\mu_A \boxplus \mu_B, \nu_A \boxdot \nu_B),$$

$$A \cdot B = (\mu_A \boxdot \mu_B, \nu_A \boxplus \nu_B).$$

Moreover, operations $\boxplus$ and $\boxdot$ are both associative, so for a finite number of IF-sets $A_1, \ldots, A_n$ we can write

$$\sum_{i=1}^{n} A_i = (\boxplus_{i=1}^{n} \mu_{A_i}, \boxdot_{i=1}^{n} \nu_{A_i}),$$

$$\prod_{i=1}^{n} A_i = A_i(\boxdot_{i=1}^{n} \mu_{A_i}, \boxplus_{i=1}^{n} \nu_{A_i}).$$

One can easily see, that there holds

$$\boxplus_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i<j}^{n} a_i \boxdot a_j + \ldots (-1)^{n+1} \boxdot_{i=1}^{n} a_i, \tag{2}$$

$$\boxdot_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i<j}^{n} a_i \boxplus a_j + \ldots (-1)^{n+1} \boxplus_{i=1}^{n} a_i, \tag{3}$$

$$\forall a_i \in \mathbb{R}, \ i = 1, \ldots, n, \ n \in \mathbb{N}.$$

Seeing that the values of the membership and non-membership function are real numbers too, we can use the formulas (2) and (3) for them as well.

We will need the following equality:

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n},$$

and hence

$$1 = \binom{n}{1} - \binom{n}{2} + \ldots + (-1)^{n+1} \binom{n}{n}.$$

That's why we will sometimes write $\begin{pmatrix} n \\ 1 \end{pmatrix}$ instead of simple n. Let us have a look at individual summands on the right site of the assertion.

$$\sum_{i=1}^{n} m(A_i) = \sum_{i=1}^{n} \int_{\Omega} \mu_{A_i} dP + \sum_{n=1}^{n} \alpha \left(1 - \int_{\Omega} (\mu_{A_i} + \nu_{A_i}) dQ\right)$$

$$\sum_{i=1}^{n} m(A_i) = \int_{\Omega} \left(\sum_{i=1}^{n} \mu_{A_i}\right) dP + \alpha \left(\begin{pmatrix} n \\ 1 \end{pmatrix} - \int_{\Omega} \left(\sum_{n=1}^{n} (\mu_{A_i} + \nu_{A_i})\right) dQ\right)$$

Similarly

$$\sum_{i<j}^{n} m(A_i \cdot A_j) = \int_{\Omega} \left(\sum_{i<j}^{n} \mu_{A_i} \boxdot \mu_{A_j}\right) dP + \alpha \left(\begin{pmatrix} n \\ 2 \end{pmatrix} - \int_{\Omega} \left(\sum_{i<j}^{n} (\mu_{A_i} \boxdot \mu_{A_j} + \nu_{A_i} \boxplus \nu_{A_j})\right) dQ\right)$$

$$\vdots$$

$$m\left(\prod_{i=1}^{n} A_i\right) = \int_{\Omega} (\boxdot_{i=1}^{n} \mu_{A_i})\, dP + \alpha \left(\begin{pmatrix} n \\ n \end{pmatrix} - \int_{\Omega} (\boxdot_{i=1}^{n} \mu_{A_i} + \boxplus_{i=1}^{n} \nu_{A_i})\, dQ\right)$$

Put $A = \sum_{i=1}^{n} A_i$ and sum the previous equalities, multiplying every second one by -1 (to get the right site of the assertion).

$$\sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i \cdot A_i) + \ldots + (-1)^{n+1} m\left(\prod_{i=1}^{n} A_i\right) =$$

$$= \int_{\Omega} \left(\sum_{i=1}^{n} \mu_{A_i} - \sum_{i<j}^{n} \mu_{A_i} \boxdot \mu_{A_j} + \ldots + (-1)^{n+1} \boxdot_{i=1}^{n} \mu_{A_i}\right) dP +$$

$$+ \alpha \left(\begin{pmatrix} n \\ 1 \end{pmatrix} - \begin{pmatrix} n \\ 2 \end{pmatrix} + \ldots + (-1)^{n+1} \begin{pmatrix} n \\ n \end{pmatrix} - \right.$$

$$\left. - \int_{\Omega} \left(\sum_{i=1}^{n} (\mu_{A_i} + \nu_{A_i}) - \sum_{i<j}^{n} (\mu_{A_i} \boxdot \mu_{A_i} + \nu_{A_i} \boxplus \nu_{A_i}) + \ldots + (-1)^{n+1} (\boxdot_{i=1}^{n} \mu_{A_i} + \boxplus_{i=1}^{n} \nu_{A_i})\right) dQ = \right.$$

$$= \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} (\mu_A + \nu_A) dQ\right) = m(A)$$

□

Corollary 2.4. *From (1.3) and (2.3) we get the assertion*

$$\mathcal{P}\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i) - \sum_{i<j}^{n} \mathcal{P}(A_i \cdot A_j) + \ldots + (-1)^{n+1} \mathcal{P}\left(\prod_{i=1}^{n} A_i\right).$$

3 Conclusion

IF-events are a generalization of fuzzy sets, by IF-event A we mean a pair (μ_A, ν_A), where μ_A is a membership function of A, μ_A is a non-membership function of A and $\mu_A + \nu_A \leq 1$. In the classic set theory the well-known inclusion-exclusion principle holds, i. e.

$$\mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i) - \sum_{i<j}^{n} \mathcal{P}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathcal{P}\left(\bigcap_{i=1}^{n} A_i\right),$$

where A_i are sets and $\mathcal{P}$ is a probability function. There was shown in [3], that the same principle holds for IF-events as well. Moreover, we can replace the union and intersection of IF-sets by the operations

$$A + B = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B),$$

$$A \cdot B = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B),$$

where $A = (\mu_A, \nu_A),\ B = (\mu_b, \nu_B)$ are IF-sets. Then the inclusion-exclusion principle will have a form

$$\mathcal{P}\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i) - \sum_{i<j}^{n} \mathcal{P}(A_i \cdot A_j) + \ldots + (-1)^{n+1}\mathcal{P}\left(\prod_{i=1}^{n} A_i\right).$$

In this contribution, the mentioned theorems have been proved using a new method and for larger set of probabilities.

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