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Numerical solution of intuitionistic fuzzy differential equations by Adams three order predictor-corrector method

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Abstract: In this paper three numerical methods to solve ”The intuitionistic fuzzy differential equations” are discussed. These methods are Adams–Bashforth, Adams–Moulton and predictor-corrector. The predictor-corrector method is generated by combining an explicit three-step method and implicit tow-step method. The Convergence and stability of the proposed methods are also presented. These methods are illustrated by solving an example.
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1 Introduction

Generalizations of fuzzy sets theory [17] is considered to be one of Intuitionistic fuzzy set (IFS). Later on Atanassov generalized the concept of fuzzy set and introduced the idea of intuitionistic fuzzy set [2, 4]. Atanassov [3] explored the concept of fuzzy set theory by intuitionistic fuzzy set (IFS) theory. They are very necessary and powerful tool in modeling imprecision, valuable applications of IFSs have been flourished in many different fields [7, 8, 9, 10, 14, 15, 16]. The numerical method for solving intuitionistic fuzzy differential equations is introduced in [6]. In this paper, intuitionistic fuzzy Cauchy problem is solved numerically by Adams three order predictor-corrector method.

The paper is organized as follows. In Section 2, some basic definitions and results are brought. The interpolation of intuitionistic fuzzy number in Section 3. Adams–Bashforth two-step and three-step methods for solving fuzzy differential equations are introduced in Section 4. In Section 5, Adams–Moulton two-step and three-step methods for solving fuzzy differential equations are proposed. Predictor-corrector three-step algorithm is discussed in Section 6. Convergence and stability of the mentioned methods are in Section 7. An example is presented in Section 8, and finally conclusion is drawn.

2 Preliminaries

2.1 Notations and definitions

Definition 2.1. *An m-step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation:*

$$y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + h\big\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) + \ldots + b_0 f(t_{i+1-m}, y_{i+1-m})\big\} \quad (1)$$

for $i = m-1, m, \ldots, N-1$
such that $a = t_0 \leq t_1 \leq \ldots \leq t_N = b,\ h = \frac{(b-a)}{N} = t_{i+1} - t_i$ and $a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_m$ are constants with the starting values $y_0 = \beta_0, \quad y_1 = \beta_1, \ldots, y_{m-1} = \beta_{m-1}$
When $b_m = 0$, the method is known as explicit, since Eq.(1) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is known as implicit, since y_{i+1} occurs on both sides of Eq.(1) and is specified only implicitly

Definition 2.2. *Associated with the difference equation*

$$y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + a_0 y_{i+1-m} + F(t_i, h, y_{i+1}, y_i, \ldots + y_{i+1-m}) \quad (2)$$

$$y_0 = \beta_0, y_1 = \beta_1, \ldots, y_{m-1} = \beta_{m-1}$$

the following, called the characteristic polynomial of the method is

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \ldots - a_1\lambda - a_0$$

If $|\lambda_i| \leq 1$ for each $i = 1, 2, ..., m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 2.1. *A multi-step method of the form (2) is stable if and only if it satisfies the root condition.*

Throughout this paper, $(\mathbb{R}, B(\mathbb{R}), \mu)$ denotes a complete finite measure space.
Let us $P_k(\mathbb{R})$ the set of all non empty compact convex subsets of $\mathbb{R}$.
we denote by

$$\mathbb{F}_1 = \mathrm{IF}(\mathbb{R}) = \left\{\langle u, v\rangle \ \ \mathbb{R} \to [0,1]^2 \,, |\forall\, x \in \mathbb{R} \ \ 0 \le u(x) + v(x) \le 1\right\}$$

An element $\langle u, v\rangle$ of $\mathbb{F}_1$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp\,\langle u, v\rangle = \mathrm{cl}\{x \in \mathbb{R} \mid v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by $\mathbb{F}_1$
For $\alpha \in [0, 1]$ and $\langle u, v\rangle \in \mathbb{F}_1$, the upper and lower α-cuts of $\langle u, v\rangle$ are defined by

$$[\langle u, v\rangle]^\alpha = \{x \in \mathbb{R} : v(x) \le 1 - \alpha\}$$

and

$$[\langle u, v\rangle]_\alpha = \{x \in \mathbb{R} : u(x) \ge \alpha\}$$

Remark 2.1. *If $\langle u, v\rangle \in IF_1$, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

Definition 2.3. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0 \end{cases}$$

Definition 2.4. *Let $\langle u, v\rangle$,$\langle u', v'\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by:*

$$\Big(\langle u, v\rangle \oplus \langle u', v'\rangle\Big)(z) = \Big(\sup_{z=x+y} \min\left(u(x), u'(y)\right), \inf_{z=x+y} \max\left(v(x), v'(y)\right)\Big)$$

$$\lambda \langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0 \end{cases}$$

According to Zadehs extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space IF_1 as follows:

$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]^\alpha = \Big[\langle u, v\rangle\Big]^\alpha + \Big[\langle z, w\rangle\Big]^\alpha, \quad \Big[\lambda \langle z, w\rangle\Big]^\alpha = \lambda\Big[\langle z, w\rangle\Big]^\alpha$$
$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]_\alpha = \Big[\langle u, v\rangle\Big]_\alpha + \Big[\langle z, w\rangle\Big]_\alpha, \quad \Big[\lambda \langle z, w\rangle\Big]_\alpha = \lambda\Big[\langle z, w\rangle\Big]_\alpha$$

where $\langle u, v\rangle$, $\langle z, w\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$.

Definition 2.5. *Let $\langle u,v\rangle$ an element of $I\!F_1$ and $\alpha \in [0,1]$, we define the following sets:*

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^+(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha) = \inf\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^-(\alpha) = \sup\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}$$

Remark 2.2.

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha), \Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle\Big]^\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha), \Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]$$

A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u,v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v:

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'} & \text{if } a_1' \leq x \leq a_2 \\ \frac{x-a_2}{a_3'-a_2} & \text{if } a_2 \leq x \leq a_3' \\ 1 & \text{otherwise} \end{cases}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$ and $u(x), v(x) \leq 0.5$ for $u(x) = v(x), \ \forall x \in \mathbb{R}$
This TIFN is denoted by $\langle u,v\rangle$=$\langle a_1, a_2, a_3; a_1', a_2, a_3'\rangle$ where,

$$[\langle u,v\rangle]_\alpha = [a_1 + \alpha(a_2-a_1), a_3 - \alpha(a_3-a_2)] \tag{3}$$

$$[\langle u,v\rangle]^\alpha = [a_1' + \alpha(a_2-a_1'), a_3' - \alpha(a_3'-a_2)] \tag{4}$$

Proposition 2.2. *For all α, $\beta \in [0,1]$ and $\langle u,v\rangle \in I\!F_1$*

(i) $\Big[\langle u,v\rangle\Big]_\alpha \subset \Big[\langle u,v\rangle\Big]^\alpha$

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are non empty compact convex sets in* $\mathbb{R}$

(iii) if $\alpha \leq \beta$ *then* $\Big[\langle u,v\rangle\Big]_\beta \subset \Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta \subset \Big[\langle u,v\rangle\Big]^\alpha$

(iv) If $\alpha_n \nearrow \alpha$ *then* $\Big[\langle u,v\rangle\Big]_\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha \in [0,1]$ we denote by

$$M_\alpha = \{x \in \mathbb{R} \ \ u(x) \geq \alpha\} \quad \text{and} \quad M^\alpha = \{x \in \mathbb{R} \ \ v(x) \leq 1-\alpha\}$$

Lemma 2.3. *[12] let* $\Big\{M_\alpha,\ \alpha\in[0,1]\Big\}$ *and* $\Big\{M^\alpha,\ \alpha\in[0,1]\Big\}$ *two families of subsets of* $\mathbb{R}$ *satisfies (i)–(iv) in proposition 2.2, if* u *and* v *define by*

$$u(x)=\begin{cases}0 & \text{if } x\notin M_0\\ \sup\{\alpha\in[0,1]\ \ x\in M_\alpha\} & \text{if } x\in M_0\end{cases}$$

$$v(x)=\begin{cases}1 & \text{if } x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ \ x\in M^\alpha\} & \text{if } x\in M^0\end{cases}$$

Then $\langle u,v\rangle\in I\!F_1$

Lemma 2.4. *Let* I *a dense subset of* $[0,1]$, *if* $\Big[\langle u,v\rangle\Big]_\alpha=\Big[\langle u',v'\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha=\Big[\langle u',v'\rangle\Big]^\alpha$, *for all* $\alpha\in I$ *then* $\langle u,v\rangle=\langle u',v'\rangle$

On the space $I\!F_1$ we will consider the following metric,

$$\begin{aligned}d_\infty\Big(\langle u,v\rangle,\langle z,w\rangle\Big)&=\frac{1}{4}\sup_{0<\alpha\le1}\Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha)-\Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big|\\&+\frac{1}{4}\sup_{0<\alpha\le1}\Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha)-\Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big|\\&+\frac{1}{4}\sup_{0<\alpha\le1}\Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha)-\Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big|\\&+\frac{1}{4}\sup_{0<\alpha\le1}\Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha)-\Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|\end{aligned}$$

Theorem 2.5. *([11])*
d_∞ *define a metric on* IF_1.

Theorem 2.6. *The metric space* (IF_1,d_∞) *is complete.*

Definition 2.6. *([12]) Let* $F:[a,b]\to IF_1$ *be an intuitionistic fuzzy valued mapping and* $t_0\in[a,b]$.
Then F *is called intuitionistic fuzzy continuous in* t_0 *iff:*

$$\forall(\varepsilon>0)(\exists\delta>0)(\forall t\in[a,b]\ \ tel\ \ que\ \ |t-t_0|<\delta)\Rightarrow d_\infty(F(t),F(t_0))<\varepsilon$$

Definition 2.7. *([12])* F *is called intuitionistic fuzzy continuous iff is intuitionistic fuzzy continuous in every point of* $[a,b]$

Definition 2.8. *Suppose* $A=[a,b]$, $F:A\to I\!F_1$ *is integrably bounded and strongly measurable for each* $\alpha\in(0,1]$ *write*

$$\begin{aligned}\left[\int_A F(t)dt\right]_\alpha&=\int_A[F(t)]_\alpha\,dt=\left\{\int_A f(t)dt|f:A\to\mathbb{R}\text{ is a measurable selection for }F_\alpha\right\}.\\\left[\int_A F(t)dt\right]^\alpha&=\int_A[F(t)]^\alpha\,dt=\left\{\int_A f(t)dt|f:A\to\mathbb{R}\text{ is a measurable selection for }F^\alpha\right\}.\end{aligned}$$

if there exists $\langle u,v\rangle\in I\!F_1$ *such that* $[\langle u,v\rangle]^\alpha=\left[\int_A F(t)dt\right]^\alpha$ *and* $[\langle u,v\rangle]_\alpha=\left[\int_A F(t)dt\right]_\alpha$ $\forall\alpha\in(0,1]$. *Then* F *is called integrable on A, write* $\langle u,v\rangle=\int_A F(t)dt$.

Theorem 2.7. *If $F : A \to I\!F_1$ is strongly measurable and integrably bounded, then F is integrable.*

Remark 2.3. *If $F : A \to I\!F_1$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0, 1]$ then*

$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s)ds$$

For more the details of continuite, measurability, integrability and differentiability we refer to [11]

Definition 2.9. *Let $\langle u, v\rangle$ and $\langle u', v'\rangle \in IF_1$, the H-difference is the IFN $\langle z, w\rangle \in IF_1$, if it exists, such that*

$$\langle u, v\rangle \ominus \langle u', v'\rangle = \langle z, w\rangle \iff \langle u, v\rangle = \langle u', v'\rangle \oplus \langle z, w\rangle$$

Definition 2.10. *A mapping $F : [a, b] \to IF_1$ is said to be Hukuhara derivable at t_0 if there exist $F'(t_0) \in IF_1$ such that both limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t}$$

and

$$\lim_{\Delta t \to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0.

2.2 Intuitionistic fuzzy Cauchy problem

In this section we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases} x'(t) &= f(t, x(t)), \;\; t \in I \\ x(t_0) &= \langle u_{t_0}, v_{t_0}\rangle \in IF_1 \end{cases} \tag{5}$$

where $x \in IF_1$ is unknown $I = [t_0, T]$ and $f : I \times IF_1 \to IF_1$.

$x(t_0)$ is an intuitionistic fuzzy number. Denote the $\alpha-$ level set

$$[x(t)]_\alpha = \Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big]$$

$$[x(t)]^\alpha = \Big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big]$$

and

$$[x(t_0)]_\alpha = \Big[[x(t_0)]_l^+(\alpha), [x(t_0)]_r^+(\alpha)\Big]$$

$$[x(t_0)]^\alpha = \Big[x(t_0)]_l^-(\alpha), [x(t_0)]_r^-(\alpha)\Big]$$

$$[f(t,x(t))]_\alpha = \Big[f_l^+(t,x(t);\alpha),[f_r^+(t,x(t);\alpha)\Big]$$

$$[f(t,x(t))]^\alpha = \Big[f_l^-(t,x(t);\alpha),[f_r^-(t,x(t);\alpha)\Big]$$

Sufficient conditions for the existence of a unique solution to Eq. (5) are:

1. Continuity of f

2. Lipschitz condition: for any pair $\big(t,\langle u,v\rangle\big),\big(t,\langle u',v'\rangle\big) \in I \times \mathbb{F}_1$, we have

$$d_\infty\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle u',v'\rangle\big)\Big) \leq K d_\infty\Big(\langle u,v\rangle,\langle u',v'\rangle\Big) \tag{6}$$

   where $K > 0$ is a given constant.

3 Interpolation of intuitionistic fuzzy number

The problem of interpolation for intuitionistic fuzzy sets is as follows:
Suppose that at various time instant x information $f(x)$ is presented as intuitionistic fuzzy set. The aim is to approximate the function $f(x)$, for all x in the domain of f. Let $x_0 < x_1 < \ldots < x_n$ be $n+1$ distinct points in $\mathbb{R}$ and let $\langle u_0, v_0\rangle, \langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle$ be $n+1$ intuitionistic fuzzy sets in IF_1. An intuitionistic fuzzy polynomial interpolation of the data is an intuitionistic fuzzy-value continuous function $f : I \longrightarrow IF_1$ satisfying:

- $f(x_i) = \langle u_i, v_i\rangle$
- If the data is crisp, then the interpolation f is a crisp polynomial.

A function f which fulfilling these condition may be constructed as follows. For each $Y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_Y such that

- $P_Y(x_i) = y_i, \quad i = 0, 1, \ldots, n$
- $P_Y(x) = \sum_{i=0}^n y_i\Big(\prod\limits_{i\neq j} \frac{x-x_j}{x_i-x_j}\Big)$

According to the extension principle, we can write the membership and non-membership function $f(x)$ for each $x \in \mathbb{R}$ as follows:

$$\mu_{f(x)}(t) = \begin{cases} \sup\limits_{\substack{y_0,y_1,\ldots,y_n \\ t=P_{y_0,y_1,\ldots,y_n}(x)}} \min\limits_{i=0,1,\ldots,n} \mu_{u_i}(y_i) & \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where μ_{u_i} is the membership function of u_i, and

$$\nu_{f(x)}(t) = \begin{cases} \inf\limits_{\substack{y_0,y_1,\ldots,y_n \\ t=P_{y_0,y_1,\ldots,y_n}(x)}} \max\limits_{i=0,1,\ldots,n} \nu_{v_i}(y_i) & \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t) \neq \emptyset \\ \\ 1 & \text{otherwise} \end{cases}$$

where ν_{v_i} is the non-membership function of v_i.
Let $J_i^+(\alpha) = [\langle u_i, v_i\rangle]_\alpha$, $J_i^-(\alpha) = [\langle u_i, v_i\rangle]^\alpha$ for any $\alpha \in [0,1]$, $i = 0, 1, ..., n$ and $[f(x)]_\alpha$, $[f(x)]^\alpha$the upper and lower α-cuts of $\langle u_i, v_i\rangle$ and $f(x)$ respectively. Hence,

$$\begin{aligned} [f(x)]_\alpha &= \big\{t \in \mathbb{R} \;\mid\; \mu_{f(x)}(t) \geq \alpha\big\} \\ &= \big\{t \in \mathbb{R} \mid \exists\, y_0, y_1, \ldots, y_n : \mu_{u_i}(y_i) \geq \alpha,\;\; i = 0, \ldots, n \;\text{ and }\; P_{y_0,y_1,\ldots,y_n}(x) = t\big\} \\ &= \big\{t \in \mathbb{R} \mid \exists\, Y \in \prod_{i=0}^{n} J_i^+(\alpha) \quad : P_{y_0,y_1,\ldots,y_n}(x) = t\big\} \end{aligned}$$

and

$$\begin{aligned} [f(x)]^\alpha &= \big\{t \in \mathbb{R} \;\mid\; \nu_{f(x)}(t) \leq 1-\alpha\big\} \\ &= \big\{t \in \mathbb{R} \mid \exists\, y_0, y_1, \ldots, y_n : \nu_{v_i}(y_i) \leq 1-\alpha,\;\; i = 0, \ldots, n \;\text{ and }\; P_{y_0,y_1,\ldots,y_n}(x) = t\big\} \\ &= \big\{t \in \mathbb{R} \mid \exists\, Y \in \prod_{i=0}^{n} J_i^-(\alpha) \quad : P_{y_0,y_1,\ldots,y_n}(x) = t\big\} \end{aligned}$$

Finally, for each $x \in \mathbb{R}$ and all $t \in \mathbb{R}$ is defined by $f(x) \in IF_1$ by:

$$f(x)(t) = \Big(\sup\big\{\alpha \in (0,1] | \exists Y \in \prod_{i=0}^{n} J_i^+(\alpha) : P_Y(x) = t\big\}, 1 - \sup\big\{\alpha \in (0,1] | \exists Y \in \prod_{i=0}^{n} J_i^-(\alpha) : P_Y(x) = t\big\}\Big),$$

where $Y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$
The interpolation polynomial can be written level set wise as

$$[f(x)]_\alpha \;=\; \{y \in \mathbb{R} : y = P_{y_0,y_1,\ldots,y_n}(x),\;\; y_i \in [\langle u_i, v_i\rangle]_\alpha \;\; i = 0, \ldots, n\},\;\; \text{for } \alpha \in (0,1]$$

and

$$[f(x)]^\alpha \;=\; \{y \in \mathbb{R} : y = P_{y_0,y_1,\ldots,y_n}(x),\;\; y_i \in [\langle u_i, v_i\rangle]^\alpha \;\; i = 0, \ldots, n\},\;\; \text{for } \alpha \in (0,1]$$

But, from Lagrange interpolation formula, we have

$$[f(x)]_\alpha = \sum_{i=0}^{n} \ell_i(x) J_i^+(\alpha)$$

and

$$[f(x)]^\alpha = \sum_{i=0}^{n} \ell_i(x) J_i^-(\alpha)$$

where $\ell_i(x)$ represents the Lagrange polynomials.
When the data $\langle u_i, v_i\rangle$ presents as triangular intuitionistic fuzzy numbers, values of the interpolation polynomial are also triangular intuitionistic fuzzy numbers. Then $f(x)$ has a particular simple form that is well suited to computation. Denote $J_i^+(\alpha) = [a_i^+(\alpha), b_i^+(\alpha)]$ and $J_i^-(\alpha) = [a_i^-(\alpha); b_i^-(\alpha)]$. Then the upper end point of $[f(x)]_\alpha$ is the solution of the optimization problem:

$$\text{Maximize} \quad P_{y_0,y_1,\ldots,y_n}(x) \quad \text{subject to} \quad a_i^+(\alpha) \le y_i \le b_i^+(\alpha) \quad i = 0, 1, \ldots, n$$

It follows that the optimal solution is

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) \ge 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \end{cases}$$

and the lower end point is obtained as the value of the interpolation polynomial associated to points

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) \ge 0 \end{cases}$$

Similarly the upper and lower end point of $[f(x)]^\alpha$ can be obtained.

Hence if $\langle u_i, v_i\rangle$ is an intuitionistic fuzzy number, for all i then also $f(x)$ is such an intuitionistic fuzzy number for each x. More precisely, if $\langle u_i, v_i\rangle = \langle u_i^l, u_i^c, u_i^r, v_i^l, u_i^c, v_i^r\rangle$ and $f(x) = \left\langle f_l(x), f^c(x), f_r(x), f^l(x), f^c(x), f^r(x)\right\rangle$, then we will have,

$$\begin{aligned} f_l(x) &= \sum_{\ell_i(x)\ge 0} \ell_i(x)u_i^l + \sum_{\ell_i(x)<0} \ell_i(x)u_i^r \\ f_r(x) &= \sum_{\ell_i(x)\ge 0} \ell_i(x)u_i^r + \sum_{\ell_i(x)<0} \ell_i(x)u_i^l \\ f^c(x) &= \sum_{i=0}^{n} \ell_i(t)u_i^c \\ f^l(x) &= \sum_{\ell_i(x)\ge 0} \ell_i(x)v_i^l + \sum_{\ell_i(x)<0} \ell_i(x)v_i^r \\ f^r(x) &= \sum_{\ell_i(x)\ge 0} \ell_i(x)v_i^r + \sum_{\ell_i(x)<0} \ell_i(x)v_i^l. \end{aligned}$$

4 Adams–Bashforth methods

Now we are going to solve intuitionistic fuzzy initial value problem $x'(t) = f(t, x(t))$ by Adams–Bashforth three-step method. Let the intuitionistic fuzzy initial values be $x(t_{i-1}),\ x(t_i),\ x(t_{i+1})$, i.e., $f(t_{i-1}, x(t_{i-1}))$, $f(t_i, x(t_i))$, $f(t_{i+1}, x(t_{i+1}))$, which are triangular intuitionistic fuzzy numbers and are shown by:

- $\left\{f_l(t_{i-1}, x(t_{i-1})), f^c(t_{i-1}, x(t_{i-1})), f_r(t_{i-1}, x(t_{i-1})), f^l(t_{i-1}, x(t_{i-1})), f^c(t_{i-1}, x(t_{i-1})), f^r(t_{i-1}, x(t_{i-1}))\right\}$

- $\Big\{f_l(t_i,x(t_i)), f^c(t_i,x(t_i)), f_r(t_i,x(t_i)), f^l(t_i,x(t_i)), f^c(t_i,x(t_i)), f^r(t_i,x(t_i))\Big\}$
- $\Big\{f_l(t_{i+1},x(t_{i+1})), f^c(t_{i+1},x(t_{i+1})), f_r(t_{i+1},x(t_{i+1})), f^l(t_{i+1},x(t_{i+1})), f^c(t_{i+1},x(t_{i+1})), f^r(t_{i+1},x(t_{i+1}))\Big\}$

also

$$x(t_{i+2}) = x(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} f(t,x(t))dt \tag{7}$$

By intuitionistic fuzzy interpolation of $f(t,x(t_{i-1})), \quad f(t,x(t_i)), \quad f(t,x(t_{i+1}))$ we have

$$\begin{aligned}
\bullet\ f_l(t,x(t)) &= \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+1} \ell_j(t) f_l(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+1} \ell_j(t) f_r(t_j,x(t_j))\\
\bullet\ f_r(t,x(t)) &= \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+1} \ell_j(t) f_r(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+1} \ell_j(t) f_l(t_j,x(t_j))\\
\bullet\ f^c(t,x(t)) &= \sum_{j=i-1}^{i+1} \ell_j(t) f^c(t_j,x(t_j))\\
\bullet\ f^l(t,x(t)) &= \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+1} \ell_j(t) f^l(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+1} \ell_j(t) f^r(t_j,x(t_j))\\
\bullet\ f^r(t,x(t)) &= \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+1} \ell_j(t) f^r(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+1} \ell_j(t) f^l(t_j,x(t_j))
\end{aligned}$$

for $t_{i+1} \leq t \leq t_{i+2}$

$$\begin{aligned}
\bullet\ \ell_{i-1}(t) &= \frac{(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} \geq 0\\
\bullet\ \ell_i(t) &= \frac{(t-t_{i-1})(t-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} \leq 0\\
\bullet\ \ell_{i+1}(t) &= \frac{(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \geq 0
\end{aligned}$$

therefore the following results will be obtained:

$$\bullet\ f_l(t,x(t)) = \ell_{i-1}(t) f_l(t_{i-1},x(t_{i-1})) + \ell_i(t) f_r(t_i,x(t_i)) + \ell_{i+1}(t) f_l(t_{i+1},x(t_{i+1})) \tag{8}$$
$$\bullet\ f_r(t,x(t)) = \ell_{i-1}(t) f_r(t_{i-1},x(t_{i-1})) + \ell_i(t) f_l(t_i,x(t_i)) + \ell_{i+1}(t) f_r(t_{i+1},x(t_{i+1})) \tag{9}$$
$$\bullet\ f^c(t,x(t)) = \ell_{i-1}(t) f^c(t_{i-1},x(t_{i-1})) + \ell_i(t) f^c(t_i,x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1},x(t_{i+1})) \tag{10}$$
$$\bullet\ f^l(t,x(t)) = \ell_{i-1}(t) f^l(t_{i-1},x(t_{i-1})) + \ell_i(t) f^r(t_i,x(t_i)) + \ell_{i+1}(t) f^l(t_{i+1},x(t_{i+1})) \tag{11}$$
$$\bullet\ f^r(t,x(t)) = \ell_{i-1}(t) f^r(t_{i-1},x(t_{i-1})) + \ell_i(t) f^l(t_i,x(t_i)) + \ell_{i+1}(t) f^r(t_{i+1},x(t_{i+1})) \tag{12}$$

From (7), (3) and (4) it follows that:

$$[x(t_{i+2})]_\alpha = \Big[[x(t_{i+2})]_l^+(\alpha), [x(t_{i+2})]_r^+(\alpha)\Big],$$

and

$$[x(t_{i+2})]^{\alpha} = \Big[[x(t_{i+2})]_l^-(\alpha), [x(t_{i+2})]_r^-(\alpha)\Big]$$

where

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, x(t)) + (1-\alpha) f_l(t, x(t))\} dt, \tag{13}$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, x(t)) + (1-\alpha) f_r(t, x(t))\} dt \tag{14}$$

and

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, x(t)) + (1-\alpha) f^l(t, x(t))\} dt, \tag{15}$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, x(t)) + (1-\alpha) f^r(t, x(t))\} dt \tag{16}$$

If (8) and (10) are situated in (13) and (9), (10) in (14) we have:

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{\ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1}))$$
$$+ (1-\alpha)\big(\ell_{i-1}(t) f_l(t_{i-1}, x(t_{i-1})) + \ell_i(t) f_r(t_i, x(t_i)) + \ell_{i+1}(t) f_l(t_{i+1}, x(t_{i+1}))\big)\Big\} dt,$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{\ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1}))$$
$$+ (1-\alpha)\big(\ell_{i-1}(t) f_r(t_{i-1}, x(t_{i-1})) + \ell_i(t) f_l(t_i, x(t_i)) + \ell_{i+1}(t) f_r(t_{i+1}, x(t_{i+1}))\big)\Big\} dt$$

and if (10) and (11) are situated in (15) and (10), (12) in (16) we have:

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{\ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1}))$$
$$+ (1-\alpha)\big(\ell_{i-1}(t) f^l(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^r(t_i, x(t_i)) + \ell_{i+1}(t) f^l(t_{i+1}, x(t_{i+1}))\big)\Big\} dt;$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{\ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1}))$$
$$+ (1-\alpha)\big(\ell_{i-1}(t) f^r(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^l(t_i, x(t_i)) + \ell_{i+1}(t) f^r(t_{i+1}, x(t_{i+1}))\big)\Big\} dt$$

The following results will be obtained by integration:

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \frac{5}{12}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f_l(t_{i-1}, x(t_{i-1}))\Big] + \frac{-16h}{12}\Big[\alpha f^c(t_i, x(t_i))$$
$$+ (1-\alpha) f_r(t_i, x(t_i))\Big] + \frac{23h}{12}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f_l(t_{i+1}, x(t_{i+1}))\Big]$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \frac{5}{12}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f_r(t_{i-1}, x(t_{i-1}))\Big] + \frac{-16h}{12}\Big[\alpha f^c(t_i, x(t_i))$$
$$+ (1-\alpha) f_l(t_i, x(t_i))\Big] + \frac{23h}{12}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f_r(t_{i+1}, x(t_{i+1}))\Big]$$

and

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \frac{5}{12}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha)f^l(t_{i-1}, x(t_{i-1}))\Big] + \frac{-16h}{12}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha)f^r(t_i, x(t_i))\Big] + \frac{23h}{12}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha)f^l(t_{i+1}, x(t_{i+1}))\Big]$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \frac{5}{12}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha)f^r(t_{i-1}, x(t_{i-1}))\Big] + \frac{-16h}{12}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha)f^l(t_i, x(t_i))\Big] + \frac{23h}{12}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha)f^r(t_{i+1}, x(t_{i+1}))\Big]$$

Thus

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \frac{h}{12}\Big[5f_l^+(t_{i-1}, x(t_{i-1});\alpha) - 16f_r^+(t_i, x(t_i);\alpha) + 23f_l^+(t_{i+1}, x(t_{i+1});\alpha)\Big] \quad (17)$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \frac{h}{12}\Big[5f_r^+(t_{i-1}, x(t_{i-1});\alpha) - 16f_l^+(t_i, x(t_i);\alpha) + 23f_r^+(t_{i+1}, x(t_{i+1});\alpha)\Big] \quad (18)$$

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \frac{h}{12}\Big[5f_l^-(t_{i-1}, x(t_{i-1});\alpha) - 16f_r^-(t_i, x(t_i);\alpha) + 23f_l^-(t_{i+1}, x(t_{i+1});\alpha)\Big] \quad (19)$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \frac{h}{12}\Big[5f_r^-(t_{i-1}, x(t_{i-1});\alpha) - 16f_l^-(t_i, x(t_i);\alpha) + 23f_r^-(t_{i+1}, x(t_{i+1});\alpha)\Big] \quad (20)$$

Therefore Adams–Bashforth three-step method is as follows:

$$\begin{cases}
[x(t_{i+2})]_l^+(\alpha) &= [x(t_{i+1})]_l^+(\alpha) + \frac{h}{12}\Big[5f_l^+(t_{i-1}, x(t_{i-1});\alpha) - 16f_r^+(t_i, x(t_i);\alpha) + 23f_l^+(t_{i+1}, x(t_{i+1});\alpha)\Big] \\
[x(t_{i+2})]_r^+(\alpha) &= [x(t_{i+1})]_r^+(\alpha) + \frac{h}{12}\Big[5f_r^+(t_{i-1}, x(t_{i-1});\alpha) - 16f_l^+(t_i, x(t_i);\alpha) + 23f_r^+(t_{i+1}, x(t_{i+1});\alpha)\Big] \\
[x(t_{i+2})]_l^-(\alpha) &= [x(t_{i+1})]_l^-(\alpha) + \frac{h}{12}\Big[5f_l^-(t_{i-1}, x(t_{i-1});\alpha) - 16f_r^-(t_i, x(t_i);\alpha) + 23f_l^-(t_{i+1}, x(t_{i+1});\alpha)\Big] \\
[x(t_{i+2})]_r^-(\alpha) &= [x(t_{i+1})]_r^-(\alpha) + \frac{h}{12}\Big[5f_r^-(t_{i-1}, x(t_{i-1});\alpha) - 16f_l^-(t_i, x(t_i);\alpha) + 23f_r^-(t_{i+1}, x(t_{i+1});\alpha)\Big] \\
[x(t_{i-1})]_l^+(\alpha) &= \beta_0,\ \ [x(t_i)]_l^+(\alpha) = \beta_1,\ \ [x(t_{i+1})]_l^+(\alpha) = \beta_2 \\
[x(t_{i-1})]_r^+(\alpha) &= \beta_3,\ \ [x(t_i)]_r^+(\alpha) = \beta_4,\ \ [x(t_{i+1})]_r^+(\alpha) = \beta_5 \\
[x(t_{i-1})]_l^-(\alpha) &= \beta_6,\ \ [x(t_i)]_l^-(\alpha) = \beta_7\ \ [x(t_{i+1})]_l^-(\alpha) = \beta_8 \\
[x(t_{i-1})]_r^-(\alpha) &= \beta_9,\ \ [x(t_i)]_r^-(\alpha) = \beta_{10},\ \ [x(t_{i+1})]_r^-(\alpha) = \beta_{11}
\end{cases} \quad (21)$$

With similar way the intuitionistic fuzzy initial value problem $x'(t) = f(t, x(t))$ can be solved by Adams–Bashforth two-step method as follows:

$$\begin{cases} [x(t_{i+2})]^+_l(\alpha) &= [x(t_{i+1})]^+_l(\alpha) - \frac{h}{2} f^+_r(t_{i-1}, x(t_{i-1}); \alpha) + \frac{3h}{2} f^+_l(t_i, x(t_i); \alpha) \\ [x(t_{i+2})]^+_r(\alpha) &= [x(t_{i+1})]^+_r(\alpha) - \frac{h}{2} f^+_l(t_{i-1}, x(t_{i-1}); \alpha) + \frac{3h}{2} f^+_r(t_i, x(t_i); \alpha) \\ [x(t_{i+2})]^-_l(\alpha) &= [x(t_{i+1})]^-_l(\alpha) - \frac{h}{2} f^-_r(t_{i-1}, x(t_{i-1}); \alpha) + \frac{3h}{2} f^-_l(t_i, x(t_i); \alpha) \\ [x(t_{i+2})]^-_r(\alpha) &= [x(t_{i+1})]^-_r(\alpha) - \frac{h}{2} f^-_l(t_{i-1}, x(t_{i-1}); \alpha) + \frac{3h}{2} f^-_r(t_i, x(t_i); \alpha) \\ [x(t_{i-1})]^+_l(\alpha) &= \beta_0, \ \ [x(t_i)]^+_l(\alpha) = \beta_1, \ \ [x(t_{i-1})]^+_r(\alpha) = \beta_2, \ \ [x(t_i)]^+_r(\alpha) = \beta_3 \\ [x(t_{i-1})]^-_l(\alpha) &= \beta_4, \ \ [x(t_i)]^-_l(\alpha) = \beta_5, \ \ [x(t_{i-1})]^-_r(\alpha) = \beta_6, \ \ [x(t_i)]^-_r(\alpha) = \beta_7 \end{cases} \tag{22}$$

5 Adams–Moulton methods

Now we are going to solve intuitionistic fuzzy initial value problem $x'(t) = f(t, x(t))$ by Adams–Moulton three-step method. Let the intuitionistic fuzzy initial values be $x(t_{i-1})$, $x(t_i)$, $x(t_{i+1})$, i.e., $f(t_{i-1}, x(t_{i-1}))$, $f(t_i, x(t_i))$, $f(t_{i+1}, x(t_{i+1}))$, which are triangular intuitionistic fuzzy numbers and are shown by:

- $\Big\{f_l(t_{i-1}, x(t_{i-1})), f^c(t_{i-1}, x(t_{i-1})), f_r(t_{i-1}, x(t_{i-1})), f^l(t_{i-1}, x(t_{i-1})), f^c(t_{i-1}, x(t_{i-1})), f^r(t_{i-1}, x(t_{i-1}))\Big\}$
- $\Big\{f_l(t_i, x(t_i)), f^c(t_i, x(t_i)), f_r(t_i, x(t_i)), f^l(t_i, x(t_i)), f^c(t_i, x(t_i)), f^r(t_i, x(t_i))\Big\}$
- $\Big\{f_l(t_{i+1}, x(t_{i+1})), f^c(t_{i+1}, x(t_{i+1})), f_r(t_{i+1}, x(t_{i+1})), f^l(t_{i+1}, x(t_{i+1})), f^c(t_{i+1}, x(t_{i+1})), f^r(t_{i+1}, x(t_{i+1}))\Big\}$

Consider the following intuitionistic fuzzy equation

$$x(t_{i+2}) = x(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} f(t, x(t))dt \tag{23}$$

By intuitionistic fuzzy interpolation of $f(t, x(t_{i-1}))$, $f(t, x(t_i))$, $f(t, x(t_{i+1}))$ we have

- $f_l(t, x(t)) = \sum_{\substack{j=i-1 \\ \ell_j(t)\geq 0}}^{i+2} \ell_j(t) f_l(t_j, x(t_j)) + \sum_{\substack{j=i-1 \\ \ell_j(t)<0}}^{i+2} \ell_j(t) f_r(t_j, x(t_j))$
- $f_r(t, x(t)) = \sum_{\substack{j=i-1 \\ \ell_j(t)\geq 0}}^{i+2} \ell_j(t) f_r(t_j, x(t_j)) + \sum_{\substack{j=i-1 \\ \ell_j(t)<0}}^{i+2} \ell_j(t) f_l(t_j, x(t_j))$
- $f^c(t, x(t)) = \sum_{\substack{j=i-1 \\ \ell_j(t)\geq 0}}^{i+2} \ell_j(t) f^c(t_j, x(t_j))$

$$\bullet f^l(t,x(t)) = \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+2} \ell_j(t) f^l(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+2} \ell_j(t) f^r(t_j,x(t_j))$$

$$\bullet\ f^r(t,x(t)) = \sum_{\substack{j=i-1\\ \ell_j(t)\geq 0}}^{i+2} \ell_j(t) f^r(t_j,x(t_j)) + \sum_{\substack{j=i-1\\ \ell_j(t)<0}}^{i+2} \ell_j(t) f^l(t_j,x(t_j))$$

for $t_{i+1} \leq t \leq t_{i+2}$

$$\bullet\ \ell_{i-1}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})} \geq 0$$

$$\bullet\ \ell_i(t) = \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})}{(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})} \leq 0$$

$$\bullet\ \ell_{i+1}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+2})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})} \geq 0$$

$$\bullet\ \ell_{i+2}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \geq 0$$

therefore the following results will be obtained:

$$\bullet f_l(t,x(t)) = \ell_{i-1}(t)f_l(t_{i-1},x(t_{i-1})) + \ell_i(t)f_r(t_i,x(t_i)) + \ell_{i+1}(t)f_l(t_{i+1},x(t_{i+1})) + \ell_{i+2}(t)f_l(t_{i+2},x(t_{i+2})) \tag{24}$$

$$\bullet f_r(t,x(t)) = \ell_{i-1}(t)f_r(t_{i-1},x(t_{i-1})) + \ell_i(t)f_l(t_i,x(t_i)) + \ell_{i+1}(t)f_r(t_{i+1},x(t_{i+1})) + \ell_{i+2}(t)f_r(t_{i+2},x(t_{i+2})) \tag{25}$$

$$\bullet f^c(t,x(t)) = \ell_{i-1}(t)f^c(t_{i-1},x(t_{i-1})) + \ell_i(t)f^c(t_i,x(t_i)) + \ell_{i+1}(t)f^c(t_{i+1},x(t_{i+1})) + \ell_{i+2}(t)f^c(t_{i+2},x(t_{i+2})) \tag{26}$$

$$\bullet f^l(t,x(t)) = \ell_{i-1}(t)f^l(t_{i-1},x(t_{i-1})) + \ell_i(t)f^r(t_i,x(t_i)) + \ell_{i+1}(t)f^l(t_{i+1},x(t_{i+1})) + \ell_{i+2}(t)f^l(t_{i+2},x(t_{i+2})) \tag{27}$$

$$\bullet f^r(t,x(t)) = \ell_{i-1}(t)f^r(t_{i-1},x(t_{i-1})) + \ell_i(t)f^l(t_i,x(t_i)) + \ell_{i+1}(t)f^r(t_{i+1},x(t_{i+1})) + \ell_{i+2}(t)f^r(t_{i+2},x(t_{i+2})) \tag{28}$$

From (23), (3) and (4) it follows that:

$$[x(t_{i+2})]_\alpha = \Big[[x(t_{i+2})]_l^+(\alpha), [x(t_{i+2})]_r^+(\alpha)\Big] \tag{29}$$

where

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,x(t)) + (1-\alpha)f_l(t,x(t))\}dt \tag{30}$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,x(t)) + (1-\alpha)f_r(t,x(t))\}dt \tag{31}$$

and

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,x(t)) + (1-\alpha)f^l(t,x(t))\}dt \tag{32}$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,x(t)) + (1-\alpha)f^r(t,x(t))\}dt \tag{33}$$

If (24) and (26) are situated in (30) and (25), (26) in (31):

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{ \ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1})) \\ + \ell_{i+2}(t) f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha)\Big(\ell_{i-1}(t) f_l(t_{i-1}, x(t_{i-1})) \\ + \ell_i(t) f_r(t_i, x(t_i)) + \ell_{i+1}(t) f_l(t_{i+1}, x(t_{i+1})) + \ell_{i+2}(t) f_l(t_{i+2}, x(t_{i+2}))\Big)\Big\} dt$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{ \ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1})) \\ + \ell_{i+2}(t) f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha)\Big(\ell_{i-1}(t) f_r(t_{i-1}, x(t_{i-1})) \\ + \ell_i(t) f_l(t_i, x(t_i)) + \ell_{i+1}(t) f_r(t_{i+1}, x(t_{i+1})) + \ell_{i+2}(t) f_r(t_{i+2}, x(t_{i+2}))\Big)\Big\} dt$$

and if (26) and (27) are situated in (32) and (26), (28) in (33) we have:

$$[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{ \ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1})) \\ + \ell_{i+2}(t) f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha)\Big(\ell_{i-1}(t) f^l(t_{i-1}, x(t_{i-1})) \\ + \ell_i(t) f^r(t_i, x(t_i)) + \ell_{i+1}(t) f^l(t_{i+1}, x(t_{i+1})) + \ell_{i+2}(t) f^l(t_{i+2}, x(t_{i+2}))\Big)\Big\} dt$$

$$[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) + \int_{t_{i+1}}^{t_{i+2}} \Big\{ \ell_{i-1}(t) f^c(t_{i-1}, x(t_{i-1})) + \ell_i(t) f^c(t_i, x(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, x(t_{i+1})) \\ + \ell_{i+2}(t) f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha)\Big(\ell_{i-1}(t) f^r(t_{i-1}, x(t_{i-1})) \\ + \ell_i(t) f^l(t_i, x(t_i)) + \ell_{i+1}(t) f^r(t_{i+1}, x(t_{i+1})) + \ell_{i+2}(t) f^r(t_{i+2}, x(t_{i+2}))\Big)\Big\} dt$$

The following results will be obtained by integration:

$$[x(t_{i+2})]_l^+(\alpha) = [x(t_{i+1})]_l^+(\alpha) + \frac{h}{24}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f_l(t_{i-1}, x(t_{i-1}))\Big] \\ + \frac{-5h}{24}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha) f_r(t_i, x(t_i))\Big] \\ + \frac{19h}{24}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f_l(t_{i+1}, x(t_{i+1})) \\ + \frac{9h}{24}\Big[\alpha f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha) f_l(t_{i+1}, x(t_{i+2}))\Big]$$

$$[x(t_{i+2})]_r^+(\alpha) = [x(t_{i+1})]_r^+(\alpha) + \frac{h}{24}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f_r(t_{i-1}, x(t_{i-1}))\Big] \\ + \frac{-5h}{24}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha) f_l(t_i, x(t_i))\Big] \\ + \frac{19h}{24}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f_r(t_{i+1}, x(t_{i+1})) \\ + \frac{9h}{24}\Big[\alpha f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha) f_r(t_{i+1}, x(t_{i+2}))\Big]$$

and

$$
\begin{aligned}
[x(t_{i+2})]_l^-(\alpha) = [x(t_{i+1})]_l^-(\alpha) &+ \frac{h}{24}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f^l(t_{i-1}, x(t_{i-1}))\Big] \\
&+ \frac{-5h}{24}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha) f^r(t_i, x(t_i))\Big] \\
&+ \frac{19h}{24}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f^l(t_{i+1}, x(t_{i+1})) \\
&+ \frac{9h}{24}\Big[\alpha f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha) f^l(t_{i+1}, x(t_{i+2}))\Big]
\end{aligned}
$$

$$
\begin{aligned}
[x(t_{i+2})]_r^-(\alpha) = [x(t_{i+1})]_r^-(\alpha) &+ \frac{h}{24}\Big[\alpha f^c(t_{i-1}, x(t_{i-1})) + (1-\alpha) f^r(t_{i-1}, x(t_{i-1}))\Big] \\
&+ \frac{-5h}{24}\Big[\alpha f^c(t_i, x(t_i)) + (1-\alpha) f^l(t_i, x(t_i))\Big] \\
&+ \frac{19h}{24}\Big[\alpha f^c(t_{i+1}, x(t_{i+1})) + (1-\alpha) f^r(t_{i+1}, x(t_{i+1})) \\
&+ \frac{9h}{24}\Big[\alpha f^c(t_{i+2}, x(t_{i+2})) + (1-\alpha) f^r(t_{i+1}, x(t_{i+2}))\Big]
\end{aligned}
$$

Therefore Adams–Moulton three-step method is obtained as follows:

$$
\begin{cases}
[x(t_{i+2})]_l^+(\alpha) = & [x(t_{i+1})]_l^+(\alpha) + \frac{h}{24}\Big[f_l^+(t_{i-1}, x(t_{i-1}); \alpha) - 5f_r^+(t_i, x(t_i); \alpha) + 19f_l^+(t_{i+1}, x(t_{i+1}); \alpha) \\
& +9f_l^+(t_{i+2}, x(t_{i+2}); \alpha)\Big] \\
[x(t_{i+2})]_r^+(\alpha) = & [x(t_{i+1})]_r^+(\alpha) + \frac{h}{24}\Big[f_r^+(t_{i-1}, x(t_{i-1}); \alpha) - 5f_l^+(t_i, x(t_i); \alpha) + 19f_r^+(t_{i+1}, x(t_{i+1}); \alpha) \\
& +9f_r^+(t_{i+2}, x(t_{i+2}); \alpha)\Big] \\
[x(t_{i+2})]_l^-(\alpha) = & [x(t_{i+1})]_l^-(\alpha) + \frac{h}{24}\Big[f_l^-(t_{i-1}, x(t_{i-1}); \alpha) - 5f_r^-(t_i, x(t_i); \alpha) + 19f_l^-(t_{i+1}, x(t_{i+1}); \alpha) \\
& +9f_l^-(t_{i+2}, x(t_{i+2}); \alpha)\Big] \\
[x(t_{i+2})]_r^-(\alpha) = & [x(t_{i+1})]_r^-(\alpha) + \frac{h}{24}\Big[f_r^-(t_{i-1}, x(t_{i-1}); \alpha) - 5f_l^-(t_i, x(t_i); \alpha) + 19f_r^-(t_{i+1}, x(t_{i+1}); \alpha) \\
& +9f_r^-(t_{i+2}, x(t_{i+2})\alpha)\Big] \\
[x(t_{i-1})]_l^+(\alpha) = & \beta_0, \ [x(t_i)]_l^+(\alpha) = \beta_1, \ [x(t_{i+1})]_l^+(\alpha) = \beta_2 \\
[x(t_i)]_r^+(\alpha) = & \beta_4, \ [x(t_{i-1})]_r^+(\alpha) = \beta_3, \ [x(t_{i+1})]_r^+(\alpha) = \beta_5 \\
[x(t_{i-1})]_l^-(\alpha) = & \beta_6, \ [x(t_i)]_l^-(\alpha) = \beta_7 \ [x(t_{i+1})]_l^-(\alpha) = \beta_8 \\
[x(t_{i-1})]_r^-(\alpha) = & \beta_9, \ [x(t_i)]_r^-(\alpha) = \beta_{10}, \ [x(t_{i+1})]_r^-(\alpha) = \beta_{11}
\end{cases}
$$

In a similar way, the initial fuzzy value problem $x'(t) = f(t, x(t))$ can be solved by Adams–Moulton two-step method as follows:

$$
\begin{cases}
[x(t_{i+2})]_l^+(\alpha) & = [x(t_{i+1})]_l^+(\alpha) - \frac{h}{12} f_r^+(t_{i-1}, x(t_{i-1}); \alpha) + \frac{2h}{3} f_l^+(t_i, x(t_i); \alpha) + \frac{5h}{12} f_l^+(t_{i+1}, x(t_{i+1}); \alpha) \\
[x(t_{i+2})]_r^+(\alpha) & = [x(t_{i+1})]_r^+(\alpha) - \frac{h}{12} f_l^+(t_{i-1}, x(t_{i-1}); \alpha) + \frac{2h}{3} f_r^+(t_i, x(t_i); \alpha) + \frac{5h}{12} f_r^+(t_{i+1}, x(t_{i+1}); \alpha) \\
[x(t_{i+2})]_l^-(\alpha) & = [x(t_{i+1})]_l^-(\alpha) - \frac{h}{12} f_r^-(t_{i-1}, x(t_{i-1}); \alpha) + \frac{2h}{3} f_l^-(t_i, x(t_i); \alpha) + \frac{5h}{12} f_l^-(t_{i+1}, x(t_{i+1}); \alpha) \\
[x(t_{i+2})]_r^-(\alpha) & = [x(t_{i+1})]_r^-(\alpha) - \frac{h}{12} f_l^-(t_{i-1}, x(t_{i-1}); \alpha) + \frac{2h}{3} f_r^-(t_i, x(t_i); \alpha) + \frac{5h}{12} f_r^-(t_{i+1}, x(t_{i+1}); \alpha) \\
[x(t_{i-1})]_l^+(\alpha) & = \beta_0, \ [x(t_i)]_l^+(\alpha) = \beta_1, \ [x(t_{i-1})]_r^+(\alpha) = \beta_2, \ [x(t_i)]_r^+(\alpha) = \beta_3 \\
[x(t_{i-1})]_l^-(\alpha) & = \beta_4, \ [x(t_i)]_l^-(\alpha) = \beta_5, \ [x(t_{i-1})]_r^-(\alpha) = \beta_6, \ [x(t_i)]_r^-(\alpha) = \beta_7
\end{cases}
\tag{34}
$$

6 Predictor-corrector three-step method

The following algorithm is based on Adams–Bashforth three-step method as a predictor and also an iteration of Adams–Moulton two-step method as a corrector.

Algorithm: To approximate the solution of following intuitionistic fuzzy initial value problem

$$
\begin{cases}
x'(t) & = f(t, x(t)), \ \ t \in I = [t_0, T] \\
[x(t_0)]_l^+(\alpha) & = \beta_0, [x(t_1)]_l^+(\alpha) = \beta_1, [x(t_2)]_l^+(\alpha) = \beta_2 \\
[x((t_0))]_r^+(\alpha) & = \beta_3, [x((t_1))]_r^+(\alpha) = \beta_4, [x((t_2))]_r^+(\alpha) = \beta_5 \\
[x((t_0))]_l^-(\alpha) & = \beta_6, [x((t_1))]_l^-(\alpha) = \beta_7, [x((t_2))]_l^-(\alpha) = \beta_8 \\
[x((t_0))]_r^-(\alpha) & = \beta_9, [x((t_1))]_r^-(\alpha) = \beta_{10}, [x((t_2))]_r^-(\alpha) = \beta_{11}
\end{cases}
$$

positive integer N is chosen

Step 1: Let $h = \frac{T - t_0}{N}$

$$
\begin{cases}
[y(t_0)]_l^+(\alpha) & = \beta_0, \ \ [y(t_1)]_l^+(\alpha) = \beta_1, \ \ [y(t_2)]_l^+(\alpha) = \beta_2 \\
[y(t_0)]_r^+(\alpha) & = \beta_3, \ \ [y(t_1)]_r^+(\alpha) = \beta_4, \ \ [y(t_2)]_r^+(\alpha) = \beta_5 \\
[y(t_0)]_l^-(\alpha) & = \beta_6, \ \ [y(t_1)]_l^-(\alpha) = \beta_7, \ \ [x(t_2)]_l^-(\alpha) = \beta_8 \\
[y(t_0)]_r^-(\alpha) & = \beta_9, \ \ [y(t_1)]_r^-(\alpha) = \beta_{10}, \ \ [y(t_2)]_r^-(\alpha) = \beta_{11}
\end{cases}
$$

Step 2: Let i=1.
Step 3: Let

$$\begin{cases} [y^{(0)}(t_{i+2})]_l^+(\alpha) &= [y(t_{i+1})]_l^+(\alpha) + \frac{h}{12}\Big[5f_l^+(t_{i-1}, y(t_{i-1}); \alpha) - 16f_r^+(t_i, y(t_i); \alpha) + 23f_l^+(t_{i+1}, y(t_{i+1}); \alpha)\Big] \\ [y^{(0)}(t_{i+2})]_r^+(\alpha) &= [y(t_{i+1})]_r^+(\alpha) + \frac{h}{12}\Big[5f_r^+(t_{i-1}, y(t_{i-1}); \alpha) - 16f_l^+(t_i, y(t_i); \alpha) + 23f_r^+(t_{i+1}, y(t_{i+1}); \alpha)\Big] \\ [y^{(0)}(t_{i+2})]_l^-(\alpha) &= [y(t_{i+1})]_l^-(\alpha) + \frac{h}{12}\Big[5f_l^-(t_{i-1}, y(t_{i-1}); \alpha) - 16f_r^-(t_i, y(t_i); \alpha) + 23f_l^-(t_{i+1}, y(t_{i+1}); \alpha)\Big] \\ [y^{(0)}(t_{i+2})]_r^-(\alpha) &= [y(t_{i+1})]_r^-(\alpha) + \frac{h}{12}\Big[5f_r^-(t_{i-1}, y(t_{i-1}); \alpha) - 16f_l^-(t_i, y(t_i); \alpha) + 23f_r^-(t_{i+1}, y(t_{i+1}); \alpha)\Big] \end{cases} \tag{35}$$

Step 4: Let $t_{i+2} = t_0 + (i+1)h$.
Step 5: Let

$$\begin{cases} [y(t_{i+2})]_l^+(\alpha) &= [y(t_{i+1})]_l^+(\alpha) - \frac{h}{12}f_r^+(t_i, y(t_i); \alpha) + \frac{2h}{3}f_l^+(t_{i+1}, y(t_{i+1}); \alpha) + \frac{5h}{12}f_l^+(t_{i+2}, y^{(0)}(t_{i+2}); \alpha) \\ [y(t_{i+2})]_r^+(\alpha) &= [y(t_{i+1})]_r^+(\alpha) - \frac{h}{12}f_l^+(t_i, y(t_i); \alpha) + \frac{2h}{3}f_r^+(t_{i+1}, y(t_{i+1}); \alpha) + \frac{5h}{12}f_r^+(t_{i+2}, y^{(0)}(t_{i+2}); \alpha) \\ [y(t_{i+2})]_l^-(\alpha) &= [y(t_{i+1})]_l^-(\alpha) - \frac{h}{12}f_r^-(t_i, y(t_i); \alpha) + \frac{2h}{3}f_l^-(t_{i+1}, y(t_{i+1}); \alpha) + \frac{5h}{12}f_l^-(t_{i+2}, y^{(0)}(t_{i+2}); \alpha) \\ [y(t_{i+2})]_r^-(\alpha) &= [y(t_{i+1})]_r^-(\alpha) - \frac{h}{12}f_l^-(t_i, y(t_i); \alpha) + \frac{2h}{3}f_r^-(t_{i+1}, y(t_{i+1}); \alpha) + \frac{5h}{12}f_r^-(t_{i+2}, y^{(0)}(t_{i+2}); \alpha) \end{cases} \tag{36}$$

Step 6: $i = i + 1$
Step 7: If $i \leq (N-2)$ go to **Step 3**.
Step 8: The algorithm ends, and $[y(T)]_l^+(\alpha),\ [y(T)]_r^+(\alpha),\ [y(T)]_l^-(\alpha),\ y(T)]_r^-(\alpha)$ approximates the value of $[x(T)]_l^+(\alpha),\ [x(T)]_r^+(\alpha),\ [x(T)]_l^-(\alpha),\ [x(T)]_r^-(\alpha)$.

7 Convergence and stability

Let the exact solutions

$$[X(t_n)]_\alpha = \Big[[X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\Big]$$

$$[X(t_n)]^\alpha = \Big[[X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\Big]$$

be approximated by

$$[x(t_n)]_\alpha = \Big[[x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\Big]$$

$$[x(t_n)]^\alpha = \Big[[x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\Big]$$

at t_n, $0 \leq n \leq N$.

The solutions are calculated by grid points at

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \;\; h = \frac{T - t_0}{N}, \;\; t_n = t_0 + nh, \;\; n = 0, 1, ...N \tag{37}$$

Our goal is to determine the convergence of the proposed methods to exact solutions, i.e. we will show

$$d_\infty\big(x(t_n), X(t_n)\big) \longrightarrow 0 \;\; when \;\;\; h \longrightarrow 0$$

Theorem 7.1. *For arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$, the Adams–Bashforth two-step approximates of Eq.(22) converge to the exact solutions $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $[X(t)]_l^-(\alpha)$, and $[X(t)]_r^-(\alpha)$ for $[X]_l^+$, $[X]_r^+$, $[X]_l^-$,$[X]_r^- \in C^3[t_0, T]$*

Proof. See [6]. □

Theorem 7.2. *For arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$, the Adams–Moulton two-step approximates of Eq.(34) converge to the exact solutions $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $[X(t)]_l^-(\alpha)$, and $[X(t)]_r^-(\alpha)$ for $[X]_l^+$, $[X]_r^+$, $[X]_l^-$,$[X]_r^- \in C^4[t_0, T]$*

Proof. See [6]. □

Remark 7.1. *The following statements hold:*

- *The convergence order of Adams–Moulton two-step method is $O(h^3)$.*
- *The convergence order of Adams–Bashforth two-step method is $O(h^2)$.*

Theorem 7.3. *The following statements hold:*

- *Adams–Bashforth two-step and three-step methods are stable.*
- *Adams–Moulton two-step and three-step methods are stable.*

Proof. For Adams–Bashforth two-step method exist only one characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$ and it is clear that satisfies the root condition then by Theorem 2.1, the Adams–Bashforth two-step method is stable.

Also, for Adams–Bashforth three-step method, exist only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda^2$ then it satisfies the root condition, therefore it is a stable method.

For the proofs of Adams–Moulton two-step and three-step methods are similar to the proof of Adams–Bashforth two-step and three-step methods. □

8 Example

Example 8.1. *Consider the intuitionistic fuzzy initial value problem*

$$\begin{cases} x'(t) &= x(t) \;\; for \;\; all \;\; t \in [0, T] \\ x_0 &= \Big(\big(\alpha - 1.0013, 1.0013 - \alpha\big), \big(-1.58\alpha, 1.58\alpha\big)\Big) \end{cases} \tag{38}$$

Applying the method of solution proposed in [12] we get

$$\begin{cases} [x(t)]_l^+(\alpha) = & (\alpha - 1.0013)\exp(t) \\ [x(t)]_r^+(\alpha) = & (1.0013 - \alpha)\exp(t) \\ [x(t)]_l^-(\alpha) = & 1.58(\alpha - 1.0013)\exp(t) \\ [x(t)]_r^-(\alpha) = & 1.58(1.0013 - \alpha)\exp(t) \end{cases}$$

Therefore the exact solutions is given by

$$[X(t)]_\alpha = \Big[(\alpha - 1.0013)\exp(t), (1.0013 - \alpha)\exp(t)\Big]$$

$$[X(t)]^\alpha = \Big[(1.58(\alpha - 1.0013)\exp(t), 1.58(1.0013 - \alpha)\exp(t)\Big]$$

which at $t = 1$ are

$$[X(1)]_\alpha = \Big[(\alpha - 1.0013)\exp(1), (1.0013 - \alpha)\exp(1)\Big]$$

$$[X(1)]^\alpha = \Big[(1.58(\alpha - 1.0013)\exp(1), 1.58(1.0013 - \alpha)\exp(1)\Big]$$

By using the Adams–Bashforth two-step method with $N = 30$ the following results are obtained:

α	$([x]_l^+, [x]_r^+)$	$([X]_l^+, [X]_r^+)$	$([x]_l^-, [x]_r^-)$	$([X]_l^-, [X]_r^-)$
0	(-2.72062357,2.72062357)	(-2.72181559,2.72181559)	(0,0)	(0,0)
0.2	(-2.17720530,2.17720530)	(-2.17815922,2.17815922)	(-0.85860086,0.85860086)	(-0.85897705,0.85897705)
0.4	(-1.63378702,1.63378702)	(-1.63450286,1.63450286)	(-1.71720173,1.71720173)	(-1.71795411,1.71795411)
0.6	(-1.09036875,1.09036875)	(-1.09084649,1.09084649)	(-2.57580260,2.57580260)	(-2.57693117,2.57693117)
0.8	(-0.54695048,0.54695048)	(-0.54719013,0.54719013)	(-3.43440346,3.43440346)	(-3.435908231,3.43590823)
1	(-0.00353221,0.00353221)	(-0.00353376,0.00353376)	(-4.29300433,4.29300433)	(-4.29488528,4.29488528)

α	$Error^+$	$Error^-$
0	(1.192024091501)1.0e-003	0
0.2	(9.53928797082)1.0e-004	(3.76190565179)1.0e-004
0.4	(7.15833502666)1.0e-004	(7.52381130359)1.0e-004
0.6	(4.77738208249)1.0e-004	(1.128571695538)1.0e-003
0.8	(2.39642913831)1.0e-004	(1.504762260717)1.0e-003
1	(1.547619414)1.0e-006	(1.880952825896)1.0e-003

and by using the predictor-corrector three-step method with N = 30 the following results are obtained:

α	$([x]_l^+,[x]_r^+)$	$([X]_l^+,[X]_r^+)$	$([x]_l^-,[x]_r^-)$	$([X]_l^-,[X]_r^-)$
0	(-2.72134120,2.72134120)	(-2.72181559,2.72181559)	(0,0)	(0,0)
0.2	(-2.17777959,2.17777959)	(-2.17815922,2.17815922)	(-0.85882734,0.85882734)	(-0.85897705,0.85897705)
0.4	(-1.63421798,1.63421798)	(-1.63450286,1.63450286)	(-1.71765469,1.71765469)	(-1.71795411,1.71795411)
0.6	(-1.09065637,1.09065637)	(-1.09084649,1.09084649)	(-2.57648203,2.57648203)	(-2.57693117,2.57693117)
0.8	(-0.54709476,0.54709476)	(-0.54719013,0.54719013)	(-3.43530938,3.43530938)	(-3.43590823,3.43590823)
1	(-0.00353315,0.00353315)	(-0.00353376,0.00353376)	(-4.29413673,4.29413673)	(-4.29488528,4.29488528)

α	$Error^+$	$Error^-$
0	(4.74387077374683)1.0e-004	0
0.2	(3.79632842404387)1.0e-004	(1.49711691250887)1.0e-004
0.4	(2.84878607435202)1.0e-004	(2.99423382501773)1.0e-004
0.6	(1.90124372465350)1.0e-004	(4.49135073755880)1.0e-004
0.8	(9.5370137496720)1.0e-005	(5.98846765003547)1.0e-004
1	(6.15902527301)1.0e-007	(7.48558456261428)1.0e-004

The exact and approximate solutions by Adams–Bashforth and predictor-corrector methods are plotted at $t = 1$ and $N = 30$ in Fig. 1.

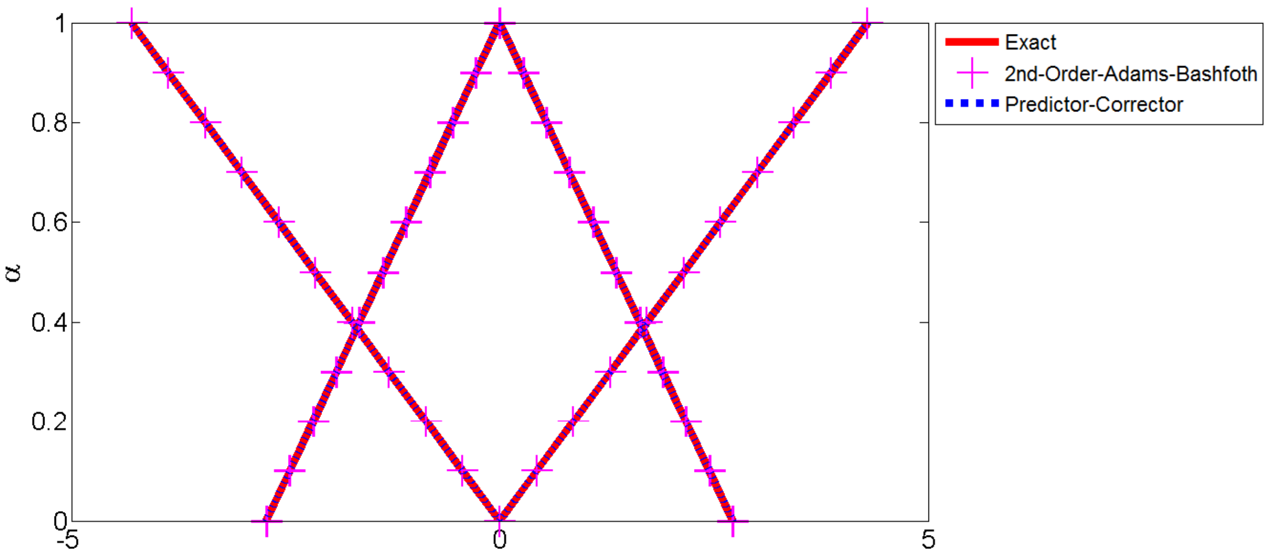

Figure 1: h=0.03

9 Conclusion

In this paper, we have applied iterative solution of Adam's predictor-corrector three order method for finding the numerical solution of intuitionistic fuzzy differential equations. Taking into account the convergence order of the Euler method is $O(h)$ (as given in [6]), a higher order of convergence is obtainable by using the methods proposed namely, that a predictor-corrector method of convergence order $O(h^m)$ be used where the Adams–Bashforth m-step method and Adams–Moulton $(m-1)$-step method are considered as predictor and corrector, respectively. For future research we can apply the predictor-corrector methods for systems and for the partial differential equations in the intuitionistic fuzzy setting, Also, we can apply these methods for solving the stiff problems.

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