On the cartesian product of intuitionistic fuzzy sets Glad Deschrijver¹ and Etienne E. Kerre

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Abstract

Cartesian products of intuitionistic fuzzy sets have been defined using the min-max and the product-probabilistic sum operations. In this paper we introduce and analyse the properties of a generalized cartesian product of intuitionistic fuzzy sets using a general triangular norm and conorm. In particular we investigate the emptiness, the commutativity, the distributivity, the interaction with respect to generalized unions and intersections, the distributivity with respect to the difference, the monotonicity and the cutting in terms of level-sets.

Keywords intuitionistic fuzzy set, generalized cartesian product

1 Preliminary definitions

Intuitionistic fuzzy sets (IFSs, for short) constitute a generalisation of the notion of a fuzzy set (FS, for short) and were introduced by K. T. Atanassov in 1983 in [1]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. In [2] intuitionistic fuzzy sets are defined as follows :

Definition 1.1 An intuitionistic fuzzy set in a universe E is an object of the form

$$A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in E \},\$$

where $\mu_A(x)$ ($\in [0,1]$) is called the "degree of membership of x in A", $\nu_A(x)$ ($\in [0,1]$) is called the "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition :

$$(\forall x \in E)(\mu_A(x) + \nu_A(x) \le 1).$$

The class of intuitionistic fuzzy sets in a universe E will be denoted $\mathcal{IF}(E)$.

Every fuzzy set can be identified with an intuitionistic fuzzy set for which the degree of non-membership equals one minus the degree of membership.

On intuitionistic fuzzy sets analogous operators as on ordinary fuzzy sets can be defined. These analogous operators are backwards compatible with fuzzy sets in the sense that, if

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applied to FSs, the fuzzy operators and their intuitionistic fuzzy counterparts give the same FS as a result. For instance, the union of two IFSs can be defined using the max-operation for the degree of membership and the min-operation for the degree of non-membership, and the result is still an IFS. Other common operators over FSs can directly be extended to IFSs, and the result of the operation is again an IFS. The set-theoretical properties that these operators establish for fuzzy sets generally still hold in the case of intuitionistic fuzzy sets. In [2] these operators and new operators which do not exist for ordinary fuzzy sets are described and their properties are investigated.

In this paper we will discuss the cartesian products of intuitionistic fuzzy sets. Before we can do so, we first need a few definitions. We will need level-sets of intuitionistic fuzzy sets which we define as follows. Let A be an IFS in E, then :

$$\begin{array}{lll} A_{\alpha} &=& \{x \mid x \in E \land \mu_{A}(x) \geq \alpha\}, \forall \alpha \in]0,1] \\ A^{\beta} &=& \{x \mid x \in E \land \nu_{A}(x) \leq \beta\}, \forall \beta \in [0,1[\\ A_{\overline{\alpha}} &=& \{x \mid x \in E \land \mu_{A}(x) > \alpha\}, \forall \alpha \in [0,1[\\ A^{\overline{\beta}} &=& \{x \mid x \in E \land \nu_{A}(x) < \beta\}, \forall \beta \in]0,1] \\ A^{\beta}_{\alpha} &=& \{x \mid x \in E \land \mu_{A}(x) \geq \alpha \land \nu_{A}(x) \leq \beta\}, \forall \alpha \in]0,1], \forall \beta \in [0,1[\\ A^{\overline{\beta}}_{\overline{\alpha}} &=& \{x \mid x \in E \land \mu_{A}(x) > \alpha \land \nu_{A}(x) < \beta\}, \forall \alpha \in [0,1[,\forall \beta \in]0,1] \end{array}$$

We recall the definition of triangular norms and conorms.

Definition 1.2 A triangular norm (t-norm, for short) T is a $[0,1]^2 - [0,1]$ map which satisfies :

$$(T.1) \quad (\forall x \in [0,1]) \ (T(x,1) = x),$$

$$(T.2) \quad (\forall (x,y) \in [0,1]^2) \ (T(x,y) = T(y,x)),$$

$$(T.3) \quad (\forall (x,y,z) \in [0,1]^3) \ (T(x,T(y,z)) = T(T(x,y),z)),$$

 $(T.4) \ (\forall (x_1, y_1, x_2, y_2) \in [0, 1]^4)$ $(x_1 \le y_1 \land x_2 \le y_2 \Rightarrow T(x_1, x_2) \le T(y_1, y_2)).$

Definition 1.3 A triangular conorm (t-conorm, for short) S is a $[0,1]^2 - [0,1]$ map which satisfies :

- $\begin{array}{l} (S.1) \ (\forall x \in [0,1]) \ (S(x,0)=x), \\ \\ (S.2) \ (\forall (x,y) \in [0,1]^2) \ (S(x,y)=S(y,x)), \\ \\ (S.3) \ (\forall (x,y,z) \in [0,1]^3) \ (S(x,S(y,z))=S(S(x,y),z)), \end{array}$
- $\begin{array}{l} (S.4) \ (\forall (x_1, y_1, x_2, y_2) \in [0, 1]^4) \\ (x_1 \leq y_1 \land x_2 \leq y_2 \Rightarrow S(x_1, x_2) \leq S(y_1, y_2)). \end{array}$

We define for further usage the following t-norm Z and t-conorm Z^* :

$$Z(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{if } \max(x,y) \neq 1; \end{cases}$$

$$Z^{*}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ 1 & \text{if } \min(x,y) \neq 0; \end{cases}$$

and the following t-norm Z_{α} and t-conorm $Z^{*\beta}$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$:

$$Z_{\alpha}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \alpha & \text{if } \alpha \leq \min(x,y) \leq \max(x,y) < 1, \\ 0 & \text{else}; \end{cases}$$
$$Z^{*\beta}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ \beta & \text{if } 0 < \min(x,y) \leq \max(x,y) \leq \beta, \\ 1 & \text{else}. \end{cases}$$

2 Cartesian products

In [2] a number of cartesian products is defined, amongst which the following two. Let A and B be intuitionistic fuzzy sets in E_1 and E_2 respectively. Then the following cartesian products are defined :

$$A \times_{3} B = \{ ((x, y), \mu_{A}(x)\mu_{B}(y), \nu_{A}(x) + \nu_{B}(y) - \nu_{A}(x)\nu_{B}(y)) \mid x \in E_{1} \land y \in E_{2} \}, A \times_{4} B = \{ ((x, y), \min(\mu_{A}(x), \mu_{B}(y)), \max(\nu_{A}(x), \nu_{B}(y)) \mid x \in E_{1} \land y \in E_{2} \}.$$

We see that in the first case the product \cdot is used for the degree of membership and the probabilistic sum $\hat{+}$ for the degree of non-membership. In the second case the minimum-function is used for the degree of membership and the maximum for the degree of non-membership. In both cases a *t*-norm is used for the degree of membership and a *t*-conorm for the degree of non-membership. We can thus generalize the above cartesian products as follows.

Definition 2.1 Let A and B be intuitionistic fuzzy sets in E_1 and E_2 respectively. Then the generalized cartesian product $A \times_{T,S} B$ is defined as follows :

$$A \times_{T,S} B = \{ ((x, y), T(\mu_A(x), \mu_B(y)), S(\nu_A(x), \nu_B(y)) \mid x \in E_1 \land y \in E_2 \},\$$

where T is a t-norm and S is a t-conorm.

For instance, the above examples can be written as $\times_3 = \times_{,,\hat{+}}$ and $\times_4 = \times_{\min,\max}$.

The generalized cartesian product $A \times_{T,S} B$ will be an intuitionistic fuzzy set in $E_1 \times E_2$ if $T(\mu_A(x), \mu_B(y)) + S(\nu_A(x), \nu_B(y)) \leq 1$. Since A and B are IFSs, we have $\nu_A(x) \leq 1 - \mu_A(x)$ and $\nu_B(y) \leq 1 - \mu_B(y)$, and because of (S.4), $S(1 - \mu_A(x), 1 - \mu_B(y)) \geq S(\nu_A(x), \nu_B(y))$, which is equivalent to $S^*(\mu_A(x), \mu_B(y)) = 1 - S(1 - \mu_A(x), 1 - \mu_B(y)) \leq 1 - S(\nu_A(x), \nu_B(y))$, where equality holds if A and B are fuzzy sets. So, if

$$(\forall (x,y) \in E_1 \times E_2) (T(\mu_A(x), \mu_B(y)) \le S^*(\mu_A(x), \mu_B(y))),$$
 (1)

then $A \times_{T,S} B$ is an intuitionistic fuzzy set. If $(\forall x \in E_1) (\nu_A(x) = 1 - \mu_A(x))$ and $(\forall y \in E_2) (\nu_B(y) = 1 - \mu_B(y))$, then $T(\mu_A(x), \mu_B(y)) + S(1 - \mu_A(x), 1 - \mu_B(y))$ $\leq 1 \Leftrightarrow T(\mu_A(x), \mu_B(y)) \leq 1 - S(1 - \mu_A(x), 1 - \mu_B(y)) = S^*(\mu_A(x), \mu_B(y)).$

So condition (1) is also necessary for the generalized cartesian product to be an IFS.

Now we investigate the properties of the newly defined cartesian product. First we introduce the following notation : for a certain universe E we define $\emptyset_E = \{(x, 0, 1) \mid x \in E\}$.

2.1 On the emptiness of the generalized cartesian product

Let $A \in \mathcal{IF}(E_1)$ and $B \in \mathcal{IF}(E_2)$. Then :

$$A = \emptyset_{E_1} \lor B = \emptyset_{E_2} \Rightarrow A \times_{T,S} B = \emptyset_{E_1 \times E_2},$$

and

$$((\exists x_0 \in A^{\overline{1}}) (\forall y \in B^{\overline{1}}) (S(\nu_A(x_0), \nu_B(y)) < 1) \lor (\exists y_0 \in B^{\overline{1}}) (\forall x \in A^{\overline{1}}) (S(\nu_A(x), \nu_B(y_0)) < 1)) \Rightarrow (A \times_{T,S} B = \emptyset_{E_1 \times E_2} \Rightarrow A = \emptyset_{E_1} \lor B = \emptyset_{E_2})$$

Proof. $A \times_{T,S} B = \emptyset_{E_1 \times E_2}$ means that for all $x \in E_1$ and for all $y \in E_2$ holds

$$T(\mu_A(x), \mu_B(y)) = 0, S(\nu_A(x), \nu_B(y)) = 1.$$

Suppose that $A = \emptyset_{E_1}$ (the case $B = \emptyset_{E_2}$ is completely analogous). Then $\mu_A(x) = 0$ and $\nu_A(x) = 1$. Since $(\forall \alpha \in [0, 1]) (T(\alpha, 0) = 0)$ and $(\forall \alpha \in [0, 1]) (S(\alpha, 1) = 1)$, it follows that

$$T(\mu_A(x), \mu_B(y)) = T(0, \mu_B(y)) = 0,$$
 and
 $S(\mu_A(x), \mu_B(y)) = S(1, \mu_B(y)) = 1.$

Hence the first implication holds.

To prove the second implication, we assume that there exists a $x_0 \in E_1$ such that $\nu_A(x_0) < 1$ and such that

$$(\forall y \in E_2) \left(\nu_B(y) < 1 \Rightarrow S(\nu_A(x_0), \nu_B(y)) < 1\right).$$
(2)

Then

$$S(\nu_A(x), \nu_B(y)) = 1, \quad \forall (x, y) \in E_1 \times E_2$$

$$\Rightarrow \quad S(\nu_A(x_0), \nu_B(y)) = 1, \quad \forall y \in E_2$$

$$\Rightarrow \quad \nu_B(y) = 1, \quad \forall y \in E_2, \quad \text{because of } (2)$$

$$\Rightarrow \quad B = \emptyset_{E_2}.$$

The last implication holds because, if $\nu_B(y) = 1$ for some $y \in E_2$, then necessarily $\mu_B(y) = 0$, because for each intuitionistic fuzzy set the sum of the two degrees is not greater than 1.

Similarly, if there exists a $y_0 \in E_2$ such that $\nu_B(y_0) < 1$ and such that

$$(\forall x \in E_1) \left(\nu_A(x) < 1 \Rightarrow S(\nu_A(x), \nu_B(y_0)) < 1\right),\tag{3}$$

then $A = \emptyset_{E_1}$.

2.2 On the commutativity of the generalized cartesian product

Let $A \in \mathcal{IF}(E_1)$ and $B \in \mathcal{IF}(E_2)$. Then :

$$A = B \Rightarrow A \times_{T,S} B = B \times_{T,S} A$$

Proof. Assume A = B; $x, y \in E$. Then

$$\mu_{A \times_{T,S} B}(x, y) = T(\mu_A(x), \mu_B(y)) = T(\mu_B(x), \mu_A(y)) = \mu_{B \times_{T,S} A}(x, y),$$

and

$$\nu_{A \times_{T,S} B}(x, y) = S(\mu_A(x), \mu_B(y)) = S(\mu_B(x), \mu_A(y)) = \nu_{B \times_{T,S} A}(x, y).$$

In classical set theory, if the cartesian product of two sets is commutative then the two sets are equal to each other. In intuitionistic fuzzy set theory this is not the case. It is possible that the cartesian product of two IFSs is commutative, but that the original IFSs are different. Consider for instance the IFSs

$$A = \{(x_1, a, b), (x_2, a, b)\}, \text{ and } B = \{(x_1, c, d), (x_2, c, d)\}.$$

Then $\forall (i,j) \in \{1,2\}^2$:

$$T(\mu_A(x_i), \mu_B(x_j)) = T(a, c) = T(\mu_B(x_i), \mu_A(x_j))$$

$$S(\nu_A(x_i), \nu_B(x_j)) = S(b, d) = S(\nu_B(x_i), \nu_A(x_j)).$$

So we obtain $A \times_{T,S} B = B \times_{T,S} A$, but $A \neq B$ if $a \neq c \lor b \neq d$.

2.3 On the distributivity of the generalized cartesian product with respect to unions and intersections

Let $A \in \mathcal{IF}(E_1)$; $B, C \in \mathcal{IF}(E_2)$. Then :

$$A \times_{T,S} (B \cap C) = (A \times_{T,S} B) \cap (A \times_{T,S} C),$$

$$A \times_{T,S} (B \cup C) = (A \times_{T,S} B) \cup (A \times_{T,S} C).$$

Proof. For the first equality, we have to prove that

$$T(\mu_A(x), \min(\mu_B(y), \mu_C(y))) = \min(T(\mu_A(x), \mu_B(y)), T(\mu_A(x), \mu_C(y)))$$
(4)

and

$$S(\nu_A(x), \max(\nu_B(y), \nu_C(y))) = \max(T(\nu_A(x), \nu_B(y)), T(\nu_A(x), \nu_C(y)))$$
(5)

hold for $(x, y) \in E_1 \times E_2$.

Let first $\mu_B(y) \leq \mu_C(y)$, then the left hand of (4) becomes $T(\mu_A(x), \mu_B(y))$. Because of (T.4) we have $T(\mu_A(x), \mu_B(y)) \leq T(\mu_A(x), \mu_C(y))$, so that the right hand of (4) also equals $T(\mu_A(x), \mu_B(y))$. If $\mu_B(y) > \mu_C(y)$, we have a similar proof of the validity of (4).

Formula (5) is proved similarly.

2.4 On the interaction of the generalized cartesian product with respect to generalized unions and intersections

Let $A \in \mathcal{IF}(E_1)$; $B, C \in \mathcal{IF}(E_2)$. In general we cannot obtain any of the following assertions :

$$A \times_{T,S} (B \cap_{T',S'} C) \subseteq (A \times_{T,S} B) \cap_{T',S'} (A \times_{T,S} C),$$
$$A \times_{T,S} (B \cup_{S',T'} C) \subseteq (A \times_{T,S} B) \cup_{S',T'} (A \times_{T,S} C),$$

$$A \times_{T,S} (B \cap_{T',S'} C) \supseteq (A \times_{T,S} B) \cap_{T',S'} (A \times_{T,S} C),$$

$$A \times_{T,S} (B \cup_{S',T'} C) \supseteq (A \times_{T,S} B) \cup_{S',T'} (A \times_{T,S} C),$$

where $B \cap_{T',S'} C = \{(y,T'(\mu_B(y),\mu_C(y)),S'(\nu_B(y),\nu_C(y))) \mid y \in E_2\}$, and $B \cup_{S',T'} C = \{(y,S'(\mu_B(y),\mu_C(y)),T'(\nu_B(y),\nu_C(y))) \mid y \in E_2\}$. Of course T', resp. S' is an arbitrary t-norm, resp. t-conorm.

Consider for instance the second and the fourth statement for the case where T = Z and $S = Z^*$. Then for an arbitrary $x \in E_1$ such that $0 < \mu_A(x) < 1$ and for an arbitrary $y \in E_2$ such that $\mu_B(y) = \mu_C(y) = 1$, we obtain $Z(\mu_A(x), S'(\mu_B(y), \mu_C(y))) = Z(\mu_A(x), S'(1, 1)) = \mu_A(x)$ and $S'(Z(\mu_A(x), \mu_B(y)), Z(\mu_A(x), \mu_C(y))) = S'(Z(\mu_A(x), 1), Z(\mu_A(x), 1)) = S'(\mu_A(x), \mu_A(x))$. Since max is the only idempotent *t*-conorm, we obtain

$$Z(\mu_A(x), S'(1,1)) < S'(Z(\mu_A(x),1), Z(\mu_A(x),1))$$

as soon as $S' \neq \max$, which contradicts the fourth formula.

For arbitrary $x \in E_1, y \in E_2$ such that $0 < \mu_A(x) < \mu_B(y) < \mu_C(y) < 1$ and $S'(\mu_B(y), \mu_C(y)) = 1$, we find $Z(\mu_A(x), S'(\mu_B(y), \mu_C(y))) = \mu_A(x)$ and $S'(Z(\mu_A(x), \mu_B(y)), Z(\mu_A(x), \mu_C(y))) = 0$. Under the given conditions, we thus find that $Z(\mu_A(x), S'(\mu_B(y), \mu_C(y))) > S'(Z(\mu_A(x), \mu_B(y)), Z(\mu_A(x), \mu_C(y)))$, which contradicts the second formula.

An analogous example shows that the first and the third inequalities don't hold either in the general case.

2.5 On the distributivity of the generalized cartesian product with respect to the difference

Let $A \in \mathcal{IF}(E_1)$; $B, C \in \mathcal{IF}(E_2)$. Then :

$$A \times_{T,S} (B \setminus C) \subseteq (A \times_{T,S} B) \setminus (A \times_{T,S} C).$$

If $B = \{(y, 1, 0) \mid y \in E_2\}, C \subseteq A, T = \min$ and $S = \max$, then equality holds. *Proof.* We need to prove

$$A \times_{T,S} (B \cap \operatorname{co} C) \subseteq (A \times_{T,S} B) \cap \operatorname{co} (A \times_{T,S} C).$$

We prove that

$$T(\mu_A(x), \min(\mu_B(y), \nu_C(y))) \le \min(T(\mu_A(x), \mu_B(y)), S(\nu_A(x), \nu_C(y))),$$
(6)

and

$$S(\nu_{A}(x), \max(\nu_{B}(y), \mu_{C}(y))) \geq \max(S(\nu_{A}(x), \nu_{B}(y)), T(\mu_{A}(x), \mu_{C}(y))).$$
(7)

Suppose for (6) that $\mu_B(y) < \nu_C(y)$, then, using (T.4) and the fact that $\forall (a, b, c) \in [0, 1]^3$: $T(a, b) \leq \min(a, b) \leq a \leq \max(a, c) \leq S(a, c)$, we obtain $T(\mu_A(x), \min(\mu_B(y), \nu_C(y))) = T(\mu_A(x), \mu_B(y)) \leq T(\mu_A(x), \nu_C(y)) \leq \nu_C(y) \leq S(\nu_A(x), \nu_C(y))$. Since the left hand of the inequality is equal to $T(\mu_A(x), \mu_B(y))$ and is smaller than $S(\nu_A(x), \nu_C(y))$, it is equal to $\min(T(\mu_A(x), \mu_B(y)), S(\nu_A(x), \nu_C(y)))$. Suppose on the contrary that $\mu_B(y) \geq \nu_C(y)$, then $T(\mu_A(x), \min(\mu_B(y), \nu_C(y))) = T(\mu_A(x), \nu_C(y)) \leq T(\mu_A(x), \mu_B(y))$ and $T(\mu_A(x), \min(\mu_B(y), \nu_C(y))) \leq \nu_C(y) \leq S(\nu_A(x), \nu_C(y))$, so also in this case the left hand of the inequality is less than or equal to the right hand. Thus (6) holds. The inequality (7) follows in a similar way.

Suppose $B = \{(y, 1, 0) \mid y \in E_2\}, C \subseteq A, T = \min$ and $S = \max$. Then $T(\mu_A(x), \min(\mu_B(y), \nu_C(y))) = T(\mu_A(x), \nu_C(y)) = \min(\mu_A(x), \nu_C(y))$, since $\mu_B(y) = 1$. The right hand of (6) equals $\min(\mu_A(x), S(\nu_A(x), \nu_C(y))) = \min(\mu_A(x), \max(\nu_A(x), \nu_C(y))) = \min(\mu_A(x), \nu_C(y))$, since $C \subseteq A$. So equality is obtained in (6). In a similar way equality is obtained in (7) under the given conditions.

2.6 On the monotonicity of the generalized cartesian product

Let $A \in \mathcal{IF}(E_1)$; $B, C \in \mathcal{IF}(E_2)$. Then :

$$B \subseteq C \Rightarrow A \times_{T,S} B \subseteq A \times_{T,S} C.$$

The reverse implication holds under one of the following conditions :

(i) $A = \{(x, 1, 0) \mid x \in E_1\}$

(ii) $(\forall (x, y) \in E_1 \times E_2) (T(\mu_A(x), \mu_B(y)) < T(\mu_A(x), \mu_C(y)) \land S(\nu_A(x), \nu_B(y)) > S(\nu_A(x), \nu_C(y))).$

Proof. Let $B \subseteq C$, then

$$(\forall y \in E_2) (\mu_B(y) \le \mu_C(y) \land \nu_B(y) \ge \nu_C(y))$$

From (T.4) it follows that

$$(\forall (x,y) \in E_1 \times E_2) (T(\mu_A(x),\mu_B(y)) \le T(\mu_A(x),\mu_C(y))$$

$$\land \quad S(\nu_A(x),\nu_B(y)) \ge S(\nu_A(x),\nu_C(y)))$$

To investigate in which cases the reverse implication holds, we consider the following facts. From T(a,b) < T(a,c) follows that $b \le c$, since $b > c \Rightarrow T(a,b) \ge T(a,c)$. So in general, from $T(a,b) \le T(a,c)$ it does not follow that $b \le c$, unless T(a,b) < T(a,c) or a = 1 (since then T(a,b) = b and T(a,c) = c). If $a \ne 1$, then from T(a,b) = T(a,c) it cannot be deduced that $b \le c$ nor $b \ge c$.

The same assertions hold for S and thus from $S(a,b) \ge S(a,c)$ we get $b \ge c$ if a = 0or S(a,b) > S(a,c). Hence we may conclude that the reverse implication only holds if $A = \{(x,1,0) \mid x \in E_1\}$, or if $T(\mu_A(x),\mu_B(y)) < T(\mu_A(x),\mu_C(y)) \land S(\nu_A(x),\nu_B(y)) > S(\nu_A(x),\nu_C(y))$.

2.7 On the cutting of the generalized cartesian product

Let $A \in \mathcal{IF}(E_1)$ and $B \in \mathcal{IF}(E_2)$. Then :

$$(A \times_{T,S} B)_{\alpha} \subseteq A_{\alpha} \times B_{\alpha}, \forall \alpha \in]0,1]$$

$$\tag{8}$$

$$(A \times_{T,S} B)_{\overline{\alpha}} \subseteq A_{\overline{\alpha}} \times B_{\overline{\alpha}}, \forall \alpha \in [0, 1[$$

$$(9)$$

$$(A \times_{T,S} B)^{\beta} \subseteq A^{\beta} \times B^{\beta}, \forall \beta \in [0, 1[$$
(10)

$$(A \times_{T,S} B)^{\overline{\beta}} \subseteq A^{\overline{\beta}} \times B^{\overline{\beta}}, \forall \beta \in]0,1]$$

$$(11)$$

$$(A \times_{T,S} B)^{\beta}_{\alpha} \subseteq A^{\beta}_{\alpha} \times B^{\beta}_{\alpha}, \forall \alpha \in]0,1], \forall \beta \in [0,1[$$
(12)

$$(A \times_{T,S} B)^{\overline{\beta}}_{\overline{\alpha}} \subseteq A^{\overline{\beta}}_{\overline{\alpha}} \times B^{\overline{\beta}}_{\overline{\alpha}}, \forall \alpha \in [0, 1[, \forall \beta \in]0, 1]$$

$$(13)$$

In (8) the equality holds if and only if $T \ge Z_{\alpha}$, or if $T(\alpha, \alpha) = \alpha$; in (9) the equality holds if and only if $T > Z_{\alpha}$, or if $(\exists V \in \mathcal{U}_{(\alpha,\alpha)}) \ (\forall (x,y) \in V \cap]\alpha, 1]^2) \ (\alpha < T(x,y))$;

in (10) the equality holds if and only if $S \leq Z^{*\beta}$, or if $S(\beta, \beta) = \beta$;

in (11) the equality holds if and only if $S < Z^{*\beta}$, or if $(\exists V \in \mathcal{U}_{(\beta,\beta)}) \ (\forall (x,y) \in V \cap [0,\beta]^2) (S(x,y) < \beta);$

in (12) the equality holds if and only if $T \ge Z_{\alpha} \land S \le Z^{*\beta}$, or if $T(\alpha, \alpha) = \alpha \land S(\beta, \beta) = \beta$; in (13) the equality holds if and only if $T > Z_{\alpha} \land S < Z^{*\beta}$, or if $(\exists V_1 \in \mathcal{U}_{(\alpha,\alpha)})$ $(\forall (x,y) \in V_1 \cap]\alpha, 1]^2)$ $(\alpha < T(x,y)) \land (\exists V_2 \in \mathcal{U}_{(\beta,\beta)})$ $(\forall (x,y) \in V_2 \cap [0,\beta]^2)$ $(S(x,y) < \beta)$,

where $\mathcal{U}_{(\alpha,\alpha)}$ denotes the class of neighbourhouds of (α, α) .

Proof. Let us first prove (8). The left hand is equal to

$$\{(x,y) \mid (x,y) \in E_1 \times E_2 \land \mu_{A \times_{T,S} B}(x,y) \ge \alpha \} \\ = \{(x,y) \mid (x,y) \in E_1 \times E_2 \land T(\mu_A(x), \mu_B(y)) \ge \alpha \}.$$

The right hand is equal to

$$\{(x,y) \mid (x,y) \in E_1 \times E_2 \land x \in A_\alpha \land y \in B_\alpha\} \\ = \{(x,y) \mid (x,y) \in E_1 \times E_2 \land \mu_A(x) \ge \alpha \land \mu_B(y) \ge \alpha\}.$$

Since $T(\mu_A(x), \mu_B(y)) \leq \min(\mu_A(x), \mu_B(y))$, we will have $\mu_A(x) \geq \alpha$ and $\mu_B(y) \geq \alpha$ as soon as $T(\mu_A(x), \mu_B(y)) \geq \alpha$.

To obtain equality in (8), we must have $\mu_A(x) \ge \alpha \land \mu_B(y) \ge \alpha \Rightarrow T(\mu_A(x), \mu_B(y)) \ge \alpha$. If $T = Z_{\alpha}$, then from $\mu_A(x) \ge \alpha$ and $\mu_B(y) \ge \alpha$ follows:

$$Z_{\alpha}(\mu_A(x), \mu_B(y)) = \min(\mu_A(x), \mu_B(y)) \ge \alpha,$$

if $\max(\mu_A(x), \mu_B(y)) = 1;$
$$Z_{\alpha}(\mu_A(x), \mu_B(y)) = \alpha, \text{ if } \max(\mu_A(x), \mu_B(y)) \ne 1.$$

Since Z_{α} is a *t*-norm, $Z_{\alpha}(a, b)$ cannot be different from $\min(a, b)$ if $\max(a, b) = 1$. Since otherwise for all $x \in E_1$ and for all $y \in E_2$ such that $\mu_A(x) \ge \alpha$ and $\mu_B(y) \ge \alpha$ holds that $Z_{\alpha}(\mu_A(x), \mu_B(y)) = \alpha$, we can conclude that Z_{α} is the smallest *t*-norm for which equality holds. Moreover, equality also holds for all *t*-norms $T > Z_{\alpha}$.

If $T(\alpha, \alpha) = \alpha$ (it cannot be greater than α since for all *t*-norms *T* there holds $T(a, b) \leq \min(a, b), \forall (a, b) \in [0, 1]^2$), then $\forall (a, b) \in [0, 1]^2$: $\alpha \leq a \land \alpha \leq b \Rightarrow \alpha = T(\alpha, \alpha) \leq T(a, b)$ because of (T.4). Thus, the condition that $T(\alpha, \alpha)$ equals α , is sufficient to have equality in (8).

The left hand of (9) is equal to

$$\{ (x,y) \mid (x,y) \in E_1 \times E_2 \land \mu_{A \times_{T,S} B}(x,y) > \alpha \}$$

= $\{ (x,y) \mid (x,y) \in E_1 \times E_2 \land T(\mu_A(x),\mu_B(y)) > \alpha \}.$

The right hand is equal to

$$\{(x,y) \mid (x,y) \in E_1 \times E_2 \land x \in A_{\overline{\alpha}} \land y \in B_{\overline{\alpha}}\} \\ = \{(x,y) \mid (x,y) \in E_1 \times E_2 \land \mu_A(x) > \alpha \land \mu_B(y) > \alpha\}.$$

Since $T(\mu_A(x), \mu_B(y)) \leq \min(\mu_A(x), \mu_B(y))$, we will have $\mu_A(x) > \alpha$ and $\mu_B(y) > \alpha$ as soon as $T(\mu_A(x), \mu_B(y)) > \alpha$.

To have equality in (9), we must have $\mu_A(x) > \alpha \land \mu_B(y) > \alpha \Rightarrow T(\mu_A(x), \mu_B(y)) > \alpha$. Since Z_{α} is the smallest *t*-norm for which a similar statement holds (see above), we can deduce that equality in (9) will hold for each *t*-norm $T > Z_{\alpha}$.

If $(\exists V \in \mathcal{U}_{(\alpha,\alpha)})$ $(\forall (a,b) \in V \cap]\alpha, 1]^2$ $(\alpha < T(a,b))$, then we already have $(\forall (a,b) \in V)$ $(a > \alpha \land b > \alpha \Rightarrow T(a,b) > \alpha)$. If $(a,b) \in]\alpha, 1]^2 \setminus V$, then, since there exist an a' and a b' such that $(a',b') \in V \cap]\alpha, 1]^2$, we obtain a' < a, b' < b. Because of (T.4) we obtain $T(a,b) > T(a',b') > \alpha$. So, for all $(a,b) \in [0,1]^2$ the implication $a > \alpha \land b > \alpha \Rightarrow T(a,b) > \alpha$ holds.

The other properties can be proved similarly.

3 Conclusion

In this paper we defined a generalized cartesian product using t-norms and t-conorms. The generalized cartesian product thus defined is an IFS if and only if $T \leq S^*$, where S^* denotes the dual t-norm with respect to S. We have investigated the set-theoretical properties of the generalized cartesian product and have found that most of them hold under certain conditions. If at least one of two IFSs is empty, then the generalized cartesian product of these IFSs is empty. The reverse holds only under certain conditions. The generalized cartesian product the equality of the original IFSs cannot be deduced. The generalized cartesian product is distributive with respect to unions and intersections, but a general distributivity law or distributivity-like laws cannot be obtained. A distributivity-like law can be obtained with respect to the difference. We have also obtained monotonicity under certain conditions. At last we have found that the level-set of the generalized cartesian product of the classical cartesian product of the level-sets of the original IFSs.

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