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Measures on crisp sets induced by measures on intuitionistic fuzzy sets

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Abstract We point out links between intuitionistic fuzzy algebras with respect to an intuitionistic fuzzy norm and classical algebras. Then we introduce classical measures induced by measures on intuitionistic fuzzy sets with respect to an intuitionistic fuzzy set and with respect to a measurable function.

1 Introduction

The first contributions on the intuitionistic fuzzy measure theory were given in [3]. Then this topic was developed in the papers [4]-[11]. In the present paper we prove that any intuitionistic fuzzy σ-algebra with respect to an intuitionistic fuzzy triangular norm induces a classical σ-algebra (Section 3) and any intuitionistic fuzzy set function induces a set function such that some properties (monotonicity, continuity from below and from above, countable and finite additivity, decomposability) are preserved (Section 4).

2 Basic Concepts

We consider the usual definitions and notations (see [1] or [2]) of the intuitionistic fuzzy set, inclusion, union, intersection, complementary of the intuitionistic fuzzy sets and we denote

$$\begin{aligned} \widetilde{0}_X &= \{\langle x,0,1\rangle ; x \in X\}, \\ \widetilde{1}_X &= \{\langle x,1,0\rangle ; x \in X\}. \end{aligned}$$

A t-norm (t-conorm) T (S) is any increasing in each argument, commutative, associative binary operation on $[0,1]$ satisfying $T(x,1) = x$ ($S(x,0) = x$), for every $x \in [0,1]$.

Let

$$\mathcal{L} = \{(x_1, x_2); x_1, x_2 \in [0,1], x_1 + x_2 \leq 1\}$$

and

$$(x_1, x_2) \leq_{\mathcal{L}} (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \geq y_2.$$

Definition 2.1 *([12], [13]) (i) An intuitionistic fuzzy t-norm is any increasing, commutative, associative mapping $\mathcal{T} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ satisfying $\mathcal{T}((1,0), x) = x$, for all $x \in \mathcal{L}$.*

(ii) An intuitionistic fuzzy t-conorm is any increasing, commutative, associative mapping $\mathcal{S} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ satisfying $\mathcal{S}((0,1), x) = x$, for all $x \in \mathcal{L}$.

The associativity allow us to extend each intuitionistic fuzzy t-norm in an unique way to an n-ary operation, for arbitrary $n \in \mathbb{N}$, by induction

$$\mathop{\mathcal{T}}_{i=1}^{n} a_i = \mathcal{T}\left(a_n, \mathop{\mathcal{T}}_{i=1}^{n-1} a_i\right)$$

then to a countable operation putting

$$\mathop{\mathcal{T}}_{n \in \mathbb{N}} a_n = \lim_{n\to\infty}{}_{\mathcal{L}} \mathop{\mathcal{T}}_{i=1}^{n} a_i,$$

where $(b_n)_{n\in\mathbb{N}} \subset \mathcal{L}, b_n = (x_n, y_n)$ is convergent if and only if the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent and

$$\lim_{n\to\infty}{}_{\mathcal{L}} b_n = \left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right).$$

It is obvious that any intuitionistic fuzzy t-conorm can be also extended to an n-ary then to a countable operation on $\mathcal{L}$.

The intuitionistic fuzzy t-norms and t-conorms induce operations between intuitionistic fuzzy sets. If

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle; x \in X\}, \\ B &= \{\langle x, \mu_B(x), \nu_B(x)\rangle; x \in X\} \end{aligned}$$

then

$$A\mathcal{T}B = \{\langle x, \mu_{A\mathcal{T}B}(x), \nu_{A\mathcal{T}B}(x)\rangle; x \in X\},$$

where

$$(\mu_{A\mathcal{T}B}(x), \nu_{A\mathcal{T}B}(x)) = \mathcal{T}((\mu_A(x), \nu_A(x)), (\mu_B(x), \nu_B(x)))$$

and

$$A\mathcal{S}B = \{\langle x, \mu_{A\mathcal{S}B}(x), \nu_{A\mathcal{S}B}(x)\rangle; x \in X\},$$

where

$$\left(\mu_{ASB}\left(x\right),\nu_{ASB}\left(x\right)\right)=\mathcal{S}\left(\left(\mu_{A}\left(x\right),\nu_{A}\left(x\right)\right),\left(\mu_{B}\left(x\right),\nu_{B}\left(x\right)\right)\right).$$

If $\mathcal{T}=\mathcal{T}_M$, that is

$$\mathcal{T}\left(\left(x_1,x_2\right),\left(y_1,y_2\right)\right)=\left(\min\left(x_1,y_1\right),\max\left(x_2,y_2\right)\right)$$

and $\mathcal{S}=\mathcal{S}_M$, that is

$$\mathcal{S}\left(\left(x_1,x_2\right),\left(y_1,y_2\right)\right)=\left(\max\left(x_1,y_1\right),\min\left(x_2,y_2\right)\right)$$

we obtain the intersection ($\cap$) and the union ($\cup$) of intuitionistic fuzzy sets, respectively. The extensions of the above operations to the finite case and countable case are natural.

The classical measure theory concepts used in this paper can be find in [14]. We present the intuitionistic fuzzy concepts.

Definition 2.2 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. A subfamily $\mathcal{I}\subseteq IFS\left(X\right)$ which satisfies*

(i) $\widetilde{0}_X\in\mathcal{I}$

(ii) $A\in\mathcal{I}$ implies $A^c\in\mathcal{I}$

(iii) $\left(A_n\right)_{n\in\mathbb{N}}\subset\mathcal{I}$ implies $\underset{n\in\mathbb{N}}{\mathcal{T}}A_n\in\mathcal{I}$,

is called intuitionistic fuzzy $\mathcal{T}-\sigma$-algebra on X.

Remark 2.3 *If $\mathcal{I}$ is an intuitionistic fuzzy $\mathcal{T}-\sigma$-algebra on X and $\mathcal{S}$ is the dual of $\mathcal{T}$, that is*

$$\mathcal{S}\left(x,y\right)=\mathcal{N}\left(\mathcal{T}\left(\mathcal{N}\left(x\right),\mathcal{N}\left(y\right)\right)\right)$$

for every $x,y\in\mathcal{L}$, where $\mathcal{N}\left(x_1,x_2\right)=\left(x_2,x_1\right)$ for every $\left(x_1,x_2\right)\in\mathcal{L}$, then $\left(A_n\right)_{n\in\mathbb{N}}\subset\mathcal{I}$ implies $\underset{n\in\mathbb{N}}{\mathcal{S}}A_n\in\mathcal{I}$ because the standard complementation A^c of an intuitionistic fuzzy set A is also based on the negation $\mathcal{N}$.

To introduce the concept of additivity of the intuitionistic fuzzy set functions the concept of disjointness of the intuitionistic fuzzy sets is need to be given.

Definition 2.4 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and $\mathcal{S}$ its dual. A finite family $\{A_1,...,A_n\}\subseteq IFS\left(X\right)$ is said to be $\mathcal{T}-$disjoint if for each $k\in\{1,...,n\}$ we have*

$$\left(\underset{i\neq k}{\mathcal{S}}A_i\right)\mathcal{T}A_k=\widetilde{0}_X.$$

An infinite family $\left(A_n\right)_{n\in\mathbb{N}}\subseteq IFS\left(X\right)$ is called $\mathcal{T}-$disjoint if for each $n\in\mathbb{N}$ the finite family $\{A_1,...,A_n\}$ is $\mathcal{T}-$disjoint.

Definition 2.5 *(see [3]-[6]) Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and $\mathcal{S}$ its dual, $\mathcal{I} \subseteq IFS(X)$ be a $\mathcal{T} - \sigma$-algebra on X and $\widetilde{m} : \mathcal{I} \to [-\infty, +\infty]$ be a mapping which assumes at most one of the values $-\infty$ and $+\infty$. The mapping $\widetilde{m}$ is called:*

(i) *$\mathcal{T}$-valuation if it satisfies the following conditions*

$$\begin{aligned} \widetilde{m}\left(\widetilde{0}_X\right) &= 0, \\ \widetilde{m}(A\mathcal{T}B) + \widetilde{m}(A\mathcal{S}B) &= \widetilde{m}(A) + \widetilde{m}(B), \end{aligned}$$

for every $A, B \in \mathcal{I}$.

(ii) *finitely $\mathcal{T}$-additive if it satisfies*

$$\begin{aligned} \widetilde{m}\left(\widetilde{0}_X\right) &= 0, \\ \widetilde{m}(A\mathcal{S}B) &= \widetilde{m}(A) + \widetilde{m}(B), \end{aligned}$$

for every $A, B \in \mathcal{I}, A\mathcal{T}B = \widetilde{0}_X$.

(iii) *countable $\mathcal{T}$-additive if it satisfies*

$$\widetilde{m}\left(\widetilde{0}_X\right) = 0$$

and for each sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ we have

$$\widetilde{m}\left(\underset{n\in\mathbb{N}}{\mathcal{S}} A_n\right) = \sum_{n\in\mathbb{N}} \widetilde{m}(A_n),$$

whenever $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{T}$-disjoint.

(iv) *monotone if it verifies*

$$\widetilde{m}(A) \leq \widetilde{m}(B),$$

whenever $A, B \in \mathcal{I}, A \subseteq B$.

(v) *fuzzy measure on $\mathcal{I}$ if it is monotone and satisfies*

$$\widetilde{m}\left(\widetilde{0}_X\right) = 0.$$

(vi) *continuous from below if*

$$\lim_{n\to\infty} \widetilde{m}(A_n) = \widetilde{m}(A),$$

whenever $(A_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ and $A_n \nearrow A$.

(vii) *continuous from above if*

$$\lim_{n\to\infty} \widetilde{m}(A_n) = \widetilde{m}(A),$$

whenever $(A_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ *and* $A_n \searrow A$.

$(viii)$ $\mathcal{T}$*-measure if it is a* $\mathcal{T}$*-valuation and continuous from below.*

(ix) $*$*-decomposable if*

$$0 \leq \widetilde{m}(A) \leq 1,$$

for every $A \in \mathcal{I}$ *and there exists the composition law* $* : [0,1] \times [0,1] \to [0,1]$ *such that*

$$\widetilde{m}(A \cup B) = \widetilde{m}(A) * \widetilde{m}(B),$$

whenever $A, B \in \mathcal{I}$ *and* $A \cap B = \widetilde{0}_X$.

The following lemma will be useful in the proof of the main results in the paper. We recall that any crisp set $M \subseteq X$ can be considered as an intuitionistic fuzzy set

$$M = \{\langle x, \chi_M(x), 1 - \chi_M(x)\rangle ; x \in X\},$$

where χ_M is the characteristic function of the set M in X.

Lemma 2.6 (i) *If* $(M_n)_{n\in\mathbb{N}}$ *are crisp sets,* $M_n \subseteq X$ *and* $A \in IFS(X)$ *then*

$$A\mathcal{T}\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \bigcup_{n\in\mathbb{N}} (A\mathcal{T}M_n)$$

and

$$A\mathcal{T}\left(\bigcap_{n\in\mathbb{N}} M_n\right) = \bigcap_{n\in\mathbb{N}} (A\mathcal{T}M_n),$$

for every intuitionistic fuzzy t-norm $\mathcal{T}$.

(ii) *Let* $(A_n)_{n\in\mathbb{N}}$ *be a family of intuitionistic fuzzy sets in* X. *If* $A_i \cap A_j = \widetilde{0}_X$ *for every* $i, j \in \mathbb{N}, i \neq j$ *then* $(A_n)_{n\in\mathbb{N}}$ *is* $\mathcal{T}_M$*-disjoint.*

Proof. (i) It is immediate because

$$\mathcal{T}((0,1), x) = (0,1)$$

and

$$\mathcal{T}((1,0), x) = x,$$

for every $x \in \mathcal{L}$.

(ii) The property of distributivity of the union with respect to intersection implies

$$\left(\bigcup_{i\neq k} A_i\right) \cap A_k = \bigcup_{i\neq k} (A_i \cap A_k) = \widetilde{0}_X.$$

Remark 2.7 *The above result of distributivity is not very interesting because* $A\mathcal{T}M = A \cap M$ *for every crisp set* M*, intuitionistic fuzzy set* A *and intuitionistic fuzzy t-norm* $\mathcal{T}$.

3 Classical algebras induced by intuitionistic fuzzy algebras

Taking into account the representation of a crisp set as an intuitionistic fuzzy set the following result can be proved.

Theorem 3.1 *If $\mathcal{I}$ is an intuitionistic fuzzy $\mathcal{T}-\sigma$-algebra on X then*

$$\mathcal{I}_c = \{M \in \mathcal{I} : M \text{ is a crisp set}\}$$

is a σ-algebra.

Proof. $\widetilde{0}_X \in \mathcal{I}$ therefore $\emptyset \in \mathcal{I}_c$.
If $M \in \mathcal{I}_c$ then

$$M^c = \{\langle x, 1-\chi_M(x), \chi_M(x)\rangle ; x \in X\} \in \mathcal{I}$$

but it is a crisp set, namely the complementation of M, therefore $M^c \in \mathcal{I}_c$.
Let $M_n \in \mathcal{I}_c, n \in \mathbb{N}$. Because $M_n \in \mathcal{I}, n \in \mathbb{N}$, we get

$$\underset{n\in\mathbb{N}}{\mathcal{T}} M_n = \left\{\left\langle x, \mu_{\underset{n\in\mathbb{N}}{\mathcal{T}} M_n}(x), \nu_{\underset{n\in\mathbb{N}}{\mathcal{T}} M_n}(x)\right\rangle ; x \in X\right\} \in \mathcal{I},$$

where

$$\begin{aligned}\mu_{\underset{n\in\mathbb{N}}{\mathcal{T}} M_n}(x) &= \begin{cases} 1, & \text{if there exists } n \in \mathbb{N} \text{ such that } x \in M_n \\ 0, & \text{otherwise}\end{cases} \\ &= \chi_{\underset{n\in\mathbb{N}}{\cup} M_n}(x)\end{aligned}$$

and

$$\nu_{\underset{n\in\mathbb{N}}{\mathcal{T}} M_n}(x) = 1 - \mu_{\underset{n\in\mathbb{N}}{\mathcal{T}} M_n}(x) = 1 - \chi_{\underset{n\in\mathbb{N}}{\cup} M_n}(x),$$

for every $x \in X$, therefore $\underset{n\in\mathbb{N}}{\cup} M_n \in \mathcal{I}_c$. ■
The σ-algebra $\mathcal{I}_c$ can be considered the classical σ-algebra induced by $\mathcal{I}$.

Example 3.2 *An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in X\}$ is called measurable with respect to σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ if the functions μ_A and ν_A are $\mathcal{A}$-measurable. In the paper [6] it is proved that*

$$\mathcal{I}_{\mathcal{A}} = \{A \in IFS(X) ; A \text{ is } \mathcal{A}\text{-measurable}\}$$

is an intuitionistic fuzzy $\mathcal{T}_M - \sigma$-algebra. It is clear that

$$(\mathcal{I}_{\mathcal{A}})_c = \mathcal{A}$$

because $M \in \mathcal{A}$ if and only if its characteristic function χ_M is $\mathcal{A}$-measurable.

Example 3.3 *The family*

$$\mathcal{H} = \{A \in IFS(X); \Omega_A \text{ or } \Lambda_A \text{ is finite or countable}\},$$

where

$$\Omega_A = \{x \in X; \mu_A(x) > 0\}$$

and

$$\Lambda_A = \{x \in X; \nu_A(x) > 0\},$$

is an intuitionistic fuzzy $\mathcal{T}_M - \sigma$-algebra (see [6]). We have

$$\mathcal{H}_c = \{M; M \subseteq X, M \text{ or } M^c \text{ is finite or countable}\}.$$

Indeed,

$$\{\langle x, \chi_M(x), 1 - \chi_M(x)\rangle; x \in X\} \in \mathcal{H}$$

if and only if

$$\{x \in X; \chi_M(x) > 0\}$$

or

$$\{x \in X; \chi_M(x) < 1\}$$

is finite or countable, if and only if

$$\{x \in X; \chi_M(x) = 1\}$$

or

$$\{x \in X; \chi_{M^c}(x) = 1\}$$

is finite or countable.

Example 3.4 *A (k_1, k_2)-intuitionistic fuzzy set on X ([15]) is an intuitionistic fuzzy set on $X, A = \{\langle x, \mu_A(x), \nu_A(x)\rangle; x \in X\}$, satisfying the property*

$$k_1 \leq \min\{\mu_A(x), \nu_A(x)\} \leq k_2, \forall x \in X,$$

where $k_1, k_2 \in [0, 0.5]$ are two constants. We denote $IFS_{k_1,k_2}(X)$ the collection of all (k_1, k_2)-intuitionistic fuzzy sets on X. In [6] it is proved that $IFS_{k_1,k_2}(X) \cup \left\{\widetilde{0}_X, \widetilde{1}_X\right\}$ is an intuitionistic fuzzy $\mathcal{T}_M - \sigma$-algebra. We immediately obtain

$$\left(IFS_{k_1,k_2}(X) \cup \left\{\widetilde{0}_X, \widetilde{1}_X\right\}\right)_c = \{\emptyset, X\}$$

if $k_1 > 0$ and

$$\left(IFS_{k_1,k_2}(X) \cup \left\{\widetilde{0}_X, \widetilde{1}_X\right\}\right)_c = \mathcal{P}(X)$$

if $k_1 = 0$ because

$$\min\{\chi_M(x), 1 - \chi_M(x)\} = 0, \forall x \in X,$$

for every crisp set M.

4 Measures on crisp sets induced by measures on intuitionistic fuzzy sets

It is obvious that the restriction of an intuitionistic fuzzy set function to corresponding crisp set function preserves properties as monotonicity, continuity from below and from above, finite and countable additivity. The below result is more general.

Theorem 4.1 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm, $\widetilde{m} : \mathcal{I} \to [-\infty, +\infty]$ be a mapping on intuitionistic fuzzy $\mathcal{T}_M - \sigma$-algebra $\mathcal{I}$ which assumes at most one of the values $-\infty$ and $+\infty, A \in \mathcal{I}$ fixed and $m_c^A : \mathcal{I}_c \to [-\infty, +\infty]$ defined by*

$$m_c^A(M) = \widetilde{m}(A\mathcal{T}M).$$

(i) If $\widetilde{m}\left(\widetilde{0}_X\right) = 0$ then $m_c^A(\emptyset) = 0$.

(ii) If $\widetilde{m}$ is monotone then m_c^A is monotone.

(iii) If $\widetilde{m}$ is continuous from below (above) then m_c^A is continuous from below (above).

(iv) If $\widetilde{m}$ is countable (finitely) $\mathcal{T}_M$-additive then m_c^A is countable (finitely) additive.

(v) If $\widetilde{m}$ is a $\mathcal{T}_M$-valuation then m_c^A is a valuation.

(vi) If $\widetilde{m}$ is $*$-decomposable then m_c^A is $*$-decomposable.

Proof. (i) Because $A\mathcal{T}\emptyset = \widetilde{0}_X$ we have

$$m_c^A(\emptyset) = \widetilde{m}\left(\widetilde{0}_X\right) = 0.$$

(ii) For any $M, N \in \mathcal{I}_c$, if $M \subseteq N$, then $A\mathcal{T}M \subseteq A\mathcal{T}N$ and

$$m_c^A(M) = \widetilde{m}(A\mathcal{T}M) \leq \widetilde{m}(A\mathcal{T}N) = m_c^A(N).$$

(iii) For any $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}_c$, if $M_n \subseteq M_{n+1}, n \in \mathbb{N}$, then $A\mathcal{T}M_n \subseteq A\mathcal{T}M_{n+1}$, the continuity from below of $\widetilde{m}$ and Lemma 2.6, (i) imply

$$\begin{aligned} m_c^A\left(\underset{n\in\mathbb{N}}{\cup} M_n\right) &= \widetilde{m}\left(A\mathcal{T}\left(\underset{n\in\mathbb{N}}{\cup} M_n\right)\right) = \\ \widetilde{m}\left(\underset{n\in\mathbb{N}}{\cup}(A\mathcal{T}M_n)\right) &= \lim_{n\to\infty} \widetilde{m}(A\mathcal{T}M_n) = \lim_{n\to\infty} m_c^A(M_n). \end{aligned}$$

(iv) We only prove the property for the countable case. Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}_c, M_i \cap M_j = \emptyset$ if $i \neq j$. Then

$$(A\mathcal{T}M_i) \cap (A\mathcal{T}M_j) = \widetilde{0}_X$$

if $i \neq j$, that is (see Lemma 2.6, (ii)) $(A\mathcal{T}M_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}_M-$disjoint family of intuitionistic fuzzy sets. We obtain

$$\begin{aligned} m_c^A\left(\underset{n\in\mathbb{N}}{\cup}M_n\right) &= \widetilde{m}\left(A\mathcal{T}\left(\underset{n\in\mathbb{N}}{\cup}M_n\right)\right) = \\ \widetilde{m}\left(\underset{n\in\mathbb{N}}{\cup}(A\mathcal{T}M_n)\right) &= \sum_{n\in\mathbb{N}}\widetilde{m}\,(A\mathcal{T}M_n) = \sum_{n\in\mathbb{N}} m_c^A\,(M_n)\,. \end{aligned}$$

(v) Let $M, N \in \mathcal{I}_c$. We have

$$\begin{aligned} m_c^A\,(M\cap N) + m_c^A\,(M\cup N) &= \widetilde{m}\,(A\mathcal{T}\,(M\cap N)) + \widetilde{m}\,(A\mathcal{T}\,(M\cup N)) \\ &= \widetilde{m}\,((A\mathcal{T}M)\cap(A\mathcal{T}N)) + \widetilde{m}\,((A\mathcal{T}M)\cup(A\mathcal{T}N)) \\ &= \widetilde{m}\,(A\mathcal{T}M) + \widetilde{m}\,(A\mathcal{T}N) \\ &= m_c^A\,(M) + m_c^A\,(N) \end{aligned}$$

and together with (i) the property is proved.

(vi) Let $M, N \in \mathcal{I}_c, M\cap N = \emptyset$. Then $(A\mathcal{T}M)\cap(A\mathcal{T}N) = \widetilde{0}_X$ and

$$\begin{aligned} m_c^A\,(M\cup N) &= \widetilde{m}\,(A\mathcal{T}(M\cup N)) \\ &= \widetilde{m}\,((A\mathcal{T}M)\cup(A\mathcal{T}N)) \\ &= \widetilde{m}\,(A\mathcal{T}M) * \widetilde{m}\,(A\mathcal{T}N) \\ &= m_c^A\,(M) * m_c^A\,(N)\,. \end{aligned}$$

■

Corollary 4.2 (i) *If $\widetilde{m}$ is a fuzzy measure on intuitionistic fuzzy sets then m_c^A is a fuzzy measure on crisp sets.*

(ii) *If $\widetilde{m}$ is a $\mathcal{T}_M$-measure then m_c^A is a measure.*

(iii) *If $\widetilde{m}$ is countable $\mathcal{T}_M$-additive then m_c^A is a measure.*

Example 4.3 *If X is a finite set then $\widetilde{m} : IFS\,(X) \to [0, +\infty]$ defined by*

$$\widetilde{m}\,(A) = \frac{1}{2}\sum_{x\in X}(\mu_A\,(x) + 1 - \nu_A\,(x))\,,$$

for any $A = \{\langle x, \mu_A\,(x)\,, \nu_A\,(x)\rangle\,; x \in X\}$, is countable $\mathcal{T}_M$-additive (see [6]). Let $M \subseteq X$. Then

$$\begin{aligned} \widetilde{m}\,(A\mathcal{T}M) &= \frac{1}{2}\sum_{x\in X}(\mu_{A\mathcal{T}M}\,(x) + 1 - \nu_{A\mathcal{T}M}\,(x)) \\ &= \frac{1}{2}\sum_{x\in M}(\mu_A\,(x) + 1 - \nu_A\,(x))\,. \end{aligned}$$

Because $(IFS(X))_c = \mathcal{P}(X)$ *we obtain*

$$\begin{aligned} m_c^A &: \mathcal{P}(X) \to [0,+\infty] \\ m_c^A(M) &= \frac{1}{2}\sum_{x\in M}(\mu_A(x) + 1 - \nu_A(x)) \end{aligned}$$

is a classical measure, for any fixed $A \in IFS(X)$. *If* $A = \widetilde{1}_X$ *we get*

$$m_c^A(M) = \sum_{x\in M} 1 = cardM$$

that is the cardinal of finite sets.

Example 4.4 *Let* $\mathcal{A} \subseteq \mathcal{P}(X)$ *be a* σ*-algebra and* $m : \mathcal{A} \to [0,+\infty]$ *be a measure. Let us denote*

$$\begin{aligned} \mathcal{I}_{\mathcal{A}} &= \{A : A = \{\langle x, \mu_A(x), \nu_A(x)\rangle ; x \in X\} \in IFS(X), \\ &\quad \mu_A \text{ and } \nu_A \text{ are } \mathcal{A}\text{-measurable}\}, \\ L_{\alpha,\beta}(A) &= \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}, \end{aligned}$$

where $\alpha, \beta \in [0,1], \alpha \neq 0$ *or* $\beta \neq 1$. *The intuitionistic fuzzy set function* $\widetilde{m}_{\mathcal{A}} : \mathcal{I}_{\mathcal{A}} \to [0,+\infty]$ *defined by*

$$\widetilde{m}_{\mathcal{A}}(A) = m(L_{\alpha,\beta}(A))$$

is countable $\mathcal{T}_M$*-additive (see [6]).*

Because

$$\begin{aligned} (\mu_{A\mathcal{T}M}(x), \nu_{A\mathcal{T}M}(x)) &= \mathcal{T}((\mu_A(x), \nu_A(x)), (\mu_M(x), \nu_M(x))) \\ &= \mathcal{T}((\mu_A(x), \nu_A(x)), (\chi_M(x), 1 - \chi_M(x))) \\ &= \begin{cases} (\mu_A(x), \nu_A(x)), & \text{if } x \in M \\ (0,1), & \text{if } x \notin M \end{cases} \end{aligned}$$

we have

$$L_{\alpha,\beta}(A\mathcal{T}M) = L_{\alpha,\beta}(A) \cap M,$$

for every $\alpha, \beta \in [0,1], \alpha \neq 0$ *or* $\beta \neq 1$. *Therefore* $(\widetilde{m}_{\mathcal{A}})_c^A : \mathcal{A} \to [0,+\infty]$ *defined by*

$$(\widetilde{m}_{\mathcal{A}})_c^A(M) = m(L_{\alpha,\beta}(A) \cap M)$$

is the measure induced by $\widetilde{m}_{\mathcal{A}}$ *and* A.

Let us denote $\mathcal{B} \subset \mathcal{P}([-\infty,+\infty])$ the family of Borel sets. Below we introduce a classical measure on $\mathcal{B}$ starting from a measure on intuitionistic fuzzy sets and a measurable function.

Let $\mathcal{I}$ be an intuitionistic fuzzy $\mathcal{T} - \sigma$-algebra and $f : X \to [-\infty,+\infty]$ be an $\mathcal{I}$-measurable function, that is $f^{-1}(M) \in \mathcal{I}$ for every Borel set $M \in \mathcal{B}$.

Theorem 4.5 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm, $\widetilde{m} : \mathcal{I} \to [-\infty, +\infty]$ be an intuitionistic fuzzy set function on intuitionistic fuzzy $\mathcal{T}_M - \sigma$-algebra $\mathcal{I}$ which assumes at most one of the values $-\infty$ and $+\infty$, $A \in \mathcal{I}$ fixed and $m_f : \mathcal{B} \to [-\infty, +\infty]$ defined by*

$$m_c^f(M) = \widetilde{m}\left(A\mathcal{T}f^{-1}(M)\right).$$

(i) *If $\widetilde{m}\left(\widetilde{0}_X\right) = 0$ then $m_c^f(\emptyset) = 0$.*

(ii) *If $\widetilde{m}$ is monotone then m_c^f is monotone.*

(iii) *If $\widetilde{m}$ is continuous from below (above) then m_c^f is continuous from below (above).*

(iv) *If $\widetilde{m}$ is countable (finitely) $\mathcal{T}_M$-additive then m_c^f is countable (finitely) additive.*

(v) *If $\widetilde{m}$ is a $\mathcal{T}_M$-valuation then m_c^f is a valuation.*

(vi) *If $\widetilde{m}$ is $*$-decomposable then m_c^f is $*$-decomposable.*

Proof. It is similar to the proof of Theorem 4.1 taking into account the properties of the preimage

$$\begin{aligned} f^{-1}(\emptyset) &= \emptyset \\ M &\subseteq N \text{ implies } f^{-1}(M) \subseteq f^{-1}(N) \\ f^{-1}\left(\underset{n\in\mathbb{N}}{\cup} M_n\right) &= \underset{n\in\mathbb{N}}{\cup} f^{-1}(M_n) \\ f^{-1}\left(\underset{n\in\mathbb{N}}{\cap} M_n\right) &= \underset{n\in\mathbb{N}}{\cap} f^{-1}(M_n). \end{aligned}$$

■

The following consequences are immediate.

Corollary 4.6 (i) *If $\widetilde{m}$ is a fuzzy measure on intuitionistic fuzzy sets then m_c^f is a fuzzy measure on $\mathcal{B}$.*

(ii) *If $\widetilde{m}$ is a $\mathcal{T}_M$-measure then m_c^f is a measure on $\mathcal{B}$.*

(iii) *If $\widetilde{m}$ is countable $\mathcal{T}_M$-additive then m_c^f is a measure on $\mathcal{B}$.*

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