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ON A HESITANCY MARGIN AND A PROBABILITY OF INTUITIONISTIC FUZZY EVENTS

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Abstract: K. Atanassov's idea of an intuitionistic fuzzy set [2] aimed at improving a formal description of phenomena diverse in meaning in relation to L. A. Zadeh's original conception of a fuzzy set [9]. In the definition of an intuitionistic fuzzy set there appears a neglected element, the so-called intuitionistic fuzzy index (hesitancy margin), being a factor introducing traces if intuition to K. Atanassov's idea. In the present paper, this factor is included in the concept of a probability of intuitionistic fuzzy events [5].
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1. Introductory remarks

L. A. Zadeh's publication [9] concerning a fuzzy set brought out a series of articles and papers making use of the idea of fuzziness in practice. At the same time, there began studies on a generalization of the concept (after all – general) of a fuzzy set. So, there appeared lattice fuzzy sets, fuzzy sets of order n, "rough" sets. One of the more successful conceptions of generalizing a Zadeh fuzzy set is K. Atabassov's idea of an intuitionistic fuzzy set [1]. It allows one to describe the cases of indefiniteness and non-univocality in life more freely and elastically, thus in a fuller manner.

Following [3], we have

DEFINITION 1. Let $X \neq \emptyset$ be a space under consideration. By an *intuitionistic fuzzy set A* we mean an object of the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \tag{1}$$

where $\mu_A, \nu_A : X \to \langle 0,1\rangle$ are, respectively, functions of the belonging and the non-belonging of the element x to the set A and, at the same time, they satisfy the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$.

The family of all intuitionistic fuzzy sets in X will be denoted by IFS(X).

Our further considerations will be preceded by a presentation and discussion of a few examples.

2. Examples

EXAMPLE 1 (Atanassov [3])

Let X be the set of all the countries whose governments are chosen by election. Assume that, for each country $x \in X$, we know the percentage of the electorate voting "for" the appropriate government. Denote this percentage by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$. Let $\nu(x) = 1 - \mu(x)$. This number concerns those electors who have voted "against" the government. At that moment, Zadeh's theory of fuzzy sets does not contribute any additional information. However, if we define the number $\nu(x)$ as the percentage of the electorate who have voted for parties or people from outside the government, then the number $1 - \mu(x) - \nu(x)$ will be connected with the group of persons who have not voted at all or their votes have been invalid. We have thus constructed the set $\{(x, \mu(x), \nu(x)) : x \in X\}$ where $0 \le \mu(x) + \nu(x) \le 1$.

In the case of a Zadeh fuzzy set, formula (1) is transformed into

$$L = \{(x, \mu_L(x), 1 - \mu_L(x)) : x \in X\}, \tag{2}$$

whereas

$$K = \{(x, \chi_K(x), 1 - \chi_K(x)) : x \in X\} \tag{3}$$

is a non-fuzzy set in the classical sense of Cantor's set theory.

In particular, the empty intuitionistic fuzzy set is the set $\emptyset = \{(x, 0, 1) : x \in X\}$, while an intuitionistic fuzzy space is the set $U = \{(x, 1, 0) : x \in X\}$.

DEFINITION 2. (Atanassov [3])

For an intuitionistic fuzzy set $A \in$ IFS(X), the difference

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{4}$$

is called an *intuitionistic fuzzy index (hesitancy margin)* of the element x in the set A.

The number $\pi_A(x)$ may be treated as the indeterminacy (indefiniteness) degree of the membership of the element *x* to the intuitionistic fuzzy set *A*. It is obvious that $\pi_A(x) \in \langle 0,1 \rangle$ for all *x*, and that, for a Zadeh fuzzy set *L*, we always have $\pi_L(x) = 1 - \mu_L(x) - [1 - \mu_L(x)] = 0$.

EXAMPLE 2

Assume that we are interested in classifying an *i*-th student from an *n*-person group *X* to the category of "gifted students". Let $\mu(x_i)$ denote the degree of the membership of the student x_i to the "gifted" (the degree of our conviction about that), $\nu(x_i)$ - the non-membership degree, $\pi(x_i)$ - the degree of our indeterminacy or hesitancy about the univocality of the classification, with that, of course, $\mu(x_i) + \nu(x_i) + \pi(x_i) = 1$.

Suppose for the moment that $\mu(x_i) = 0.2,\ \nu(x_i) = 0.5$, thus $\pi(x_i) = 0.3$. Under the circumstances favourable to the student (such as tests, oral or written examinations, additional achievements), the greatest degree of classifying him to the set of "gifted students" is $\mu_{\max}(x_i) = \mu(x_i) + \pi(x_i) = 0.5$ (then $\nu(x_i) = 0.5,\ \pi(x_i) = 0$). On the other hand, the circumstances may turn out to be unfavourable and, then, $\mu(x_i) = 0.2$ remains unchanged, whereas $\nu_{\max}(x_i) = \nu(x_i) + \pi(x_i) = 0.8$. With such data, the student has very faint chances to be classified among the "gifted".

Suppose now that $\mu(x_i) = \nu(x_i) = 0.5$ (then $\pi(x_i) = 0$ – a complete lack of hesitancy about classifying him). Such a situation should be interpreted as follows: the student is a man of average ability and nothing will change our opinion about him.

Let us go to extremes and suppose that $\mu(x_i) = \nu(x_i) = 0$ and $\pi(x_i) = 1$. Such a situation means that, subject to the influx of information, everything may happen and we may freely change our decision on the numbers $\mu(x_i)$ and $\nu(x_i)$, respectively.

Finally, suppose that we have $\mu(x_i) = 0.5,\ \nu(x_i) = 0.2$ and $\pi(x_i) = 0.3$. Then $\mu_{\max}(x_i) = \mu(x_i) + \pi(x_i) = 0.8$ and $\nu_{\max}(x_i) = \nu(x_i) + \pi(x_i) = 0.5$. Such values mean that this student has very considerable chances to be reckoned as a "gifted" man.

The above example shows the essence of the value $\pi(x)$ in interpreting an intuitionistic fuzzy set and, at the same time, gives a freedom and a possibility of manoeuvring the values $\mu(x)$ and $\nu(x)$ according to the influx of information and the evolution of the knowledge of the person examining a concrete problem.

It is clear that, in order to give a full description of an intuitionistic fuzzy set, it suffices to use two of the three functions – membership, non-membership, index. The distinction of the function $\pi(x)$ in the description of the set allows one to interpret the description of the problem situation faster and more explicitly. Therefore it will be more convenient to define an intuitionistic fuzzy set in the form

$$A = \{(x, \mu_A(x), \pi_A(x)) \ : x \in X\} \tag{5}$$

or, equivalently,

$$A = \{\mu_A(x), \pi_A(x) / x \ : x \in X\}$$

instead of (1).

3. Probability

Let (*E, F, P*) be a probability space in the ordinary sense, with the σ-field *F* of subsets of a set *E* and with a probability function *P*.

DEFINITION 3 (Gerstenkorn and Mańko [5])

By an intuitionistic fuzzy event in *E* we mean any intuitionistic fuzzy set whose functions $\mu(x)$ and $\nu(x)$ (thus $\pi(x)$) are measurable.

The family of *intuitionistic fuzzy events* will be denoted by IFM(*E*).

Making use of [5], we then introduce

DEFINITION 4

For $A \in$ IFM(*E*), the number

$$P(A) = \int_E [\mu_A(x) + 0.5\pi_A(x)] P(dx) \tag{6}$$

is called a *probability of the event A*.

The function so defined satisfies the classical properties of the Kolmogorov probability, that is [5],

1) $P(A) \geq 0$
2) $P(U) = 1$
3) $P(A \cup B) = P(A) + P(B)$ for $A \cap B = \emptyset$
4) $P(\emptyset) = 0$
5) $P(A) \leq 1$
6) $P(A') = 1 - P(A)$
7) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for $A, B \in$ IFM(*E*),

where the operations $\cup, \cap, '$ are set operations in the intuitionistic sense [3].

When $\pi_A(x) \equiv 0$, formula (6) reduces to the well-known formula for the probability of a fuzzy event, proposed in [10] by L. A. Zadeh.

Assume now that $E = \{x_1, x_2, ..., x_n\}$ is a finite set. Let a probability function $P = \{p_1, p_2, ..., p_n\}$ be defined in *E* and let $A \in$ IFM(*E*) be a random event. Then formula (6) takes the form

$$P(A) = \sum_{i=1}^{n} [\mu_A(x_i) + 0.5\pi_A(x_i)] p_i \ . \tag{7}$$

We shall now consider the case of the classical Laplace probability by using the notion of the cardinality of a set.

DEFINITION 5

By the *cardinality* of a set $A \in$ IFS(*E*) we mean the number

$$\text{card}\, A = \sum_{i=1}^{n} \left(\mu_A(x_i) + 0.5\pi_A(x_i)\right). \tag{8}$$

Formula (8) is a natural generalization of the formula for the cardinality of a fuzzy set [7] and has been modified in comparison with [6].

Suppose now that the probability distribution in the set *E* is $P = \left\{\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right\}$, i. e. that each elementary event is equally privileged. Then, following [6], we propose

DEFINITION 6

By the *probability* of an event $A \in$ IFM(*E*) we mean the number

$$P(A) = \frac{\text{card}\, A}{\text{card}\, E} = \sum [\mu_A(x_i) + 0.5 \cdot \pi_A(x_i)] \cdot \frac{1}{n} . \tag{9}$$

The last expression is a special case of formula (7) and represents the classical Laplace probability carried over to the ground of the theory of intuitionistic fuzzy events.

EXAMPLE 3

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of five students in whose midst we define a set *A* of "gifted" students and let $P(\{x_i\}) = \frac{1}{5}$ for $i = 1, 2, 3, 4, 5$. Let $A = \{(x; \mu_A(x); \pi_A(x)\}$ be an intuitionistic fuzzy set of the form:

$$A = \{(x_1; 0.6, 0.1), (x_2; 0.6, 0.3), (x_3; 0.5, 0.2), (x_4; 0.8, 0.1), (x_5; 0.2, 0.3)\}.$$

From *X* we choose at random a "gifted" student. Then, by (9),

$$P(A) = \sum_{i=1}^{5} [\mu_A(x_i) + 0.5\pi_A(x_i)] \cdot \frac{1}{5} = 0.64 .$$

Conclusion

In our paper we have emphasized the importance of the hesitancy margin – the element (underestimated as yet) defining an intuitionistic fuzzy set. This parameter constitutes a subtle elasticity of the notion of such a set. We have modified the formulae for the probability of an intuitionistic fuzzy event by using the hesitancy margin. The formulae obtained are some generalizations of those from paper [5]. Simultaneously, they constitute a certain element of building the probability theory proposed in [8].

We also draw the reader's attention to the fact that a completely different conception of the probability of intuitionistic fuzzy events was presented in papers [4] and [5].

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