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On generalized double statistical convergence of order α in intuitionistic fuzzy n-normed spaces

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Abstract: In this work, we introduce the notion $[V,\lambda]_2(\mathcal{I})$-summability and ideal λ-double statistical convergence of order α with respect to the intuitionistic fuzzy n-normed (μ, v). In addition, we present a series of inclusion theorems associated with these new definitions.
Keywords: Ideal, Filter, $\mathcal{I}$-double statistical convergence, $\mathcal{I}_\lambda$-double statistical convergence order α, $[V,\lambda]_2(\mathcal{I})$-summability, closed subspace.
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1 Introduction

Intuitionistic fuzzy set (IFNS) is one of the generalizations of fuzzy sets theory [41]. Out of several higher-order fuzzy sets, IFNS first introduced by Atanassov [1] have been found to be compatible to deal with vagueness. The conception of IFNS can be viewed as an appropriate and alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set theory. In fuzzy sets the degree of acceptance is considered only but IFNS is characterized by a membership function and a non-membership function such that the sum of both values is less than one. Presently intuitionistic fuzzy sets are being studied and used in different field of science and and engineering, e.g., population dynamics, chaos control, computer programming, nonlinear dynamical systems, fuzzy physics, fuzzy topology etc. The concept of an intuitionistic fuzzy metric space was introduced by Park [24]. Furthermore, Saadati and Park [25] gave the notion of an intuitionistic fuzzy normed space.

The theory of 2-norm and n-norm on a linear space was introduced by Gähler [9, 10], which was developed by Kim and Cho [13], Malceski [14], and Misiak [18], Gunawan and Mashadi [11] and Vijayabalaji and Narayanan [40] extended n-normed linear space to fuzzy n-normed linear space.

In recent years fuzzy topology proves to be a very useful tool to deal with such situation where the use of classical theories breaks down. The most popular application of fuzzy topology in quantum particle physics aries in string and ε^{∞}-theory of El-Nashie [6] who presented the relation of fuzzy Kähler interpolation of ε^{∞} to the recent work on cosmo-topology. In [33], E.Savas introduced λ-double sequence spaces of fuzzy real numbers defined by Orlicz function.

The term "*statistical convergence*" was first defined by Fast [7] and Schoenberg [38] independently, which is generalization of the concept of ordinary convergence. Statistical convergence appears in many fields such as in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [8], Šalát [26], Cakalli [2], Di Maio and Kocinac [17] and many others.

Definition 1. *Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the $asymptotic\ density$ of K denoted by $\delta(K)$, is defined as*

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 2. *A number sequence $x = (x_k)$ is said to be $statistically\ convergent$ to the number L if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| > \varepsilon\}$ has asymptotic density zero, i.e.*

$$\lim_{n} \frac{1}{n} |\{k \leq n : |x_k - L| > \varepsilon\}| = 0.$$

In this case we write $st - \lim x = L$ (see, [7, 8]).

Note that every convergent sequence is statistically convergent to the same limit, but converse need not be true.

On the other hand Mursaleen [19] introduced the concept of λ-statistical convergence as a generalization of the statistical convergence and studied its relation to statistical convergence, Cesàro summability and strong (V, λ)-summability.

Furthermore, in [3, 4] a different direction was given to the study of these important summability methods where the notions of statistical convergence of order α and λ-statistical convergence of order α were introduced and studied.

In [15], P. Kostyrko et al. introduced and investigated $\mathcal{I}$-convergence of sequences in a metric space which is an interesting generalization of statistical convergence and studied some properties of such convergence. Subsequently, more investigations and more applications of ideals were introduced and studied in different directions, for instances, see in [5, 27, 28, 29, 30, 31, 32].

Also, in [23] Mohiuddine and Lohani introduced the notion of the generalized statistical convergence in intuitionistic fuzzy normed spaces. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in [21, 22, 23].

Quite recently, $\mathcal{I}$-double statistical convergence has been established as a better study than double statistical convergence. It is found very interesting that some results on sequences, series and summability can be proved by replacing the double statistical convergence by $\mathcal{I}$-double statistical convergence. Also, it should be note that the results of $\mathcal{I}_\lambda$-double statistical convergence in an intuitionistic fuzzy normed linear space happen to be stronger than those proved for λ-double statistical convergence in an intuitionistic fuzzy normed linear space.

In this paper, we intend to use ideals to introduce the concept of $\mathcal{I}_\lambda$-double statistical convergence of order α with respect to the intuitionistic fuzzy n-normed space (μ, v), and study some of its consequences.

It should be noted that throughout the paper, $\mathbb{N}$ will denote the set of all natural numbers.

The following definitions and notions will be needed in the sequel.

Definition 3. *([38]) A triangular norm (t-norm) is a continuous mapping* $* : [0,1]\times[0,1] \to [0,1]$ *such that* $(S, *)$ *is an abelian monoid with unit one and* $c * d \le a * b$ *if* $c \le a$ *and* $d \le b$ *for all* $a, b, c, d \in [0,1]$.

Definition 4. *([38]) A binary operation* $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* t*-conorm if it satisfies the following conditions:*

(i) $\diamondsuit$ *is associate and commutative,*

(ii) $\diamondsuit$ *is continuous,*

(iii) $a\diamondsuit 0 = a$ *for all* $a \in [0,1]$,

(iv) $a\diamondsuit b \le c\diamondsuit d$ *whenever* $a \le c$ *and* $b \le d$ *for each* $a, b, c, d \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [25] has recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 5. *([25])The five-tuple* $(X, \mu, v, *, \diamondsuit)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamondsuit$ *is a continuous* t*-conorm, and* μ, v *are fuzzy sets on* $X \times (0,\infty)$ *satisfying the following conditions for every* $x, y \in X$*, and* $s, t > 0$*:*

(a) $\mu(x,t) + v(x,t) \le 1,$

(b) $\mu(x,t) > 0,$

(c) $\mu(x,t) = 1$ *if and only if* $x = 0,$

(d) $\mu(\alpha x,t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \ne 0,$

(e) $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s),$

(f) $\mu(x,.) : (0,\infty) \to [0,1]$ *is continuous,*

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0,$

(h) $v(x,t) < 1,$

(i) $v(x,t) = 0$ *if and only if* $x = 0$,

(j) $v(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(k) $v(x,t) \lozenge v(y,s) \geq v(x+y, t+s)$,

(l) $v(x,.) : (0,\infty) \to [0,1]$ *is continuous,*

(m) $\lim_{t\to\infty} v(x,t) = 0$ *and* $\lim_{t\to 0} v(x,t) = 1$.

In this case (μ, v) is called an intuitionistic fuzzy norm. Then observe that $(X, \mu, v, *, \lozenge)$ is an intuitionistic fuzzy normed space.

We also recall that the concept of double convergence in an intuitionistic fuzzy normed space is studied in [16].

Before proceeding further, we should recall some notations. In [9], Gähler introduced the following concept of n-normed space

Let $n \in N$ and X be a real linear space of dimension $d \geq n \geq 2$. A real valued function $||., ..., .|| : X^n \to R$ satisfying the following four properties:

(N_1) $||x_1, x_{2,}..., x_n|| = 0$ if and only if $x_1, x_{2,}..., x_n$ are linearly dependent vectors,

(N_2) $||x_1, x_{2,}..., x_n|| = \left|\left|x_{j_1}, x_{_{j_2}},..., x_{j_n}\right|\right|$ for every permutation $(j_1, j_2, ..., j_n)$ of $(1, 2, ..., n)$,

(N_3) $||\alpha x_1, x_{2,}..., x_n|| = |\alpha| \, ||x_1, x_{2,}..., x_n||$ for all $\alpha \in R$,

(N_4) $\left|\left|x, x'_, x_2, ..., x_n\right|\right| \leq ||x, x_2, ..., x_n|| + \left|\left|x'_, x_2, ..., x_n\right|\right|$ for $x, x'_, x_2, ..., x_n \in X$

is called an n-norm on X and the pair $(X, ||., ..., .||)$ is called a linear n-normed space. The concept of a 2-normed space was developed by Gähler [9] in the mid of 1960's while that of an n-normed space can be found in Misiak [18]. Since then, many others have studied this concept and obtained various results; see for instance Gunawan [11].

We now have

Definition 6. *An IFnNLS is the five-tuple ($X, \mu, v, *, \circ$), where X is a linear space over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, v are fuzzy sets on $X^n \times (0,\infty)$, μ denotes the degree of membership and v denotes the degree of non-membership of $(x_1, x_2, ..., x_n, t) \in X^n \times (0,1)$ satisfying the following conditions for every $(x_1, x_2, ..., x_n) \in X^n$ and $s, t > 0$:*

(i) $\mu(x_1, x_2, ..., x_n, t) + v(x_1, x_2, ..., x_n, t) \leq 1$,

(ii) $\mu(x_1, x_2, ..., x_n, t) > 0$,

(iii) $\mu(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

(iv) $\mu(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(v) $\mu(x_1, x_2, ..., cx_n, t) = \mu(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,

(vi) $\mu(x_1, x_2, ..., x_n, s) * \mu(x_1, x_2, ..., x'_n, t) \leq \mu(x_1, x_2, ..., x_n + x'_n, s+t)$

(vii) $\mu(x_1, x_2, ..., x_n, t) : (0,\infty) \to [0,1]$ is continuous in t,

(viii) $\lim_{t\to\infty}\mu(x_1,x_2,...,x_n,t)=1$ and $\lim_{t\to 0}\mu(x_1,x_2,...,x_n,t)=0$

(ix) $v(x_1,x_2,...,x_n,t)<1,$

(x) $v(x_1,x_2,...,x_n,t)=0$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

(xi) $v(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1,x_2,...,x_n$

(xii) $v(x_1,x_2,...,cx_n,t)=v(x_1,x_2,...,x_n,\frac{t}{|c|})$ if $c\neq 0, c\in F,$

(xiii) $v(x_1,x_2,...,x_n,s)\circ v(x_1,x_2,...,x'_n,t)\geq v(x_1,x_2,...,x_n+x'_n,s+t)$

(xiv) $v(x_1,x_2,...,x_n,t):(0,\infty)\to[0,1]$ is continuous in $t,$

(xv) $\lim_{t\to\infty}v(x_1,x_2,...,x_n,t)=0$ and $\lim_{t\to 0}v(x_1,x_2,...,x_n,t)=1$

Example 1. *Let $(X,||.||)$ be an n-normed space. Also let $a*b=ab$ and $a\circ b=\min\{a+b,1\}$ for all $a,b\in[0,1]$, $\mu(x_1,x_2,...,x_n,t)=\frac{t}{t+||x_1,x_2,...,x_n||}$ and $v(x_1,x_2,...,x_n,t)=\frac{||x_1,x_2,...,x_n||}{t+||x_1,x_2,...,x_n||}$. Then $(X,\mu,v,*,\circ)$ is an IFnNS.*

The following definition has given in [36].

Definition 7. *Let $(X,\mu,v,*,\circ)$ be an IFnNS. We say that a sequence $x=\{x_k\}$ in X is convergent to $L\in X$ with respect to the intuitionistic fuzzy n-norm (μ,v) if, for every $\varepsilon>0$, $t>0$ and $y_1,y_2,...,y_{n-1}\in X$, there exists $k_0\in\mathbb{N}$ such that $\mu(y_1,y_2,...,y_{n-1},x_k-L,t)>1-\varepsilon$ and $v(y_1,y_2,...,y_{n-1},x_k-L,t)<\varepsilon$ for all $k\geq k_0$. It is denoted by $(\mu,v)_n-\lim x=L$ or $x_k\overset{(\mu,v)_n}{\to}L$ as $k\to\infty$.*

The following definitions and notions will be needed.

Definition 8. *A family $\mathcal{I}\subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A,B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$,
(b) $A\in\mathcal{I},\ B\subset A$ implies $B\in\mathcal{I}$,

Definition 9. *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\}\in\mathcal{I}$ for each $n\in\mathbb{N}$.*

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 10. *(See [15]) Let $\mathcal{I}\subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.*

The sequence $x=(x_k)$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L\in\mathbb{R}$ if for each $\epsilon>0$ the set $A(\epsilon)=\{n\in\mathbb{N}:|x_k-L|\geq\epsilon\}\in\mathcal{I}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal in $\mathbb{N}\times\mathbb{N}$.

2 $\mathcal{I}_\lambda$-double statistical convergence on IFnNS

In this section we deal with the relation between these two new methods as also the relation between $\mathcal{I}_\lambda$-double statistical convergence and $\mathcal{I}$-double statistical convergence in an intuitionistic fuzzy n-normed space (μ, v). First we should recall some notation on the $\mathcal{I}$-double statistical convergence and ideal convergence.

The idea of λ-statistical convergence of single and double sequences of fuzzy numbers has been studied by [33] and Savas [34], respectively.

Statistical convergence of double sequences $x = (x_{kl})$ has been defined and studied by Mursaleen and Edely [20]; and for fuzzy numbers by Savas and Mursaleen [35].

Now, it would be helpful to give some definitions.

Let $K \subseteq \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of (k,l) in K such that $k \leq m$ and $l \leq n$. Then the two-dimensional analogue of natural density can be defined as follows [20].

The *lower asymptotic density* of the set $K \subseteq \mathbb{N}\times\mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{m,n} \frac{K(m,n)}{mn}.$$

In case the sequence $(K(m,n)/mn)$ has a limit then we say that K has a *double natural density* and is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

Definition 11 ([20]). *A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$, the set*

$$\{(k,l), k \leq m \text{ and } l \leq n : \mid x_{k,l} - \ell \mid \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{k,l} x_{k,l} = \ell$.

We define the concept of double λ-density:
Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers both of which tends to ∞ as m and n approach ∞, respectively. Also let $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 0$ and $\mu_{n+1} \leq \mu_n + 1, \mu_1 = 0$. The collection of such sequence (λ, μ) will be denoted by Δ.

Let $K \subseteq \mathbb{N}\times\mathbb{N}$. The number

$$\delta_\lambda(K) = \lim_{mn} \frac{1}{\lambda_{mn}} |\{k \in I_n, l \in J_m : (k,l) \in K\}|,$$

where $I_m = [m - \lambda_m + 1, m]$ and $J_n = [n - \mu_n + 1, n]$ and $\lambda_{mn} = \lambda_m \mu_n$, is said to be the λ-*density* of K, provided the limit exists.

We have

Definition 12. *Let $\mathcal{I}$ be a nontrivial admissible ideal in $\mathbb{N}\times\mathbb{N}$. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. Then, a sequence $x = (x_{k,l})$ is said to be $\mathcal{I}$ - double statistically convergent of order α to $L \in X$, where $0 < \alpha \leq 1$, with respect to the intuitionistic fuzzy n- normed space (μ, v), if for every $\varepsilon > 0$, and every $\delta > 0$, $t > 0$ and $y_1, y_2, ..., y_{n-1} \in X$,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(mn)^{\alpha}}\left|\left\{\begin{array}{c}k\leq m \text{ and } l\leq n:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

In this case we write $x_{k,l}\overset{(\mu,v)}{\to}L\left(S_n^{\alpha}(\mathcal{I})^{(\mu,v)_n}\right)$.

Remark 1. *For* $\mathcal{I}=\mathcal{I}_{fin}, S_2^{\alpha}(\mathcal{I})$*-convergence coincides with statistical convergence of order* α*, with respect to the intuitionistic fuzzy normed space* (μ,v)*. For an arbitrary ideal* $\mathcal{I}$ *and for* $\alpha=1$ *it coincides with* $\mathcal{I}$*-double statistical convergence, with respect to the intuitionistic fuzzy* n*-normed space* (μ,v) *, (see, [21]). When* $\mathcal{I}=\mathcal{I}_{fin}$ *and* $\alpha=1$ *it becomes only double statistical convergence with respect to the intuitionistic fuzzy* n*-normed space* (μ,v)*.*

We write the generalized double de la Valée–Poussin mean by

$$t_{m,n}(x)=\frac{1}{\lambda_m\mu_n}\sum_{k\in I_m,l\in J_n}x_{k,l}.$$

A sequence $x=(x_{k,l})$ is said to be $[V,\lambda]_2(\mathcal{I})$-summable to a number $L\in X$, if

$$\mathcal{I}-limt_{m,n}(x)=L,$$

i.e. for any $\delta>0$,

$$\{(m,n)\in\mathbf{N}\times\mathbf{N}:|t_{m,n}(x)-L|\geq\delta\}\in\mathcal{I}.$$

Throughout this paper we shall denote $\lambda_m\mu_n$ by $\lambda_{m,n}$ and $(k\in I_m, l\in J_n)$ by $(k,l)\in I_{mn}$.

We are now ready to obtain our main results.

Definition 13. *Let* $(X,\mu,v,*,\Diamond)$ *be an IFnNS. Then* $x=(x_{k,l})$ *is said to be* $[V,\lambda]_2(\mathcal{I})$*-summable of order* α *to* $L\in X$ *with respect to the intuitionistic fuzzy* n*-normed space* (μ,v) *, if for any* $\delta>0$ *and* $t>0$ *and* $y_1,y_2,...,y_{n-1}\in X$*,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left\{\begin{array}{c}\mu\left(y_1,y_2,...,y_{n-1},t_{m,n}(x)-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},t_{m,n}(x)-L,t\right)\geq\varepsilon\end{array}\right\}\geq\delta\right\}\in\mathcal{I}.$$

In this case we write $[V,\lambda]_2^{\alpha}(\mathcal{I})^{(\mu,v)_n}-\lim x=L$.

Definition 14. *Let* $(X,\mu,v,*,\Diamond)$ *be an IFnNS. A sequence* $x=(x_{k,l})$ *is said to be* $\mathcal{I}_{\lambda}$*-double statistically convergent of order* α *or* $S_{\lambda}^{\alpha}(\mathcal{I})_2$*-convergent to* $L\in X$ *with respect to the intuitionistic fuzzy* n*-normed space*(μ,v) *, if for every* $\varepsilon>0$, $\delta>0$ *and* $t>0$ *and* $y_1,y_2,...,y_{n-1}\in X$*,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}^{\alpha}}\left|\left\{\begin{array}{c}(k,l)\in I_{mn}:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

In this case we write $S_{\lambda}^{\alpha}(\mathcal{I})_2^{(\mu,v)_n}-\lim x=L$ *or* $x_{k,l}\to L(S_{\lambda}^{\alpha}(\mathcal{I})_2^{(\mu,v)_n})$.

Remark 2. *For $\mathcal{I}=\mathcal{I}_{fin}$, $S_\lambda^\alpha(\mathcal{I})_2-$convergence with respect to the intuitionistic fuzzy n-normed space (μ,v), coincides with λ-double statistical convergence of order α, with respect to the intuitionistic fuzzy n-normed space (μ,v). For an arbitrary ideal $\mathcal{I}$ and for $\alpha=1$ it coincides with $\mathcal{I}_\lambda$-double statistical convergence with respect to the intuitionistic fuzzy n-normed space (μ,v). Finally for $\mathcal{I}=\mathcal{I}_{fin}$ and $\alpha=1$ it becomes λ-double statistical convergence with respect to the intuitionistic fuzzy n-normed space $(\mu,v)_n$. Also note that taking $\lambda_{mn}=mn$ we get Definition 13 from Definition 14.*

We denote by $S_\lambda^\alpha(\mathcal{I})_2^{(\mu,v)_n}$ and $[V,\lambda]_2^\alpha(\mathcal{I})^{(\mu,v)_n}$ the collections of all $S_\lambda(\mathcal{I})_2$-convergent of order α and $[V,\lambda]_2(\mathcal{I})$ -convergent of order α sequences respectively.

We begin by stating the following result.

Theorem 1. *Let $(X,\mu,v,*,\Diamond)$ be an IFnNS, $\lambda=(\lambda_{mn})\in\Delta$. Then $x_{k,l}\to L([V,\lambda]_2^\alpha(\mathcal{I})^{(\mu,v)_n})$ $\Rightarrow x_{k,l}\to L(S_\lambda^\alpha(\mathcal{I})_2^{(\mu,v)_n})$.*

Proof. By hypothesis, for every $\varepsilon>0$, $\delta>0$, $t>0$ and $y_1,y_2,...,y_{n-1}\in X$ let $x_{k,l}\to L\,[V,\lambda]_2^\alpha(I)^{(\mu,v)_n}$. We have

$$\sum_{(k,l)\in I_{mn}}\left(\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon \text{ or } v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\right)$$

$$\ge \sum_{\substack{(k,l)\in I_{mn}\ \&\ \mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)<1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)>\varepsilon}}\begin{pmatrix}\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon \text{ or}\\ v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\end{pmatrix}$$

$$\ge\varepsilon\left|\{(k,l)\in I_{mn}:\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon \text{ or } v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\}\right|.$$

Then observe that

$$\frac{1}{\lambda_{mn}^\alpha}\left|\{(k,l)\in I_{mn}:\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon \text{ or } \nu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\}\right|\ge\delta$$

$$\Rightarrow \frac{1}{\lambda_{mn}^\alpha}\sum_{(k,l)\in I_{mn}}\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le(1-\varepsilon)\delta$$

or

$$\frac{1}{\lambda_{mn}^\alpha}\sum_{(k,l)\in I_{mn}}v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\delta$$

which implies

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}^\alpha}\left|\left\{\begin{array}{c}(k,l)\in I_{mn}:\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon\\ \text{or } v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\end{array}\right\}\right|\ge\delta\right\}$$

$$\subset\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}^\alpha}\left\{\begin{array}{c}\sum\limits_{(k,l)\in I_{mn}}\mu(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\le 1-\varepsilon\\ \text{or } \sum\limits_{(k,l)\in I_{mn}}v(y_1,y_2,...,y_{n-1},x_{k,l}-L,t)\ge\varepsilon\end{array}\right\}\ge\varepsilon\delta\right\}.$$

Since $x_{k,l}\to L\left([V,\lambda]_2^\alpha(\mathcal{I})^{(\mu,v)_n}\right)$, we immediately see that $x_{k,l}\to L\left(S_\lambda^\alpha(\mathcal{I})_2^{(\mu,v)_n}\right)$, this completed the proof of the theorem. $\square$

We now state and prove the following result.

Theorem 2. *Let* $(X, \mu, v, *, \Diamond)$ *be an IFnNS.* $S^{\alpha}(\mathcal{I})_2^{(\mu,v)_n} \subset S_{\lambda}^{\alpha}(\mathcal{I})_2^{(\mu,v)_n}$ *if* $\lim \inf_{mn\to\infty} \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} > 0$.

Proof. For given $\varepsilon > 0$ and every $t > 0$ and $y_1, y_2, ..., y_{n-1} \in X$ we have

$$\begin{aligned} & \frac{1}{(mn)^{\alpha}} \left| \left\{ \begin{array}{c} k \leq m \text{ and } l \leq n : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \right| \\ \geq \quad & \frac{1}{(mn)^{\alpha}} \left| \left\{ \begin{array}{c} (k,l) \in I_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \right| \\ = \quad & \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} \frac{1}{\lambda_{mn}^{\alpha}} \left| \left\{ \begin{array}{c} (k,l) \in I_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \right|. \end{aligned}$$

If $\lim \inf_{n\to\infty} \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} = \alpha$ then from definition $\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} < \frac{\alpha}{2}\right\}$ is finite. For every $\delta > 0$,

$$\begin{aligned} & \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}^{\alpha}} \left| \left\{ \begin{array}{c} (k,l) \in I_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \right| \geq \delta \right\} \\ \subset & \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^{\alpha}} \left| \left\{ \begin{array}{c} (k,l) \in I_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \right| \geq \frac{\alpha}{2}\delta \right\} \\ & \cup \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{\lambda_{mn}^{\alpha}}{(mn)^{\alpha}} < \frac{\alpha}{2} \right\}. \end{aligned}$$

Since $\mathcal{I}$ is admissible, the set on the right-hand side belongs to $\mathcal{I}$ and this completed the proof of the theorem. □

Finally we conclude this paper by giving the following theorem.

Theorem 3. *Let* $\lambda = (\lambda_{mn})$ *and* $\mu = (\mu_{mn})$ *be two sequences in* Λ *such that* $\lambda_{mn} \leq \mu_{mn}$ *for all* $(m,n) \in \mathbb{N} \times \mathbb{N}$ *and let* α *and* β *be fixed real numbers such that* $0 < \alpha \leq \beta \leq 1$,

If

$$\lim_{n\to\infty} \inf \frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}} > 0 \tag{1}$$

then $S_{\mu}^{L}(\mathcal{I})_2^{\beta(\mu,v)_n} \subseteq S_{\lambda}^{L}(\mathcal{I})_2^{\alpha(\mu,v)_n}$.

Proof. Suppose that $\lambda_{mn} \leq \mu_{mn}$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$ and let (1) be satisfied. For given $\varepsilon > 0$, $t > 0$ and $y_1, y_2, ..., y_{n-1} \in X$ we have

$$\begin{aligned} & \left\{ \begin{array}{c} (k,l) \in J_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\} \\ \supseteq & \left\{ \begin{array}{c} (k,l) \in I_{mn} : \mu(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \leq 1 - \varepsilon \\ \text{or } v(y_1, y_2, ..., y_{n-1}, x_{k,l} - L, t) \geq \varepsilon \end{array} \right\}. \end{aligned}$$

Therefore we can write

$$\frac{1}{\mu_{mn}^{\beta}}\left|\left\{\begin{array}{c}(k,l)\in J_{mn}:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|$$

$$\geq\frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}}\frac{1}{\lambda_{mn}^{\alpha}}\left|\left\{\begin{array}{c}(k,l)\in I_{mn}:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|$$

and so for all $(m,n)\in\mathbb{N}\times\mathbb{N}$ we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}^{\alpha}}\left|\left\{\begin{array}{c}(k,l)\in I_{mn}:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|\geq\delta\right\}$$

$$\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\mu_{mn}^{\beta}}\left|\left\{\begin{array}{c}(k,l)\in J_{mn}:\mu\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\leq 1-\varepsilon\\ \text{or } v\left(y_1,y_2,...,y_{n-1},x_{k,l}-L,t\right)\geq\varepsilon\end{array}\right\}\right|\geq\delta\frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}}\right\}\in\mathcal{I}$$

Hence

$$S_{\mu}^{L}(\mathcal{I})_2^{\beta(\mu,v)_n}\subseteq S_{\lambda}^{L}(\mathcal{I})_2^{\alpha(\mu,v)_n}. \qquad \square$$

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