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# Componentwise decomposition of intuitionistic L-fuzzy integrals and interval-valued intuitionistic fuzzy integrals

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**Abstract** We prove a componentwise decomposition theorem of an intuitionistic $L$-fuzzy integral to its $L$- fuzzy integrals components, where $L$ is a complete lattice with negation, and a componentwise decomposition theorem of an interval-valued intuitionistic fuzzy integral to its interval-valued fuzzy integrals components.

## 1 Introduction

A fuzzy measure is a normal monotone set function which vanishes at the empty set. Fuzzy integral is a general term for integrals with respect to fuzzy measures. Fuzzy integrals are proved to be important tools for decision making under risk and uncertainty, multicriteria decision making, subjective evaluation, information fusion, classification, image processing, etc. (see [5]). Extending the concepts and results from the interval $[0,1]$ to other sets we obtain general results which can be used to model more complicated problems in the previous topics at least. In the book [6] (see also the paper [16] on this topic) the case of the set of intuitionistic fuzzy values, i. e.,

$$\{(\alpha,\beta)\,;\alpha,\beta\in[0,1]\,,\alpha+\beta\leq 1\}\,,$$

is studied. In the present paper we consider the set of the intuitionistic $L$-fuzzy values, i. e.

$$\left\{(\alpha,\beta)\,;\alpha,\beta\in L,\alpha\leq\overline{\beta}\right\},$$

where $L$ is a lattice with a negation $\bar{\cdot}$, and the set of the interval-valued intuitionistic fuzzy values, i. e.,

$$\{[\alpha_1,\beta_1]\,,[\alpha_2,\beta_2]\,;\alpha_1,\beta_1,\alpha_2,\beta_2\in[0,1]\,,\beta_1+\beta_2\leq 1\}\,.$$

The main result in the very recent paper [5] is used to obtain two componentwise decomposition theorems: of an intuitionistic $L$-fuzzy integral to its $L$-fuzzy integrals components (in Section 3) and of an interval-valued intuitionistic fuzzy integral to its interval-valued fuzzy integrals components (in Section 4). The results help us to calculate in a simple way these integrals.

Throughout on this paper $X$ is a nonempty set and $\mathcal{A}\subseteq\mathcal{P}(X)$ is a $\sigma$-algebra.

## 2 Basic concepts and results

Let $\Lambda = (\Lambda, \vee, \wedge, 0_\Lambda, 1_\Lambda)$ be a complete lattice and $\leq$ its natural order. Basic concepts of fuzzy measure theory (see [14], [18]) can be extended in the following natural way ([11], [12], [19]):

**Definition 2.1** *A lattice-valued fuzzy measure or a $\Lambda$-fuzzy measure on $\mathcal{A}$ is a lattice-valued set function $v : \mathcal{A} \to \Lambda$ with the following properties:*

*(i) $v(\emptyset) = 0_\Lambda$;*

*(ii) $v(X) = 1_\Lambda$;*

*(iii) $A, B \in \mathcal{A}, A \subseteq B$ implies $v(A) \leq v(B)$.*

**Definition 2.2** *A lattice-valued mapping $f : X \to \Lambda$ is called $\mathcal{A}$-measurable if $F_\alpha \in \mathcal{A}$ and $F^\alpha \in \mathcal{A}$, for every $\alpha \in \Lambda$, where*

$$F_\alpha = \{x \in X; f(x) \geq \alpha\}$$

*and*

$$F^\alpha = \{x \in X; \alpha \geq f(x)\}.$$

**Definition 2.3** *Let $A \in \mathcal{A}$ and $f : X \to \Lambda$ be an $\mathcal{A}$-measurable function. The lattice-valued fuzzy integral (or $\Lambda$-fuzzy integral) of $f$ on $A$ with respect to $\Lambda$-fuzzy measure $v : \mathcal{A} \to \Lambda$ is defined by*

$$\int_A^\Lambda f\, dv = \bigvee_{\alpha \in \Lambda} (\alpha \wedge v(F_\alpha \cap A)).$$

Let $L = (L, \vee, \wedge, 0, 1)$ be a complete lattice with the natural order $\leq$ and the negation $\bar{\cdot}$ (i. e., $\bar{\cdot} : L \to L$ satisfies $\overline{0} = 1, \overline{1} = 0, \alpha \leq \beta$ implies $\overline{\beta} \leq \overline{\alpha}$ and $\overline{\overline{\alpha}} = \alpha$, for every $\alpha \in L$). The lattice $\mathcal{L} = (\mathcal{L}, \curlyvee, \curlywedge, \mathbf{0}, \mathbf{1})$ introduced by

$$\mathcal{L} = \left\{(\alpha, \beta); \alpha, \beta \in L, \ \alpha \leq \overline{\beta}\right\}$$

$$\begin{aligned}
(\alpha_1, \beta_1) \curlywedge (\alpha_2, \beta_2) &= (\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2), \\
(\alpha_1, \beta_1) \curlyvee (\alpha_2, \beta_2) &= (\alpha_1 \vee \alpha_2, \beta_1 \wedge \beta_2), \\
\mathbf{0} &= (0, 1), \\
\mathbf{1} &= (1, 0)
\end{aligned}$$

and

$$\widetilde{(\alpha, \beta)} = (\beta, \alpha)$$

is a complete lattice with negation too and the natural order $\preceq$ on $\mathcal{L}$ is given by

$$(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2) \Leftrightarrow \alpha_1 \leq \alpha_2 \text{ and } \beta_2 \leq \beta_1.$$

We present the main result of the recent paper [5].

**Theorem 2.4** *Let $v : \mathcal{A} \to \mathcal{L}, v(A) = (v_1(A), v_2(A))$ be an $\mathcal{L}$-valued fuzzy measure and $f : X \to \mathcal{L}, f(x) = (g(x), h(x))$ be an $\mathcal{A}$-measurable $\mathcal{L}$-valued mapping. Then*

$$\int_X^{\mathcal{L}} f\, dv = \left( \int_X^L g\, dv_1\,,\ \overline{\left( \int_X^L \overline{h}\, d\overline{v_2} \right)} \right),$$

*where $\overline{h}(x) = \overline{h(x)}$, for every $x \in X$ and $\overline{v_2}(A) = \overline{v_2(A)}$, for every $A \in \mathcal{A}$.*

The important case $L = [0,1]$ is presented in the paper [5] and other particular cases are studied in the following sections.

# 3 Componentwise decomposition of intuitionistic L-fuzzy integrals

Given a complete lattice $L$, Goguen [9] introduced the concept of $L$-fuzzy set as a mapping $\mu : X \to L$. The concept of intuitionistic $L$-fuzzy set was introduced in [3].

**Definition 3.1** *Let $L = (L, \leq)$ be a complete lattice with a non-increasing involution operation $N$. An intuitionistic $L$-fuzzy set in the universe $X$ is a mapping $\widetilde{\mu} : X \to L \times L$ such that if $\widetilde{\mu}(x) = (\alpha, \beta)$ then $\alpha \leq N(\beta)$, for all $x$ in $X$.*

Therefore the underlying lattice of intuitionistic $L$-fuzzy set theory is the lattice $\mathcal{L}$ introduced after Definition 2.3.

To increase the importance of the results in this section we mention that the fuzzy rough sets in the sense in [13] are intuitionistic $L$-fuzzy sets (result proved in [7]) and the concepts of intuitionistic fuzzy set, intuitionistic $L$-fuzzy set and $L$-fuzzy set are mathematically equivalent (see [17]).

The intuitionistic $L$-fuzzy integral ($\mathcal{L}$-fuzzy integral, with the above notations) of an intuitionistic $L$-fuzzy valued measurable mapping can be introduced as in Definition 2.3. The following result, a componentwise decomposition theorem of intuitionistic $L$-fuzzy integrals, is an obvious consequence of Theorem 2.4.

**Theorem 3.2** *Let $(L, \leq)$ be a complete lattice with a non-increasing involution operation $N$ such that $N(0) = 1$ and $N(1) = 0$. If $v$ is an intuitionistic $L$-fuzzy measure on $\mathcal{A}, v = (v_1, v_2)$ and $f$ is an $\mathcal{A}$-measurable intuitionistic $L$-fuzzy valued mapping on the universe $X, f(x) = (g(x), h(x))$ then*

$$\int_X f\, dv = \left( \int_X g\, dv_1\,,\ N\left( \int_X N(h)\ dN(v_2) \right) \right),$$

*where $(N(h))(x) = N(h(x))$, for every $x \in X$ and $(N(v_2))(A) = N(v_2(A))$, for every $A \in \mathcal{A}$.*

**Remark 3.3** *In the case $L = [0,1]$ and $N(\alpha) = 1 - \alpha$ the result was proved in [6].*

# 4 Componentwise decomposition of interval-valued intuitionistic fuzzy integrals

Let us consider the complete lattice $\widetilde{L} = \left(\widetilde{L}, \vee_{\widetilde{L}}, \wedge_{\widetilde{L}}, 0_{\widetilde{L}}, 1_{\widetilde{L}}\right)$ introduced by (see e. g. [8])

$$\widetilde{L} = \{[x_1, x_2]\,; (x_1, x_2) \in [0,1] \times [0,1]\,, x_1 \leq x_2\}\,,$$

$$\begin{aligned}
[x_1, x_2] \vee_{\widetilde{L}} [y_1, y_2] &= [\max\{x_1, y_1\}\,, \max\{x_2, y_2\}]\,,\\
[x_1, x_2] \wedge_{\widetilde{L}} [y_1, y_2] &= [\min\{x_1, y_1\}\,, \min\{x_2, y_2\}]\,,\\
0_{\widetilde{L}} &= [0,0]\,,\\
1_{\widetilde{L}} &= [1,1]\,,
\end{aligned}$$

with the natural order $\leq_{\widetilde{L}}$ and the negation $\overline{\cdot}^{\widetilde{L}}$ given by

$$\begin{aligned}
[x_1, x_2] \leq_{\widetilde{L}} [y_1, y_2] &\Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2,\\
\overline{[x_1, x_2]}^{\widetilde{L}} &= [1 - x_2, 1 - x_1]\,.
\end{aligned}$$

The lattice $\widetilde{L}$ is the underlying lattice of interval-valued fuzzy set theory (see e. g. [1], [8]).

We repeat the construction in [5] and given in the present paper after Definition 2.3, in the case of lattice $\widetilde{L}$. We obtain the complete lattice

$$\widetilde{\mathcal{L}} = \left(\widetilde{\mathcal{L}}, \vee_{\widetilde{\mathcal{L}}}, \wedge_{\widetilde{\mathcal{L}}}, 0_{\widetilde{\mathcal{L}}}, 1_{\widetilde{\mathcal{L}}}\right),$$

where

$$\widetilde{\mathcal{L}} = \{([x_1, x_2]\,, [y_1, y_2])\,; x_1, x_2, y_1, y_2 \in [0,1]\,, x_1 + y_2 \leq 1, x_2 + y_1 \leq 1\}\,,$$

$$\begin{aligned}
&([x_1, x_2]\,, [y_1, y_2]) \vee_{\widetilde{\mathcal{L}}} ([z_1, z_2]\,, [t_1, t_2])\\
&= ([\max\{x_1, z_1\}\,, \max\{x_2, z_2\}]\,, [\min\{y_1, t_1\}\,, \min\{y_2, t_2\}])\,,\\
&([x_1, x_2]\,, [y_1, y_2]) \wedge_{\widetilde{\mathcal{L}}} ([z_1, z_2]\,, [t_1, t_2])\\
&= ([\min\{x_1, z_1\}\,, \min\{x_2, z_2\}]\,, [\max\{y_1, t_1\}\,, \max\{y_2, t_2\}])\,,
\end{aligned}$$

$$\begin{aligned}
0_{\widetilde{\mathcal{L}}} &= ([0,0]\,, [1,1])\,,\\
1_{\widetilde{\mathcal{L}}} &= ([1,1]\,, [0,0])\,,
\end{aligned}$$

with the natural order $\leq_{\widetilde{\mathcal{L}}}$ and the negation $\overline{\cdot}^{\widetilde{\mathcal{L}}}$ given by

$$\begin{aligned}
([x_1, x_2]\,, [y_1, y_2]) \leq_{\widetilde{\mathcal{L}}} ([z_1, z_2]\,, [t_1, t_2]) &\Leftrightarrow x_1 \leq z_1, x_2 \leq z_2\\
&\quad \text{and } t_1 \leq y_1, t_2 \leq y_2,\\
\overline{([x_1, x_2]\,, [y_1, y_2])}^{\widetilde{\mathcal{L}}} &= ([y_1, y_2]\,, [x_1, x_2])\,.
\end{aligned}$$

The interval-valued intuitionistic fuzzy sets were introduced in [4] as extensions of the intuitionistic fuzzy sets [2] and of the interval valued fuzzy sets (see, e. g., [10]).

**Definition 4.1** *([4], see also [1]). An interval-valued intuitionistic fuzzy set in an universe $X$ is a mapping $g : X \to \widetilde{L} \times \widetilde{L}$ such that*

$$g(u) = \left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right)$$

*and $x_1^h + x_2^h \leq 1$, for all $u$ in $X$.*

Therefore the underlying lattice of interval-valued intuitionistic fuzzy sets is

$$\widetilde{\mathcal{L}}_0 = \left(\widetilde{\mathcal{L}}_0, \vee_{\widetilde{\mathcal{L}}_0}, \wedge_{\widetilde{\mathcal{L}}_0}, 0_{\widetilde{\mathcal{L}}_0}, 1_{\widetilde{\mathcal{L}}_0}\right),$$

where ([4], see also [1] or [15])

$$\widetilde{\mathcal{L}}_0 = \left\{\left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right); x_1^l, x_1^h, x_2^l, x_2^h \in [0,1], x_1^h + x_2^h \leq 1\right\},$$

$$\begin{aligned}
&\left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right) \vee_{\widetilde{\mathcal{L}}_0} \left(\left[y_1^l, y_1^h\right], \left[y_2^l, y_2^h\right]\right) \\
&= \left(\left[\max\left\{x_1^l, y_1^l\right\}, \max\left\{x_1^h, y_1^h\right\}\right], \left[\min\left\{x_2^l, y_2^l\right\}, \min\left\{x_2^h, y_2^h\right\}\right]\right), \\
&\left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right) \wedge_{\widetilde{\mathcal{L}}_0} \left(\left[y_1^l, y_1^h\right], \left[y_2^l, y_2^h\right]\right) \\
&= \left(\left[\min\left\{x_1^l, y_1^l\right\}, \min\left\{x_1^h, y_1^h\right\}\right], \left[\max\left\{x_2^l, y_2^l\right\}, \max\left\{x_2^h, y_2^h\right\}\right]\right),
\end{aligned}$$

$$\begin{aligned}
0_{\widetilde{\mathcal{L}}_0} &= ([0,0], [1,1]), \\
1_{\widetilde{\mathcal{L}}_0} &= ([1,1], [0,0]),
\end{aligned}$$

with the natural order $\leq_{\widetilde{\mathcal{L}}_0}$ and the negation $\overline{\cdot}^{\widetilde{\mathcal{L}}_0}$ given by

$$\begin{aligned}
&\left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right) \leq_{\widetilde{\mathcal{L}}_0} \left(\left[y_1^l, y_1^h\right], \left[y_2^l, y_2^h\right]\right) \Leftrightarrow x_1^l \leq y_1^l, x_1^h \leq y_1^h \\
&\qquad\qquad \text{and } y_2^l \leq x_2^l, y_2^h \leq x_2^h, \\
&\overline{\left(\left[x_1^l, x_1^h\right], \left[x_2^l, x_2^h\right]\right)}^{\widetilde{\mathcal{L}}_0} = \left(\left[x_2^l, x_2^h\right], \left[x_1^l, x_1^h\right]\right).
\end{aligned}$$

According to Theorem 2.4 a componentwise decomposition theorem of $\widetilde{\mathcal{L}}$-valued fuzzy integrals to its $\widetilde{L}$-valued fuzzy integrals components can be given. Because $\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}$ and, in addition, $\widetilde{\mathcal{L}}_0$ is a complete sublattice of the complete lattice $\widetilde{\mathcal{L}}$, the following result of componentwise decomposition of interval-valued intuitionistic fuzzy integrals to interval-valued fuzzy integrals is immediate.

**Theorem 4.2** *Let $v = (v_1, v_2)$ be an interval-valued intuitionistic fuzzy measure on $\sigma$-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ (that is $v : \mathcal{A} \to \widetilde{\mathcal{L}}_0$ which satisfies $v(\varnothing) = 0_{\widetilde{\mathcal{L}}_0}, v(X) = 1_{\widetilde{\mathcal{L}}_0}$ and $A, B \in \mathcal{A}, A \subseteq B$ implies $v(A) \leq_{\widetilde{\mathcal{L}}_0} v(B)$) and $f : X \to \widetilde{\mathcal{L}}_0, f(x) = (g(x), h(x))$ be an $\mathcal{A}$-measurable interval-valued intuitionistic fuzzy valued mapping such that $f : X \to \widetilde{\mathcal{L}}, f(x) = (g(x), h(x))$ is $\mathcal{A}$-measurable too. Then*

$$\int_X^{\widetilde{\mathcal{L}}_0} f\, dv = \left(\int_X^{\widetilde{L}} g\, dv_1, \overline{\left(\int_X^{\widetilde{L}} \overline{h}^{\widetilde{L}}\, d\overline{v_2}^{\widetilde{L}}\right)}^{\widetilde{L}}\right),$$

*where $\overline{h}^{\widetilde{L}}(x) = \overline{h(x)}^{\widetilde{L}}$, for every $x \in X$ and $\overline{v_2}^{\widetilde{L}}(A) = \overline{v_2(A)}^{\widetilde{L}}$, for every $A \in \mathcal{A}$.*

**Remark 4.3** *Because $\widetilde{\mathcal{L}}_0$ is a complete sublattice of the complete lattice $\widetilde{\mathcal{L}}$, the right side in the above formula of decomposition is indeed an element of $\widetilde{\mathcal{L}}_0$.*

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