Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy modal operators

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Abstract: In 1965, Fuzzy Set Theory was introduced by Zadeh as an extension of crisp sets [10]. K. T. Atanassov defined the concept of Intuitionistic Fuzzy Sets, in 1983 [1]. Some operations and operators on intuitionistic fuzzy sets, like modal operators, level operators, topological operators, etc., was defined by same author [2]. In later times, new operators were defined on IFSs and several properties of these operators were studied by different authors [3, 5, 6, 7, 8, 9]. In this study, we examine some relationships between new modal operators with topological operators.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, Intuitionistic fuzzy topological operators.
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1 Introduction

Fuzzy Set Theory was introduced by Zadeh [10] as an extension of crisp sets. Atanassov introduced the concept of Intuitionistic Fuzzy Sets [1], form an extension of fuzzy sets by expanding the truth value set to the lattice $[0, 1] \times [0, 1]$ is defined as following.

Definition 1. *Let* $L = [0, 1]$ *then* $L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$ *is a lattice with* $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".

For $(x_1, y_1), (x_2, y_2) \in L^*$,the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as following;
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$

For each $J \subseteq L^*$

sup $J = (sup\{x : (x, y \in [0,1]), ((x,y) \in J)\}, inf\{y :(x, y \in [0,1])((x,y) \in J)\})$ and
inf $J = (inf\{x : (x, y \in [0,1])((x,y) \in J)\}, sup\{y : (x, y \in [0,1])((x,y) \in J)\})$.

Intuitionistic fuzzy modal operators were introduced by Atanassov [1, 2].Then several extensions of these operators were studied by different authors [2, 8, 5, 6]. Some algebraic and characteristic properties of these operators were examined by several authors. New modal operators $L^{\omega}_{\alpha,\beta}$, $K^{\omega}_{\alpha,\beta}$, $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ were defined in [9, 7] and operator $\otimes_{\alpha,\beta,\gamma,\delta}$ defined in [3]. In 1986, Atanassov defined two operators called topological operators and extensions of these operators defined by same author [1, 2].

2 Preliminaries

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x)$, $(\nu_A : X \to [0,1])$ is called the " degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.
The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 3. *[1] An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1] Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the following set is called the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$$

Some special Intuitionistic Fuzzy Sets on X are defined as following;

$$O^* = \{\langle x, 0, 1\rangle : x \in X\}$$

$$X^* = \{< x, 1, 0 >: x \in X\}$$

In 1986, Atanassov introduced topological operators and the extensions of these operators were defined by same author in 2001 as following,

Definition 5. *[2] Let X be a set and $A \in IFS(X)$.*

$$C(A) = \{\langle x, K, L\rangle : x \in X\}$$

where $K = \sup_{y\in X} \mu_A(y)$, $L = \inf_{y\in X} \nu_A(y)$ *and*

$$I(A) = \{\langle x, k, l\rangle : x \in X\}$$

where $k = \inf_{y\in X} \mu_A(y)$, $l = \sup_{y\in X} \nu_A(y)$.

Definition 6. *[2] Let X be a set and $A \in IFS(X)$. Let K, L, k and l be as above forms,*

1. $C_\mu(A) = \{\langle x, K, \min(1-K, \nu_A(x))\rangle : x \in X\}$
2. $C_\nu(A) = \{\langle x, \mu_A(x), L\rangle : x \in X\}$
3. $I_\mu(A) = \{\langle x, k, \nu_A(x)\rangle : x \in X\}$
4. $I_\nu(A) = \{\langle x, \min(1-l, \mu_A(x)), l\rangle : x \in X\}$

Intuitionistic fuzzy modal operators $L^\omega_{\alpha,\beta}$ and $K^\omega_{\alpha,\beta}$ defined in [9] and some of their properties were examined.

Definition 7. *[9] Let X be a set and $A \in IFS(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$.*

1. $L^\omega_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle : x \in X\}$
2. $K^\omega_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle : x \in X\}$

In [3], an intuitionistic fuzzy modal operator, represented by $\otimes_{\alpha,\beta,\gamma,\delta}$, was introduced.

Definition 8. *[3] Let X be a set and $A \in IFS(X), \alpha, \beta, \gamma, \delta \in [0,1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.*

$$\otimes_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma\nu_A(x), \beta\mu_A(x) + \delta\nu_A(x)\rangle\}$$

New intuitionistic fuzzy modal operators $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$ are defined by the authors as following,

Definition 9. *[7] Let X be a set and $A \in IFS(X), \alpha, \beta, \alpha + \beta \in [0,1]$.*

1. $S_{\alpha,\beta}(A) = \{\langle x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha)\rangle : x \in X\}$
2. $T_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x))\rangle : x \in X\}$

3 Main results

After the definition of new modal operators, some properties of them were studied by authors [4, 7, 9]. In this study, we examined properties of new modal operators with topological operators.

Theorem 1. *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \gamma, \delta \in [0,1]$, $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ *then*

1. $\otimes_{\alpha,\beta,\gamma,\delta}(C(A)) \sqsubseteq C(\otimes_{\alpha,\beta,\gamma,\delta}(A))$
2. $I(\otimes_{\alpha,\beta,\gamma,\delta}(A)) \sqsubseteq \otimes_{\alpha,\beta,\gamma,\delta}(I(A))$

Proof. (1) It is clear that

$$\inf_{y\in X} \nu_A(y) \leq \sup_{y\in X} \nu_A(y)$$

and

$$\inf_{y\in X} \mu_A(y) \leq \sup_{y\in X} \mu_A(y)$$

So

$$\begin{aligned} \gamma \inf_{y\in X} \nu_A(y) &\leq \gamma \sup_{y\in X} \nu_A(y) \\ &\Rightarrow \alpha \sup_{y\in X} \mu_A(y) + \gamma \inf_{y\in X} \nu_A(y) \leq \alpha \sup_{y\in X} \mu_A(y) + \gamma \sup_{y\in X} \nu_A(y) \\ &\Rightarrow \alpha \sup_{y\in X} \mu_A(y) + \gamma \inf_{y\in X} \nu_A(y) \leq \sup_{y\in X} (\alpha\mu_A(y) + \gamma\nu_A(y)) \end{aligned}$$

on the other hand

$$\begin{aligned} \delta \inf_{y\in X} \mu_A(y) &\leq \delta \sup_{y\in X} \mu_A(y) \\ &\Rightarrow \beta \inf_{y\in X} \nu_A(y) + \delta \inf_{y\in X} \mu_A(y) \leq \beta \inf_{y\in X} \nu_A(y) + \delta \sup_{y\in X} \mu_A(y) \\ &\Rightarrow \inf_{y\in X} (\beta\nu_A(y) + \delta\mu_A(y)) \leq \beta \inf_{y\in X} \nu_A(y) + \delta \sup_{y\in X} \mu_A(y) \end{aligned}$$

So, $\otimes_{\alpha,\beta,\gamma,\delta}(C(A)) \sqsubseteq C(\otimes_{\alpha,\beta,\gamma,\delta}(A))$.
(2) can be prooven in with same way. □

Theorem 2. *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \gamma, \delta \in [0,1]$, $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ *then*

1. $I_\mu(\otimes_{\alpha,\beta,\gamma,\delta}(A)) \sqsubseteq \otimes_{\alpha,\beta,\gamma,\delta}(I_\mu(A))$
2. $\otimes_{\alpha,\beta,\gamma,\delta}(C_\nu(A)) \sqsubseteq C_\nu(\otimes_{\alpha,\beta,\gamma,\delta}(A))$

Proof. (1) If we use inequality $\inf_{y\in X} \nu_A(y) \leq \nu_A(x)$, for all $x \in X$ then

$$\begin{aligned} \gamma \inf_{y\in X} \nu_A(y) &\leq \gamma\nu_A(x) \Rightarrow \gamma \inf_{y\in X} \nu_A(y) + \alpha \inf_{y\in X} \mu_A(y) \leq \alpha \inf_{y\in X} \mu_A(y) + \gamma\nu_A(x) \\ &\Rightarrow \inf_{y\in X} \gamma\nu_A(y) + \alpha\mu_A(y) \leq \alpha \inf_{y\in X} \mu_A(y) + \gamma\nu_A(x) \end{aligned}$$

and also, if we use $\inf_{y\in X}\mu_A(y) \le \mu_A(x)$,for all $x \in X$ then

$$\delta \inf_{y\in X}\mu_A(y) + \beta\nu_A(x) \le \delta\mu_A(x) + \beta\nu_A(x)$$

We obtain that $I_\mu(\otimes_{\alpha,\beta,\gamma,\delta}(A)) \sqsubseteq \otimes_{\alpha,\beta,\gamma,\delta}(I_\mu(A))$.
(2) If we use same inequalities then,

$$\inf_{y\in X}\nu_A(y) \;\; \le \nu_A(x) \Rightarrow \gamma \inf_{y\in X}\nu_A(y) + \; \alpha\mu_A(x) \le \gamma\nu_A(x) + \alpha\mu_A(x)$$

and

$$\begin{aligned}\inf_{y\in X}\mu_A(y) \le \mu_A(x) \quad &\Rightarrow \quad \delta \inf_{y\in X}\mu_A(y) + \beta \inf_{y\in X}\nu_A(y) \le \delta\mu_A(x) + \beta \inf_{y\in X}\nu_A(y) \\ &\Rightarrow \quad \inf_{y\in X}(\delta\mu_A(y) + \beta\nu_A(y)) \le \delta\mu_A(x) + \beta \inf_{y\in X}\nu_A(y)\end{aligned}$$

So,$\otimes_{\alpha,\beta,\gamma,\delta}(C_\nu(A)) \sqsubseteq C_\nu(\otimes_{\alpha,\beta,\gamma,\delta}(A))$. □

Theorem 3. *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \omega \in [0, 1]$, $\alpha + \beta \le 1$ *then*

1. $L^\omega_{\alpha,\beta}(C(A)) = C(L^\omega_{\alpha,\beta}(A))$
2. $L^\omega_{\alpha,\beta}(I(A)) = I(L^\omega_{\alpha,\beta}(A))$

Proof. (1)

$$\sup_{y\in X}(\alpha\mu_A(y) + \omega(1-\alpha)) = \alpha \sup_{y\in X}\mu_A(y) + \omega(1-\alpha)$$

and

$$\inf_{y\in X}\alpha(1-\beta)\nu_A(y) + \alpha\beta(1-\omega) = \alpha(1-\beta)\inf_{y\in X}\nu_A(y) + \alpha\beta(1-\omega)$$

So, $L^\omega_{\alpha,\beta}(C(A)) = C(L^\omega_{\alpha,\beta}(A))$.
(2)

$$\inf_{y\in X}\alpha\mu_A(y) + \omega(1-\alpha) = \alpha \inf_{y\in X}\mu_A(y) + \omega(1-\alpha)$$

and

$$\sup_{y\in X}\alpha(1-\beta)\nu_A(y) + \alpha\beta(1-\omega) = \alpha(1-\beta)\sup_{y\in X}\nu_A(y) + \alpha\beta(1-\omega)$$

Therefore, $L^\omega_{\alpha,\beta}(I(A)) = I(L^\omega_{\alpha,\beta}(A))$. □

Theorem 4. *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \omega \in [0, 1]$, $\alpha + \beta \le 1$ *then*

1. $K^\omega_{\alpha,\beta}(C(A)) = C(K^\omega_{\alpha,\beta}(A))$
2. $K^\omega_{\alpha,\beta}(I(A)) = I(K^\omega_{\alpha,\beta}(A))$

Theorem 5. *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \omega \in [0, 1]$, $\alpha + \beta \le 1$ *then*

1. $L^\omega_{\alpha,\beta}(C_\nu(A)) = C_\nu(L^\omega_{\alpha,\beta}(A))$

2. $L^{\omega}_{\alpha,\beta}(I_\mu(A)) = I_\mu(L^{\omega}_{\alpha,\beta}(A))$

Proof. (1)

$$\begin{aligned} C_\nu(L^{\omega}_{\alpha,\beta}(A)) &= \left\{\left\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \inf_{y\in X}\alpha(1-\beta)\nu_A(y) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\inf_{y\in X}\nu_A(y) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\} \\ &= L^{\omega}_{\alpha,\beta}(C_\nu(A)) \end{aligned}$$

(2)

$$\begin{aligned} I_\mu(L^{\omega}_{\alpha,\beta}(A)) &= \left\{\left\langle x, \inf_{y\in X}(\alpha\mu_A(y) + \omega(1-\alpha)), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \alpha\inf_{y\in X}\mu_A(y) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\} \\ &= L^{\omega}_{\alpha,\beta}(I_\mu(A)) \end{aligned}$$

□

Theorem 6. *Let $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0, 1]$, $\alpha + \beta \leq 1$ then*

1. $K^{\omega}_{\alpha,\beta}(C_\nu(A)) = C_\nu(K^{\omega}_{\alpha,\beta}(A))$
2. $K^{\omega}_{\alpha,\beta}(I_\mu(A)) = I_\mu(K^{\omega}_{\alpha,\beta}(A))$

Theorem 7. *Let $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0, 1]$, $\alpha + \beta \leq 1$ then*

1. $L^{\omega}_{\alpha,\beta}(I_\nu(A)) \sqsubseteq I_\nu(L^{\omega}_{\alpha,\beta}(A))$
2. $K^{\omega}_{\alpha,\beta}(I_\nu(A)) \sqsubseteq I_\nu(K^{\omega}_{\alpha,\beta}(A))$

Proof. (1)

$$L^{\omega}_{\alpha,\beta}(I_\nu(A)) = \left\{\left\langle x, \alpha\min(1-\sup_{y\in X}\nu_A(y), \mu_A(x)) + \omega(1-\alpha), \right.\right.$$
$$\left.\left.\alpha(1-\beta)\sup_{y\in X}\nu_A(y) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\}$$

and

$$I_\nu(L^{\omega}_{\alpha,\beta}(A)) = \left\{\left\langle x, \min(1-\alpha(1-\beta)\sup_{y\in X}\nu_A(y) - \alpha\beta(1-\omega), \alpha\mu_A(x) + \omega(1-\alpha)), \right.\right.$$
$$\left.\left.\alpha(1-\beta)\sup_{y\in X}\nu_A(y) + \alpha\beta(1-\omega)\right\rangle : x \in X\right\}.$$

Let us call $\min(\alpha - \alpha\sup_{y\in X}\nu_A(y) + \omega(1-\alpha), \alpha\mu_A(x) + \omega(1-\alpha)) = \min(a, c)$ and $\min(1-\alpha(1-\beta)\sup_{y\in X}\nu_A(y) - \alpha\beta(1-\omega), \alpha\mu_A(x) + \omega(1-\alpha)) = \min(b, c)$.

If $\min(b, c) = b$

$$\begin{aligned}\alpha+\alpha\beta &\le 1 \Rightarrow \alpha(1-\omega)+\alpha\beta(1-\omega)\le 1-\omega\\ &\Rightarrow \alpha+\omega(1-\alpha)\le 1+\alpha\beta\sup_{y\in X}\nu_A(y)-\alpha\beta+\alpha\beta\omega\\ &\Rightarrow \alpha-\alpha\sup_{y\in X}\nu_A(y)+\omega(1-\alpha)\le 1-\alpha(1-\beta)\sup_{y\in X}\nu_A(y)-\alpha\beta(1-\omega)\end{aligned}$$

then we obtain that $a \le b$ so, $\min(a,c)=a$.

If $\min(b,c)=c$ then $\min(a,c)=c \vee \min(a,c)=a$.

Therefore, $L^{\omega}_{\alpha,\beta}(I_\nu(A)) \sqsubseteq I_\nu(L^{\omega}_{\alpha,\beta}(A))$.

(2) It can be proved similarly. □

Theorem 8. *Let $A \in IFS(X)$ and $\alpha,\beta,\omega \in [0,1]$, $\alpha+\beta\le 1$ then*

1. $C_\mu(L^{\omega}_{\alpha,\beta}(A)) \sqsubseteq L^{\omega}_{\alpha,\beta}(C_\mu(A))$
2. $C_\mu(K^{\omega}_{\alpha,\beta}(A)) \sqsubseteq K^{\omega}_{\alpha,\beta}(C_\mu(A))$

Proof. (1)

$$C_\mu(L^{\omega}_{\alpha,\beta}(A))=\Big\{\Big\langle x,\sup_{y\in X}\alpha\mu_A(y)+\omega(1-\alpha),$$

$$\min\big(1-\sup_{y\in X}(\alpha\mu_A(y)+\omega(1-\alpha)),\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega)\big)\Big\rangle : x\in X\Big\}$$

and

$$L^{\omega}_{\alpha,\beta}(C_\mu(A))=\Big\{\Big\langle x,\alpha\sup_{y\in X}\mu_A(y)+\omega(1-\alpha),$$

$$\alpha(1-\beta)\min\big(1-\sup_{y\in X}\mu_A(y),\nu_A(x)\big)+\alpha\beta(1-\omega)\Big\rangle : x\in X\Big\}$$

Let us call

$$\min(\alpha(1-\beta)-\alpha(1-\beta)\sup_{y\in X}\mu_A(y)+\alpha\beta(1-\omega),\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega))=\min(a,c)$$

and

$$\min(1-\alpha\sup_{y\in X}\mu_A(y)+\omega(1-\alpha),\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega))=\min(b,c).$$

If $\min(b,c)=b$,

$$\begin{aligned}\alpha-\alpha\beta\omega &< 1+\omega-\alpha\omega\\ &\Rightarrow \alpha-\alpha\beta\omega-\alpha(1-\beta)\sup_{y\in X}\mu_A(y)\le 1+\omega-\alpha\omega-\alpha\sup_{y\in X}\mu_A(y)\end{aligned}$$

then we obtain that $a\le b$ so, $\min(a,c)=a$.

If $\min(b,c)=c$ then $\min(a,c)=c\vee\min(a,c)=a$.

Thus, $C_\mu(L^{\omega}_{\alpha,\beta}(A)) \sqsubseteq L^{\omega}_{\alpha,\beta}(C_\mu(A))$.

(2) It is straightforward. □

Theorem 9. *Let* $A \in IFS(X)$ *and* $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$ *then*

1. $S_{\alpha,\beta}(C(A)) \sqsubseteq C(S_{\alpha,\beta}(A))$
2. $I(S_{\alpha,\beta}(A)) \sqsubseteq S_{\alpha,\beta}(I(A))$

Proof. (1) If we use $\inf_{y\in X} \nu_A(y) \leq \sup_{y\in X} \nu_A(y)$ and $\inf_{y\in X} \mu_A(y) \leq \sup_{y\in X} \mu_A(y)$ then

$$\begin{aligned} \alpha(1-\beta)\inf_{y\in X}\nu_A(y) &\leq \alpha(1-\beta)\sup_{y\in X}\nu_A(y) \\ &\Rightarrow \alpha\sup_{y\in X}\mu_A(y) + \alpha(1-\beta)\inf_{y\in X}\nu_A(y) \\ &\leq \alpha\sup_{y\in X}\mu_A(y) + \alpha(1-\beta)\sup_{y\in X}\nu_A(y) \end{aligned}$$

and

$$\begin{aligned} \beta(1-\alpha)\inf_{y\in X}\mu_A(y) &\leq \beta(1-\alpha)\sup_{y\in X}\mu_A(y) \\ &\Rightarrow \beta\inf_{y\in X}\nu_A(y) + \beta(1-\alpha)\inf_{y\in X}\mu_A(y) + \alpha\beta \\ &\leq \beta\inf_{y\in X}\nu_A(y) + \beta(1-\alpha)\sup_{y\in X}\mu_A(y) + \alpha\beta \end{aligned}$$

We obtain $S_{\alpha,\beta}(C(A)) \sqsubseteq C(S_{\alpha,\beta}(A))$.
(2) Proved by analogy. □

Theorem 10. *Let* $A \in IFS(X)$ *and* $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$ *then*

1. $S_{\alpha,\beta}(C_\nu(A)) \sqsubseteq C_\nu(S_{\alpha,\beta}(A))$
2. $I_\mu(S_{\alpha,\beta}(A)) \sqsubseteq S_{\alpha,\beta}(I_\mu(A))$

Theorem 11. *Let* $A \in IFS(X)$ *and* $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$ *then*

1. $I_\mu(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(I_\mu(A))$
2. $T_{\alpha,\beta}(C_\nu(A)) \sqsubseteq C_\nu(T_{\alpha,\beta}(A))$

Proof. (1) It is clear that $\inf_{y\in X} \nu_A(y) \leq \nu_A(x)$, for all $x \in X$, so

$$\begin{aligned} \beta(1-\alpha)\inf_{y\in X}\nu_A(y) &\leq \beta(1-\alpha)\nu_A(x) \\ &\Rightarrow \beta\inf_{y\in X}\mu_A(y) + \beta(1-\alpha)\inf_{y\in X}\nu_A(y) + \alpha\beta \\ &\leq \beta\inf_{y\in X}\mu_A(y) + \beta(1-\alpha)\nu_A(x) + \alpha\beta \end{aligned}$$

while on the other hand, $\inf_{y\in X} \mu_A(y) \leq \mu_A(x)$, for all $x \in X$ then,

$$\begin{aligned} \alpha(1-\beta)\inf_{y\in X}\mu_A(y) &\leq \alpha(1-\beta)\mu_A(x) \\ &\Rightarrow \alpha\nu_A(x) + \alpha(1-\beta)\inf_{y\in X}\mu_A(y) \\ &\leq \alpha\nu_A(x) + \alpha(1-\beta)\mu_A(x) \end{aligned}$$

We show that $I_\mu(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(I_\mu(A))$.

(2) It is straightforward. □

Theorem 12. *Let* $A \in IFS(X)$ *and* $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$ *then*

1. $T_{\alpha,\beta}(C(A)) \sqsubseteq C(T_{\alpha,\beta}(A))$
2. $I(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(I(A))$

4 Conclusion

In this paper, new relationships between new modal operators with topological operators are given. Thus, we obtained some properties of the latest modal operators.

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