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Modified ranking of intuitionistic fuzzy numbers

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Abstract: The notion of fuzzy subsets was introduced by Zadeh [19] and it was generalised to intuitionistic fuzzy subsets by Atanassov [1]. After the invention of intuitionistic fuzzy subsets, many real life problems are studied accurately. The ranking of intuitionistic number plays a main role in modeling many real life problems involving intuitionistic fuzzy decision making, intuitionistic fuzzy clustering. In this paper, a new method of intuitionistic fuzzy scoring to intuitionistic fuzzy number has been introduced and studied. The significance of the proposed intuitionistic fuzzy scoring method has been discussed. The aim of this paper is to introduce a new technique for clustering based on intuitionistic fuzzy number. The proposed scoring method has been applied to clustering problem where the data collected is in terms of intuitionistic fuzzy linguistic term which is converted into intuitionistic fuzzy number. The intuitionistic fuzzy number is converted to intuitionistic fuzzy scoring using the defined scoring method. A distance measure has been applied to intuitionistic fuzzy score and the similarity measure can be calculated with the help of obtained distance measure. Now we find that the association matrix is tolerance relation. By using the algorithm, the tolerance relation is converted to fuzzy equivalence relation. By fixing alpha cut, the data are clustered in to different groups. The new intuitionistic fuzzy scoring method has wide application in various fields.

1 Introduction

Decision making is a most important scientific, social and economic endeavor. In any decision process we weigh the information about an issue or outcome and choose among two or more alternatives for subsequent action. Let $M_1, M_2, ..., M_n$ be n alternatives available and $C_1, C_2, ..., C_m$ be m criteria's involved in the measurement of alternative. Let A_{ij} be the performance of alternative M_i with respect to criteria C_j and w_j be the relative importance of criteria. Then the decision making problem is the selection of the best alternative with respect to criteria.

In Classical decision making problems, A_{ij} and w_j are real numbers. In reality, A_{ij} and w_j are not necessarily real numbers. They may be linguistic terms like good, poor which are fuzzy in nature. In [2], [3], [9], a fuzzy version of Saaty's AHP [11] method was developed by using triangular fuzzy numbers for linguistic terms. In Fuzzy Decision making, the ranking of fuzzy numbers play a main role. There are many methods available in literature. Among them the ranking of fuzzy numbers by crisp score is introduced by Chen and Hwang in [2]. There are many real life problems in which A_{ij} are fuzzy linguistic terms with hesitation like good with little hesitation, bad with strong hesitation. Those problems can be studied by intuitionistic fuzzy decision making. The ranking of intuitionistic fuzzy numbers plays a main role in modeling many real life problems involving intuitionistic fuzzy decision making, intuitionistic fuzzy clustering. Mitchell introduced a method of ranking intuitionistic fuzzy numbers in [10]. In this paper, a new method of intuitionistic fuzzy number that generalizes Chen and Hwang's scoring method has been introduced for ranking of intuitionistic fuzzy numbers and is illustrated by an example.

This paper is organized as follows: In section 2, the preliminaries of fuzzy and intuitionistic fuzzy sets, fuzzy number and intuitionistic fuzzy number, Chen and Hwang's scoring method, intuitionistic fuzzy linguistic terms and the conversion of intuitionistic linguistic terms has been discussed. In section 3, a new score of fuzzy number is defined which inturn has been generalized to intuitionistic fuzzy numbers which involves the scores of both membership and non-membership. Some proposition has been defined and proved. In section 4, the significance of the proposed scoring method has been discussed. Some results of the proposed scoring method and some remarks of the existing scoring method has been discussed. In section 5, the proposed scoring method has been applied to clustering problem. Finally in section 6, the conclusions are drawn.

First we give a brief review of preliminaries.

2.1 Preliminaries

Definition 2.1.1[4] : A fuzzy number M is a fuzzy subset of the set of real numbers R which satisfies the following properties

i. M must be a normal fuzzy subset of R
ii. Each α - cut α_M must be a closed interval for every $\alpha \in (0, 1]$
iii. The support of M must be bounded.

Definition 2.1.2 [4]: A fuzzy number M is defined to be a triangular fuzzy number if its membership function $\mu_M : R \to [0 , 1]$ is equal to

$$\mu_M(z) = \begin{cases} \dfrac{z-a}{b-a} & if \;\; z \in [a,b] \\ \dfrac{z-c}{b-c} & if \;\; z \in [b,c] \\ 0 & otherwise \end{cases}$$

where $a \le b \le c$; a and c stand for the lower and upper values of the support of the fuzzy number M respectively and b for the modal value. This fuzzy number is denoted by (a,b,c). Now $\dfrac{z-a}{b-a}$ and $\dfrac{z-c}{b-c}$ are known as left and right legs of the triangular fuzzy number (a,b,c).

Definition 2.1.3 [17]: The complement μ^c of a fuzzy subset μ of a set X is a fuzzy subset given by $\mu^c(x)=1-\mu(x),\ x\in X$.

Note 2.1.1 [17] : Let $M=(a,b,c)$ be a triangular fuzzy number. The complement M^c of a triangular fuzzy number M is defined by $\mu_M^c(x)=1-\mu_M(x)$. Hence the membership function μ_M^c [see fig(i)] is defined by

$$\mu_M^c(z)=\begin{cases}\dfrac{b-z}{b-a} & if\ z\in[a,b]\\ \dfrac{b-z}{b-c} & if\ z\in[b,c]\\ 1 & otherwise\end{cases}$$

Definition 2.1.4 [4] : Let $\mu_{\max}$ and $\mu_{\min}$ be a maximizing fuzzy subset and minimizing fuzzy subset of R defined by

$$\mu_{\max}=\begin{cases}x & if\ x\in[0,1]\\ 0 & otherwise\end{cases}$$

and

$$\mu_{\min}=\begin{cases}1-x & if\ x\in[0,1]\\ 0 & otherwise\end{cases}$$

respectively. Then the right score of a triangular fuzzy number M can be determined by $R(M)=\sup(\mu_A(x)\wedge\mu_{\max}(x))$ and the left score of a triangular fuzzy number M can be determined by $L(M)=\sup(\mu_A(x)\wedge\mu_{\min}(x))$. Then the membership score of the fuzzy number M can be computed by $T(M)=\dfrac{[R(M)+1-L(M)]}{2}$.

Equivalently $R(M)$ is the ordinate of the intersecting point of $\mu_{\max}$ and right leg of μ_A and similarly $L(M)$ is the ordinate of the intersecting point of $\mu_{\min}$ and left leg of μ_A.

Note 2.1.2 [4] : Let (a,b,c) be a triangular fuzzy number. Then the total score of M is given by $T(M)=\dfrac{[R(M)+1-L(M)]}{2}$, where left score $L(M)$ and right score of $R(M)$ of this fuzzy number are given by $L(M)=\dfrac{1-a}{1+b-a}$ and $R(M)=\dfrac{c}{1+c-b}$ [see fig 2].

Definition 2.1.5 [1] : Let X be a nonempty set. An intuitionistic fuzzy set(IFS) A in X is defined by $A=(\mu_A,\nu_A)$, where $\mu_A:X\to[0\ ,1]$ and $\nu_A:X\to[0\ ,1]$ with the conditions $0\le\mu_A(x)+\nu_A(x)\le1,\ \forall x\in X$. The numbers $\mu_A(x),\nu_A(x)\in[0\ ,1]$ denote the degree of membership and non-membership of x to lie in A respectively. For each intuitionistic fuzzy subset A in X, $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ is called hesitancy degree of x to lie in A.

Definition 2.1.6 [18] : A mapping $S:IFS(X)\ x\ IFS(X)\to[0\ ,1]$ is said to be the similarity measure if it satisfies the following properties:

S1: $0 \le S(A,B) \le 1$, for every $A,B \in IFS\ (X)$

S2: *If* $A = B$, $S(A,B) = 1$

S3: $S(A,B) = S(B,A)$

S4: If $A \subseteq B \subseteq C$, $A, B, C \in IFS(X)$ then $S(A,C) \le S(A,B)$ and $S(A,C) \le S(B,C)$.

Definition 2.1.7 [18] : A mapping $d : IFS(X)\ x\ IFS(X) \to [0\ ,1]$ is said to be the distance measure if $d(A,B)$ satisfies the following properties:

D1: $0 \le d(A,B) \le 1$

D2: $d(A,B) = 0$ *iff* $A = B$

D3: $d(A,B) = d(B,A)$

D4: If $A \subseteq B \subseteq C$, then $d(A,C) \ge d(A,B)$ and $d(A,C) \ge d(B,C)$. Equivalently

D5 : If $A \subseteq B \subseteq C \subseteq D$, $A, B, C, D \in IFS(X)$ then $d(A,D) \ge d(B,C)$.

Definition 2.1.8 [16] : A fuzzy relation R defined on X is said to be reflexive if $\mu_R(x,x) = 1$ for every $x \in X$. A fuzzy relation R defined on X is said to be symmetric if $\mu_R(x,y) = \mu_R(y,x)$ for every $x, y \in X$. A fuzzy relation R defined on X is said to be transitive if $\mu_R(x,z) \ge \min\{\mu_R(x,y), \mu_R(y,z)\}$ for every $x, y, z \in X$.

Definition 2.1.9 [21] : Let $A_j\,(j = 1,2,........\ m)$ be m IFSs, then $C = \left(c_{ij}\right)_{m\times m}$ is called an association matrix, where $c_{ij} = c\left(A_i, A_j\right)$ is the association coefficient of A_i and A_j, which has the following properties:

(1) $0 \le c_{ij} \le 1$, for all $i, j = 1,\ 2,\ ....\ m$;

(2) $c_{ij} = 1$ if and only if $A_i = A_j$;

(3) $c_{ij} = c_{ji}$, for all $i, j = 1,\ 2,\ ....\ m$.

Note 2.1.3 [16] : A fuzzy relation R defined on X is said to be a fuzzy tolerance relation if R is reflexive and symmetric. A fuzzy relation R defined on X is said to be fuzzy equivalence relation if R is a fuzzy tolerance relation and transitive.

Note 2.1.4 [16] : Every fuzzy equivalence relation is a fuzzy tolerance relation. But the converse need not be true. A fuzzy tolerance relation can be extended to a fuzzy equivalence relation by the following algorithm.

Algorithm 2.1.1 [21] : Let $C = (c_{ij})_{m\times m}$ be an association matrix, if $C^2 = C \circ C = \left(\bar{c_{ij}}\right)_{m\times m}$, then C^2 is called a composition matrix of C, where $\bar{c_{ij}} = \underset{k}{Max}\{Min\{c_{ik}, c_{kj}\}\}$, $i, j = 1, 2, ... m$.

2.2 Intuitionistic fuzzy numbers

In this section, the notion of intuitionistic fuzzy numbers are introduced and studied.

Definition 2.2.1 [1]: An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of R is said to be an intuitionistic fuzzy number if μ_A and ν_A are fuzzy numbers with $\nu_A \le \mu_A^c$, where μ_A^c denotes the complement of μ_A.

Definition 2.2.2 [17]: A triangular intuitionistic fuzzy number A is defined by $A = \{(\mu_A, \nu_A) \mid x \in R\}$, where μ_A and ν_A are triangular fuzzy numbers with $\nu_A(x) \le \mu_A^c(x)$. So a triangular intuitionistic fuzzy number A is given by $A = \{(a,b,c),(e,f,g)\}$ with $(e,f,g) \le (a,b,c)^c$ i.e., either $e \ge b$ and $f \ge c$ or $f \le a$ and $g \le b$ where (a,b,c) and (e,f,g) are membership and non-membership fuzzy numbers of A.

An intuitionistic fuzzy numbers $\{(a,b,c),(e,f,g)\}$ with $e \ge b$ and $f \ge c$ is shown in fig 3.

2.3 Intuitionistic fuzzy linguistic terms

Whenever we collect data, we can't study the expert's exact thinking because he may have some hesitancy. It is not possible to consider the non membership grade by his opinion. For example a student says that a particular staff can be given 0.6 membership grade for punctuality but if we may ask him whether the particular staff can be given 0.4 non-membership grade for punctuality, then he definitely shows some hesitation. The problem can be dealt effectively, if we calculate the non membership grade using hesitation. So in real life problems, one can model an expert's opinion easily by intuitionistic fuzzy subsets.

Definition 2.3.1: An intuitionistic fuzzy linguistic term is given by fuzzy linguistic term along the level of hesitation. It is denoted by (linguistic term, level of hesitation), where the level of hesitation is also a fuzzy linguistic term. In the above example the student can give his opinion as good with little hesitation or good with strong hesitation or good with no hesitation according to his hesitation or confidence. This way of approach generalizes ordinary fuzzy linguistic terms by giving no hesitation. Some of the possible linguistic terms are (good, little) , (good, Moderate) , (Excellent , no hesitation). In this paper, the linguistic terms for the level of hesitation are very little, little, moderate, strong and very strong.

2.4 Conversion of intuitionistic linguistic terms

An intuitionistic fuzzy term can be converted into triangular intuitionistic fuzzy number using triangular fuzzy numbers. The conversion of linguistics terms for both the membership and hesitancy into triangular fuzzy numbers is given by table 3 and 4.

3 A new score of fuzzy numbers

In this section, a new method of scoring is introduced and studied.

Definition 3.1 : A right score $R(M)$ of fuzzy number M is defined as the ordinate of the intersecting point of $\mu_{\min}$ and right leg of μ_M and a left score $L(M)$ of fuzzy number M is defined as the ordinate of the intersecting point of $\mu_{\max}$ and left leg of μ_M. Let (a,b,c) be a triangular fuzzy number. Then the total score of M is given by $T(M) = \frac{[L(M)+1-R(M)]}{2}$, where left score $L(M)$ and right score of $R(M)$ of this fuzzy number are given by $L(M) = \frac{a}{1-b+a}$ and $R(M) = \frac{1-c}{1-c+b}$ as shown in fig 4.

3.1 Non-membership score of fuzzy numbers

Let $\mu_M = (a,b,c)$ be a fuzzy number. Then the complement $\mu_M^c : R \to [0,1]$ of μ_M is given by $\mu_M^c(z) = \begin{cases} \frac{z-b}{c-b} & if\ z \in [b,c] \\ \frac{z-a}{b-a} & if\ z \in [a,b] \\ 1 & otherwise \end{cases}$

Here the lines $y = \frac{z-b}{c-b}$ in $[b,c]$ and $y = \frac{z-a}{b-a}$ in $[a,b]$ are called the right and left legs of μ_M^c. Now define R_c score and NL_c score are the ordinates of the intersection of $\mu_{\max}$ with the right leg of μ_M^c and left leg of μ_M^c respectively. Similarly define NR_c score and L_c score are the ordinates of the intersection of $\mu_{\min}$ with the right leg of μ_M^c and left leg of μ_M^c respectively. Define the scores T_c of μ_M^c by $T_c = \frac{1-R_c(a,b,c)+L_c(a,b,c)}{2}$ and NT_c by $NT_c(e,f,g) = \frac{1-NL_c(e,f,g)+NR_c(e,f,g)}{2}$.

Proposition 3.1 : The score T_c and NT_c are the membership scores of μ_M^c. Equivalently $T_c = NT$ and $NT_c = T$.

Let $M = \{(a,b,c),(e,f,g)\}$ with $(e,f,g) \le (a,b,c)^c$ i.e., either $e \ge b$ and $f \ge c$ or $f \le a$ and $g \le b$, then we shall prove the following proposition.

Proposition 3.2 : Let $M = \{(a,b,c),(e,f,g)\}$ with $e \ge b$ and $f \ge c$. The non-membership total score NT_c of the non-membership fuzzy number (e,f,g) is less than or equal to the non-membership total score T_c of the membership fuzzy number (a, b, c) i.e., $NT_c(e, f, g) < T_c(a, b, c)$.

Proof: By definition, $NT_c(e,f,g) = \dfrac{1 - NL_c(e,f,g) + NR_c(e,f,g)}{2}$

$$\text{and } T_c = \frac{1 - R_c(a,b,c) + L_c(a,b,c)}{2}.$$

First we prove the following claim.

Claim: $R(e,f,g) \geq R_c(a,b,c)$ and $L(e,f,g) \leq L_c(a,b,c)$ see fig 5.

Proof of the claim:

To prove that $NT_c(e,f,g) = \dfrac{1 - NL_c + NR_c}{2} < \dfrac{1 - R_c + L_c}{2} = T_c(a,b,c)$

i.e., to prove that $1 - \dfrac{1 + NL_c - NR_c}{2} < 1 - \dfrac{1 - L_c + R_c}{2}$

i.e., to prove that $\dfrac{1 + NL_c - NR_c}{2} > \dfrac{1 - L_c + R_c}{2}$

i.e., to prove that $\dfrac{1 + R - L}{2} > \dfrac{1 - L_c + R_c}{2}$

i.e., to prove that $R(e,f,g) > R_c(a,b,c)$ and $L(e,f,g) < L_c(a,b,c)$

By definition $R(e,f,g) = \dfrac{1-c}{1+b-c}$ and $R_c(a,b,c) = \dfrac{1-b}{1+c-b}$. Since $g \leq b$ and clearly $1 - g \geq 1 - b$. If $1 + f - g \leq 1 + c - b$, by definition $\dfrac{1}{1+f-g} \geq \dfrac{b}{1+c-b}$ and hence $\dfrac{1-g}{1+f-g} \geq \dfrac{1-b}{1+c-b}$. Hence $R(e,f,g) \geq R_c(a,b,c)$. If $1 + f - g \geq 1 + c - b$, we have to prove $R(e,f,g) \geq R_c(a,b,c)$.

The corresponding abscissa $x_{R(e,f,g)}$ of $R(e,f,g)$ is given by substituting $R(e,f,g)$ in $\mu_{\min}$. So $x_{R(e,f,g)} = \dfrac{f}{1+f-g}$. Similarly the corresponding abscissa $x_{R_c(a,b,c)}$ of $R_c(a,b,c)$ is given by substituting $R_c(a,b,c)$ in $\mu_{\max}$. So $x_{R_c(a,b,c)} = \dfrac{c}{1+c-b}$. Since $f \leq c$ and $1 + f - g \geq 1 + c - b$, then $\dfrac{f}{1+f-g} \geq \dfrac{c}{1+c-b}$. Hence $x_{R(e,f,g)} \leq x_{R_c(a,b,c)}$. By the definition of $\mu_{\min}$, it is a continuous function in [0, 1]. By differentiating $\dfrac{d\mu_{\min}}{dx} = -1 < 0$ and hence $\mu_{\min}$ is monotonically decreasing. Hence $R(e,f,g) > R_c(a,b,c)$.

By monotonic increasing property of $\mu_{\max}$ and by corresponding abscissa of $L(e, f, g)$ and $L_c(a,b,c)$, we can prove that $L(e, f, g) \leq L_c(a,b,c)$. By definition $L(e, f, g) = \dfrac{e}{1-f+e}$ and $L_c(a, b, c) = \dfrac{b}{1-a+b}$. Since $e \leq b$ and if $1-f+e \geq 1-a+b$ then $\dfrac{1}{1-f+e} \leq \dfrac{1}{1-a+b}$ by definition $\dfrac{e}{1-f+e} \leq \dfrac{b}{1-a+b}$ and hence $L(e, f, g) < L_c(a,b,c)$. Hence the claim is proved.
Hence $NT_c(e, f, g) < T_c(a, b, c)$ is proved.

Proposition 3.3 : Let $M = \{(a, b, c), (e, f, g)\}$ with $e \geq b$ and $f \geq c$. If T is the membership score of intuitionistic fuzzy number M and NT_c is the non-membership score of intuitionistic fuzzy number M, then $\dfrac{NT_c(e, f, g) + T(a,b,c)}{2} \leq 1$.

Definition 3.2: Let $M = \{(a, b, c), (e, f, g)\}$ be an intuitionistic triangular fuzzy number. If $e \geq b$ and $f \geq c$, then the score of the intuitionistic fuzzy number M is defined by (T, NT_c), where T is the membership score of M which is obtained from (a,b,c) and NT_c is the non-membership score of M which is obtained from (e, f, g).

$$
\begin{aligned}
NT_c(e, f, g) &= \frac{1 - NL_c(e, f, g) + NR_c(e, f, g)}{2} \\
&= \frac{1 + L(a,b,c) - R(a,b,c)}{2} \\
&= T(a,b,c).
\end{aligned}
$$

4 Significance of the proposed method

Remarks 4.1: The intuitionistic fuzzy score of the given intuitionistic fuzzy number defined in [17] fails to get the score for intuitionistic fuzzy number $M = \{(0, 0, 0), (1, 1, 1)\}$. The score of T is 0 and the score for NT_c is also 0.

Remarks 4.2: The ranking method defined in [17] extension of Chen and Hwang fails to rank the intuitionistic fuzzy numbers in some situations. So we define the total score T for membership function different from the total score T defined in section 3. From the defined total score, we define the total score NT_c for non-membership function.

Remarks 4.3: Chen and Hwang's score are $T(0.25, 0.25, 0.25) = 0.25$, $T(0.15, 0.25, 0.35) = 0.2727$ and $T(0.1, 0.25, 0.4) = 0.2826$. Hence $T(0.25, 0.25, 0.25) < T(0.15, 0.25, 0.35) < T(0.1, 0.25, 0.4)$ which contradicts human intuition.

But the defined total score for $T(0.25,\ 0.25,\ 0.25) = 0.25$, $T(0.15,\ 0.25,\ 0.35) = 0.223$ and $T(0.1,\ 0.25,\ 0.4) = 0.206$. Hence $T(0.25,\ 0.25,\ 0.25) > T(0.15,\ 0.25,\ 0.35) > T(0.1,\ 0.25,\ 0.4)$.

Remarks 4.4: In Chen and Hwang's method the crisp score of $T(0,\ 0.2,\ 0.244) = 0.2002$ and $T(0.2,\ 0.2,\ 0.2) = 0.2$. Hence $T(0,\ 0.2,\ 0.244) > T(0.2,\ 0.2,\ 0.2)$ which is anti-intuitive. Also human intuitive says $T(0.8,\ 0.9, 1.0) > T(0.864,\ 0.864,\ 0.864)$ .But in Chen's $T(0.8,\ 0.9, 1.0) = 0.836 < 0.864 = T(0.864,\ 0.864,\ 0.864)$. But the new total score of $T(0,\ 0.2,\ 0.244) = 0.105$ and $T(0.2,\ 0.2,\ 0.2) = 0.2$ (ie.,) $T(0,\ 0.2,\ 0.244) < T(0.2,\ 0.2,\ 0.2)$ and $T(0.8,\ 0.9, 1.0) = 0.944 > 0.864 = T(0.864,\ 0.864,\ 0.864)$ which favors human intuition.

4.1 Results of the proposed method

Results 4.1.1 : If $f \le c$ and $f \le a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then $NT_c(e, f, g) \le T(a, b, c)$.
Results 4.1.2 : If $f \le c$ and $f \le a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then $NT_c(e, f, g) \le T(a, b, c)$.
Results 4.1.3 : If $f \ge c$ and $f \ge a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \le 1 - b + a$, then $NT_c(e, f, g) \ge T(a, b, c)$.
Results 4.1.4 : If $f \ge c$ and $f \le a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \le 1 - b + a$, then the results cannot be predicted.
Results 4.1.5 : If $f \ge c$ and $f \ge a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then $NT_c(e, f, g) \ge T(a, b, c)$.
Results 4.1.6 : If $f \le c$ and $f \le a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \le 1 - b + a$, then the results cannot be predicted.
Results 4.1.7 : If $f \le c$ and $f \le a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \le 1 - b + a$, then $NT_c(e, f, g) \le T(a, b, c)$.
Results 4.1.8 : If $f \ge c$ and $f \ge a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \le 1 - b + a$, then $NT_c(e, f, g) \ge T(a, b, c)$.
Results 4.1.9 : If $f \ge c$ and $f \ge a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then $NT_c(e, f, g) \ge T(a, b, c)$.
Results 4.1.10 : If $f \ge c$ and $f \le a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then $NT_c(e, f, g) \le T(a, b, c)$.
Results 4.1.11 : If $f \ge c$ and $f \le a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then the results cannot be predicted.
Results 4.1.12 : If $f \ge c$ and $f \le a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \le 1 - b + a$, then the results cannot be predicted.
Results 4.1.13 : If $f \le c$ and $f \ge a$ and if $1 - f + e \le 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then then the results cannot be predicted.
Results 4.1.14 : If $f \le c$ and $f \ge a$ and if $1 - f + e \ge 1 - b + a$ and $1 + f - g \ge 1 - b + a$, then then the results cannot be predicted.

Results 4.1.15 : If $f \leq c$ and $f \geq a$ and if $1-f+e \geq 1-b+a$ and $1+f-g \leq 1-b+a$, then then the results cannot be predicted.
Results 4.1.16 : If $f \leq c$ and $f \geq a$ and if $1-f+e \leq 1-b+a$ and $1+f-g \leq 1-b+a$, then then the results cannot be predicted.

5 Intuitionistic fuzzy clustering

The intuitionistic fuzzy clustering is the problem of obtaining the group of clusters from data points $X_1, X_2, X_3, \ldots.., X_m$ based on the features $p_1, p_2, p_3, \ldots.., p_n$ which are intuitionistic fuzzy in nature. The data are evaluated by features in terms of intuitionistic fuzzy linguistic terms and they are converted into intuitionistic fuzzy scores. Using the new intuitionistic fuzzy distance formula, a tolerance relation is defined and the tolerance relation is converted into equivalence relation by using the algorithm [2.2.1]. By reasonable alpha cuts, the data are clustered.

5.1 Distance measure

Let $A = (TA(x_i), NT_c A(x_i), \pi_A)$ and $B = (TB(x_i), NT_c B(x_i), \pi_B)$ be two intuitionistic fuzzy sets. Let TA, TB be the membership scores of intuitionistic fuzzy numbers corresponding to A and B. Let $NT_c A$, $NT_c B$ be the non - membership scores of intuitionistic fuzzy numbers corresponding to A and B. Let πA, πB be the hesitancy scores of intuitionistic fuzzy numbers corresponding to A and B. Distance between A and B is given by

$$d(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[(TA(x_i) - TB(x_i))^2 + (NT_c A(x_i) - NT_c B(x_i))^2 + (\pi A(x_i) - \pi B(x_i))^2\right]}$$

5.2 Similarity measure

Let $A = (TA(x_i), NT_c A(x_i), \pi A)$ and $B = (TB(x_i), NT_c B(x_i), \pi B)$ be two intuitionistic fuzzy sets. Let TA , $NT_c A$, πA be the membership , non-membership and hesitancy respectively for the intuitionistic fuzzy set for A. Similarly TB , $NT_c B$, πB be the membership , non-membership and hesitancy respectively for the intuitionistic fuzzy set for B.

The similarity measure between A and B is calculated as $S(A,B) = 1 - d(A,B)$ where $d(A,B)$ is the proposed distance measure.

5.3 Clustering technique

The intuitionistic fuzzy clustering is the problem of obtaining the group of clusters from data points $A_1, A_2, A_3, \ldots.., A_n$ based on the features $X_1, X_2, X_3, \ldots.., X_m$ which are intuitionistic fuzzy in nature. By using similarity measure defined in [ ], the fuzzy tolerance relation is obtained. By using algorithm by Zeshui Xu (2008), fuzzy equivalence relation $R = (r_{ij})$ on the set of data points $A_1, A_2, A_3, \ldots.., A_n$ can be found where $r_{ij} = S(X_i, X_j)$. By the acceptable level of alpha cuts, the data are clustered.

5.4 Illustration

The structure of the typical intuitionistic fuzzy clustering problem considered here consists of ten students $X_1, X_2, X_3, \ldots\ldots, X_m$ who give feedback to three teachers $S_1, S_2, \ldots\ldots\ldots, S_3$ based on six criteria such as Punctuality, Innovative teaching, Communication skills, Given practical examples, Completion of Syllabus, counseling. The students are clustered by taking their skill into account based on their psychological opinions on their staff. Let A_{ij} denote the feedback given to the i^{th} staff based on the j^{th} criterion which is an intuitionistic fuzzy linguistic term (as shown in table 1). The intuitionistic fuzzy linguistic terms are converted to intuitionistic fuzzy numbers (as shown in table 2 and 3) which are in turn converted to intuitionistic fuzzy score (as shown in table 5) using new technique defined in the section 3. For eg. let the data collected from the expert be (Fairly good, Very little). The fuzzy linguistic term fairly good for the membership function has the triangular fuzzy numbers (0.6, 0.7, 0.8) as shown in table 2. Similarly the fuzzy linguistic term Very little for hesitancy has the triangular fuzzy number (0.1, 0.1, 0.2). Now the intuitionistic fuzzy linguistic term (Fairly good, Very little) has the intuitionistic fuzzy number ((0.6, 0.7, 0.8), (0.1, 0.1, 0.2)). The intuitionistic fuzzy score for (Fairly good, Very little) is (0.722, 0.106). Similarly the intuitionistic fuzzy score for the other intuitionistic fuzzy numbers can be found. By using the normalized distance formula for the intuitionistic fuzzy score, we get the distance between any pair of students. Distance between X_1 and X_2 i.e.,

$$d(X_1, X_2) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[(TA(x_i) - TB(x_i))^2 + (NT_cA(x_i) - NT_cB(x_i))^2 + (\pi A(x_i) - \pi B(x_i))^2\right]}$$

From the obtained distance, we can get the similarity between the corresponding pair of students. The similarity measure between X_1 and X_2 can be found as $s(X_1, X_2) = 1 - d(X_1, X_2) = 0.127$. The similarity between each pair of students is as follows.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0.127	0.390	0.685	0.135	0.333	0.700	0.152	0.342	0.698
X_2	0.127	0	0.381	0.671	0.158	0.313	0.691	0.141	0.347	0.687
X_3	0.390	0.381	0	0.417	0.386	0.188	0.452	0.386	0.255	0.457
X_4	0.685	0.671	0.417	0	0.683	0.428	0.221	0.661	0.421	0.205
X_5	0.135	0.158	0.386	0.683	0	0.318	0.690	0.181	0.339	0.684
X_6	0.333	0.313	0.188	0.428	0.318	0	0.442	0.315	0.202	0.437
X_7	0.700	0.691	0.452	0.221	0.690	0.442	0	0.676	0.428	0.180
X_8	0.152	0.141	0.386	0.661	0.181	0.315	0.676	0	0.331	0.676
X_9	0.342	0.347	0.255	0.421	0.339	0.202	0.428	0.331	0	0.417
X_{10}	0.698	0.687	0.457	0.205	0.684	0.437	0.180	0.676	0.417	0

Hence the similarity measure for other pair of data is obtained and it seems to be a tolerance relation. Applying algorithm [2.1] to the tolerance relation, we get fuzzy equivalence relation.

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{array} \\
\begin{array}{c} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{array} &
\left(\begin{array}{cccccccccc}
1 & 0.873 & 0.687 & 0.583 & 0.865 & 0.687 & 0.583 & 0.859 & 0.687 & 0.583 \\
0.873 & 1 & 0.687 & 0.583 & 0.865 & 0.687 & 0.583 & 0.859 & 0.687 & 0.583 \\
0.687 & 0.687 & 1 & 0.583 & 0.687 & 0.812 & 0.583 & 0.687 & 0.798 & 0.583 \\
0.583 & 0.583 & 0.583 & 1 & 0.583 & 0.583 & 0.795 & 0.583 & 0.583 & 0.795 \\
0.865 & 0.865 & 0.687 & 0.583 & 1 & 0.687 & 0.583 & 0.859 & 0.687 & 0.583 \\
0.687 & 0.687 & 0.812 & 0.583 & 0.687 & 1 & 0.583 & 0.687 & 0.798 & 0.583 \\
0.583 & 0.583 & 0.583 & 0.795 & 0.583 & 0.583 & 1 & 0.583 & 0.583 & 0.820 \\
0.859 & 0.859 & 0.687 & 0.583 & 0.859 & 0.687 & 0.583 & 1 & 0.687 & 0.583 \\
0.687 & 0.687 & 0.798 & 0.583 & 0.687 & 0.798 & 0.583 & 0.687 & 1 & 0.583 \\
0.583 & 0.583 & 0.583 & 0.795 & 0.583 & 0.583 & 0.820 & 0.583 & 0.583 & 1
\end{array}\right)
\end{array}$$

Now we can fix some reasonable alpha cuts, so that the data are clustered in to different groups. At $\alpha = 0.798$, the data are clustered in to three groups as $\{X_1, X_2, X_5, X_8\}$, $\{X_3, X_6, X_9\}$, and $\{X_4, X_7, X_{10}\}$ as shown below:

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{array} \\
\begin{array}{c} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{array} &
\left(\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right)
\end{array}$$

6 Conclusions

In this paper, a new intuitionistic fuzzy scoring method has been defined for the intuitionistic fuzzy number in which hesitation is greater than membership fuzzy number. Similarly, we can study the intuitionistic fuzzy number in which the hesitation is less than membership fuzzy number. This new method includes the concept of both membership and non-membership function of an intuitionistic fuzzy number. By this defined method, we can study the problems of involving hesitation easily. In this paper, the defined intuitionistic fuzzy scoring method has been applied to clustering problem.

Figure:

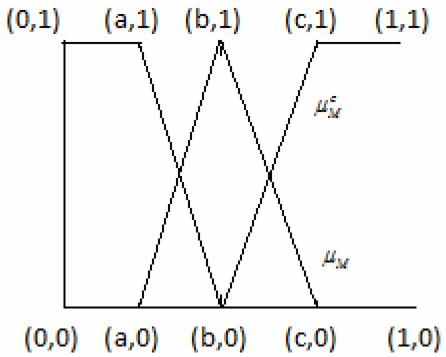

Fig (1) - Triangular fuzzy number
(a, b, c) and its complement

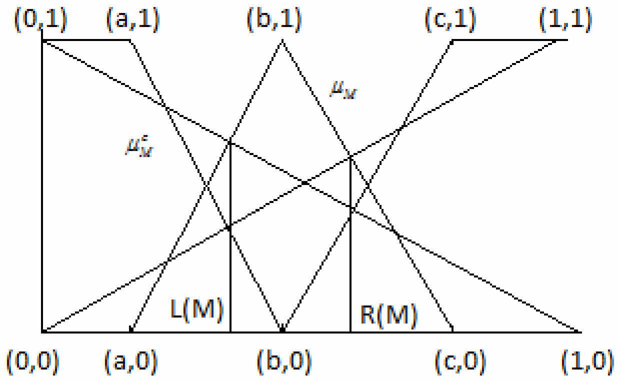

Fig (2) - Chen and Hwang scores of
a fuzzy number M= (a, b, c)

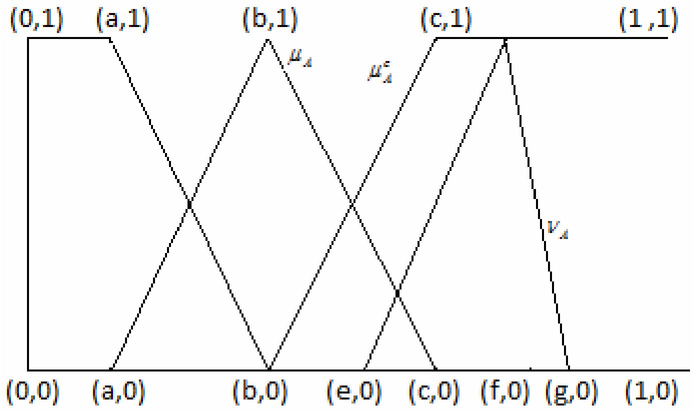

Fig 3 - Triangular intuitionistic fuzzy number
A = { (a,b,c),(e,f,g)}

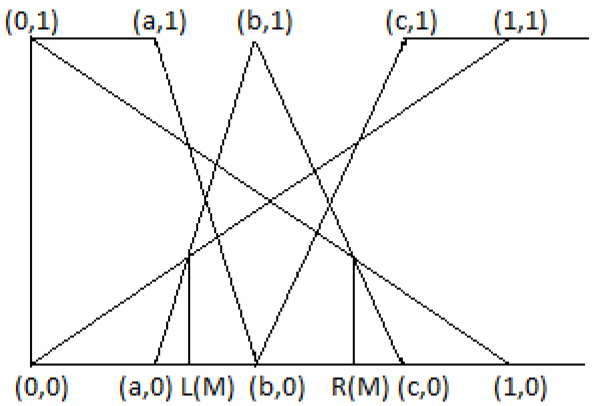

Fig 4 - The proposed scores of a fuzzy
number M = (a,b,c)

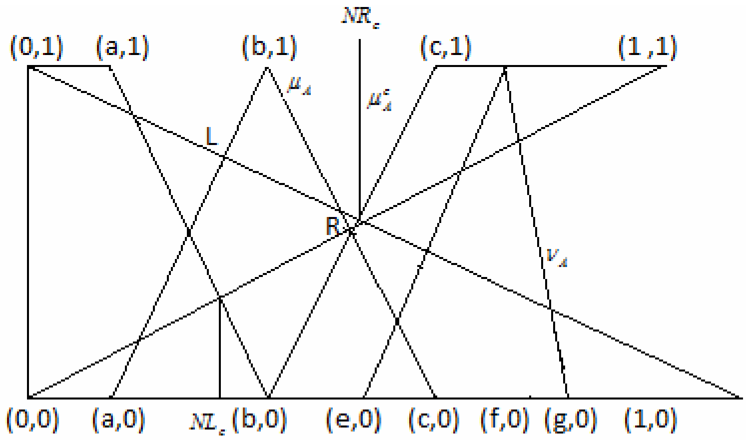

Fig 5: Membership and non-Membership score of a intuitionistic fuzzy number M={(a,b,c),(e,f,g)}

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Table 1: Opinions of students about features on staff in terms of Intuitionistic linguistic term

X_1	S_1	S_2	S_3
Punctuality	(VG, VL)	(FVG, VL)	(VG, VL)
Innovative Teaching	(FVG, VL)	(VG, M)	(FVG, M)
Communication Skills	E	(FE, VL)	E
Given Practical Examples	(VG, M)	E	(FE, VL)
Completion of Syllabus	E	(VG, L)	E
Counseling	(VG, L)	E	(FVG, M)

X_2	S_1	S_2	S_3
Punctuality	(FVG, L)	FE	E
Innovative Teaching	(VG, VL)	E	(VG, L)
Communication Skills	E	(VG, VL)	FE
Given Practical Examples	(FVG, L)	(FVG, M)	(FVG, VL)
Completion of Syllabus	E	(FVG, M)	E
Counseling	FE	E	FE

X_3	S_1	S_2	S_3
Punctuality	(N, M)	(FG, L)	(FG, L)
Innovative Teaching	(G, L)	(N, L)	(G, M)
Communication Skills	(FG, S)	(FG, M)	(N, VL)
Given Practical Examples	(N, L)	(FVG, M)	(FN, VL)
Completion of Syllabus	(FVG, S)	(N, VL)	L
Counseling	(FG, S)	(G, M)	(FG, L)

X_4	S_1	S_2	S_3
Punctuality	(FN, L)	L	P
Innovative Teaching	(FN, VL)	P	(FN, L)
Communication Skills	(FN, VL)	P	L
Given Practical Examples	L	(FN, L)	P
Completion of Syllabus	P	(FN, VL)	P
Counseling	P	(FN, L)	(FN, VL)

X_5	S_1	S_2	S_3
Punctuality	(FG, L)	(VG, VL)	FE
Innovative Teaching	(FVG, L)	(FVG, L)	E
Communication Skills	FE	(VG, VL)	(VG, L)
Given Practical Examples	E	FE	E
Completion of Syllabus	(VG, L)	E	(VG, L)
Counseling	E	FE	(FVG, L)

X_6	S_1	S_2	S_3
Punctuality	(FG, L)	(FVG, M)	(FG, L)
Innovative Teaching	(FVG, VL)	(G, L)	(FVG, L)
Communication Skills	(FG, L)	(FG, M)	(FG, VL)
Given Practical Examples	(N, L)	(G, L)	(N, VL)
Completion of Syllabus	(FG, L)	(FG, VL)	(N, L)
Counseling	(FVG, VL)	(N, VL)	(FG, M)

X_7	S_1	S_2	S_3
Punctuality	P	P	L
Innovative Teaching	(FN, VL)	(N, VL)	P
Communication Skills	L	L	P
Given Practical Examples	P	P	(FN, VL)
Completion of Syllabus	(N, L)	(FN, VL)	L
Counseling	L	L	(N, L)

X_8	S_1	S_2	S_3
Punctuality	FE	E	(FVG, L)
Innovative Teaching	E	(FVG, L)	(VG, L)
Communication Skills	(VG, VL)	E	E
Given Practical Examples	(FVG, M)	(FG, VL)	(VG, VL)
Completion of Syllabus	(FE, L)	FE	(FVG, VL)
Counseling	(VG, VL)	(VG, VL)	E

X_9	S_1	S_2	S_3
Punctuality	(G, L)	(N, M)	(FG, L)
Innovative Teaching	(N, VL)	(G, L)	(N, VL)
Communication Skills	(G, L)	(FG, L)	(G, L)
Given Practical Examples	(FG, L)	(N, VL)	(FG, L)
Completion of Syllabus	(N, VL)	FVG	(G, M)
Counseling	(G, L)	(G, L)	(N, L)

X_{10}	S_1	S_2	S_3
Punctuality	L	P	(FN, L)
Innovative Teaching	(N, L)	L	L
Communication Skills	L	(FN, L)	P
Given Practical Examples	P	L	L
Completion of Syllabus	L	(N, L)	(FN, VL)
Counseling	(FN, VL)	L	P

E – Excellent FE – Fairly Excellent VG – Very Good FVG – Fairly Very Good
G – Good FG – Fairly Good N – Normal FN – Fairly Normal L – Low P – Poor
L – Little VL – Very Little M – Moderate S – Strong VS – Very Strong

Table 2: Conversion of linguistic terms for membership

Linguistic terms	Fuzzy Numbers
Poor	(0.1, 0.1, 0.2)
Low	(0.1, 0.2, 0.3)
Fairly Normal	(0.2, 0.3, 0.4)
Normal	(0.3, 0.4, 0.5)
Fairly Good	(0.4, 0.5, 0.6)
Good	(0.5, 0.6, 0.7)
Fairly Very Good	(0.6, 0.7, 0.8)
Very Good	(0.7, 0.8, 0.9)
Fairly Excellent	(0.8, 0.9, 1.0)
Excellent	(0.9, 0.95, 0.95)

Table 3: Conversion of linguistic terms for hesitancy.

Linguistic terms	Fuzzy Numbers
Very Little	(0.1, 0.1, 0.2)
Little	(0.1, 0.2, 0.3)
Moderate	(0.2, 0.3, 0.4)
Strong	(0.3, 0.4, 0.5)
Very Strong	(0.4, 0.5, 0.6)

Table 4: Opinions of X_1 student about features on staff in terms of IFN.

X_1	S_1	S_2	S_3
Punctuality	((0.7, 0.8, 0.9), (0.1, 0.1, 0.2))	((0.6, 0.7, 0.8), (0.1, 0.1, 0.2))	((0.7, 0.8, 0.9), (0.1, 0.1, 0.2))
Innovative Teaching	((0.6, 0.7, 0.8), (0.1, 0.1, 0.2))	((0.7, 0.8, 0.9), (0.2, 0.3, 0.4))	((0.6, 0.7, 0.8), (0.2, 0.3, 0.4))
Communication Skills	((0.9, 0.95, 0.95), (0, 0, 0))	((0.8, 0.9, 1.0), (0.1, 0.1, 0.2))	((0.9, 0.95, 0.95), (0, 0, 0))
Given Practical Examples	((0.7, 0.8, 0.9), (0.2, 0.3, 0.4))	((0.9, 0.95, 0.95), (0, 0, 0))	((0.8, 0.9, 1.0), (0.1, 0.1, 0.2))
Completion of Syllabus	((0.9, 0.95, 0.95), (0, 0, 0))	((0.7, 0.8, 0.9), (0.1, 0.2, 0.3))	((0.9, 0.95, 0.95), (0, 0, 0))
Counseling	((0.7, 0.8, 0.9), (0.1, 0.1, 0.2))	((0.9, 0.95, 0.95), (0, 0, 0))	((0.6, 0.7, 0.8), (0.1, 0.2, 0.3))

Table 5: Intuitionistic fuzzy scores of X_1 using IFN.

X_1	S_1	S_2	S_3
Punctuality	(0.833, 0.106)	(0.722, 0.106)	(0.833, 0.106)
Innovative Teaching	(0.722, 0.106)	(0.833, 0.278)	(0.722, 0.278)
Communication Skills	(0.95, 0)	(0.944, 0.106)	(0.95, 0)
Given Practical Examples	(0.833, 0.278)	(0.95, 0)	(0.944, 0.106)
Completion of Syllabus	(0.95, 0)	(0.833, 0.167)	(0.95, 0)
Counseling	(0.833, 0.106)	(0.95, 0)	(0.722, 0.167)