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Gromov products and sums on intuitionistic fuzzy sets

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Abstract: The article presents a definition of Gromov sum and researches its essential characteristics. This concept is crucial for defining of Gromov sums and products about finite metrics spaces constituted from intuitionistic fuzzy sets.
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1 Introduction

The main idea of this work is that developing an account of Gromov product allows us to define Gromov sum. The study addresses the problems raised by metric spaces which are composed by intuitionistic fuzzy sets.

2 Preliminaries

Definition 1. *[1] Let a (crisp) set E be fixed and let $A \subset E$ be a fixed set. IFS A^* in E is the following:*

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\, x \in E\},$$

where the functions $\mu_A(x), \nu_A(x) : E \to [0,1]$ *define respectively, the degree of membership and degree of nonmembership of the element* $x \in E$ *, to the set A, and for every* $x \in E$ *,*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Definition 2. *Let* X *be nonempty. Let* $x, y, z \in X$*. The distance measure* d *between* x *and* y *is a mapping* $d : X \times X \to R^+$ *which satisfies the following axioms:*

A1) $0 \leq d(x, y)$,

A2) $d(x, y) = 0$ *if and only if* $x = y$,

A3) $d(x, y) = d(y, x)$,

A4) $d(x, z) + d(z, y) \geq d(x, y)$.

Definition 3. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | \, x \in X\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x), \rangle | \, x \in X\}$ *are two intuitionistic fuzzy sets with universum* $X = \{x_1, x_2, \ldots, x_n\}$*. Szmidt [3] proposes the following four distance measures between* A *and* B*:*

The Hamming distance;

$$d^1_{IFS}(A, B) = \tfrac{1}{2} \textstyle\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$

The Euclidean distance;

$$e^1_{IFS}(A, B) = \sqrt{\tfrac{1}{2} \textstyle\sum_{i=1}^{n} [(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2]}$$

The Normalized Hamming distance;

$$l^1_{IFS}(A, B) = \tfrac{1}{2n} \textstyle\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$

The Normalized Euclidean distance;

$$q^1_{IFS}(A, B) = \sqrt{\tfrac{1}{2n} \textstyle\sum_{i=1}^{n} [(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2]}$$

3 Gromov products and sums

Given a metric space (X, d) with a fixed point $x_0 \in X$, the Gromov product similarity [2] (or Gromov product, covariance, overlap function) $(.)_{x_0}$ is a similarity on X defined by

$$(x.y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)).$$

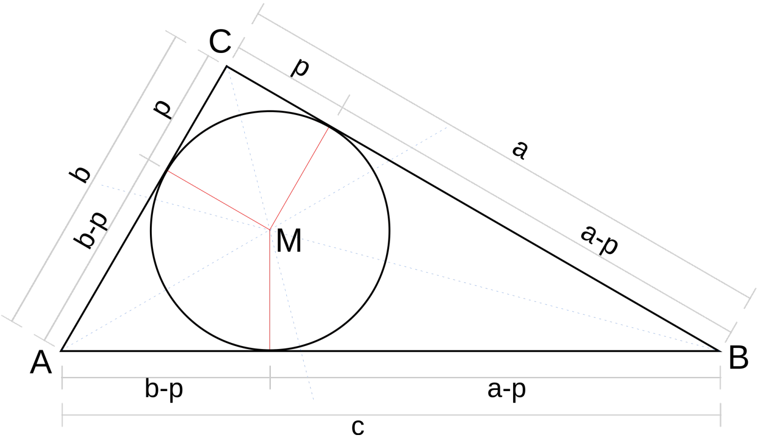

Figure 1: Illustration of the Gromov products

In the Euclidean plane, the Gromov product $(A.B)_C$ equals the distance p between C and the point where the incircle of the geodesic triangle ABC touches the edge CB or CA. Indeed from the diagram $c = (a-p) + (b-p)$, so that $p = \frac{(a+b-c)}{2} = (A.B)_C$.

The triangle inequality for d implies

$$(x.y)_{x_0} \geq (x.z)_{x_0} + (y.z)_{x_0} - (z.z)_{x_0}.$$

Definition 4. *Gromov sum we nominate*

$$(x+y)_{x_0} = \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,y)).$$

Following the diagram we have $(A+B)_C = \frac{(a+b+c)}{2}$. When $\triangle ABC$ is right triangle we have $S_{\triangle ABC} = (A.B)_C.(A+B)_C$

Theorem 1. *Gromov sum has the following listed properties*

$$\begin{array}{rl}
(1) & (x+y)_{x_0} + (x.y)_{x_0} = d(x,x_0) + d(y,x_0), \\
(2) & (x+y)_{x_0} - (x.y)_{x_0} = d(x,y), \\
(3) & (x+y)_{x_0} \leq (x+z)_{x_0} + (y+z)_{x_0} - (z+z)_{x_0}, \\
(4) & (x+y)_{x_0} = (y+x)_{x_0}, \\
(5) & |(x+y)_p - (x+z)_p| \leq d(y,z), \\
(6) & |(x+y)_p - (x+y)_q| \leq d(p,q), \\
(7) & (y+z)_x \geq d(y,z), \\
(8) & (x+z)_y = (y+z)_x = (x+y)_z, \\
(9) & (y+z)_y = (y+z)_z = d(z,y), \\
(10) & (x+y)_z = (x.y)_z + (x.z)_y + (y.z)_x.
\end{array}$$

Proof. (1) $(x+y)_{x_0} + (x.y)_{x_0}$

$$= \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,y)) + \frac{1}{2}(d(x,x_0) + d(y,x_0) - d(x,y))$$

$$= d(x,x_0) + d(y,x_0);$$

(2) $(x+y)_{x_0} - (x.y)_{x_0}$

$$= \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,y)) - \frac{1}{2}(d(x,x_0) + d(y,x_0) - d(x,y)) = d(x,y)$$

(3) $(x+z)_{x_0} + (y+z)_{x_0} - (z+z)_{x_0}$

$$= \frac{1}{2}(d(x,x_0)+d(z,x_0)+d(x,z))+\frac{1}{2}(d(y,x_0)+d(z,x_0)+d(y,z))-\frac{1}{2}(d(z,x_0)+d(z,x_0)+d(z,z))$$

$$= \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,z) + d(y,z)) \geq \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,y)) = (x+y)_{x_0}$$

Here we use the triangle inequality for d.

(4) $(x+y)_{x_0} = \frac{1}{2}(d(x,x_0) + d(y,x_0) + d(x,y)) = (y+x)_{x_0}$

(5)

$$(x+y)_p = \frac{1}{2}(d(x,p) + d(y,p) + d(x,y))$$
$$(x+z)_p = \frac{1}{2}(d(x,p) + d(z,p) + d(x,z))$$

Then from the triangle inequality for d we have

$$(x+y)_p - (x+z)_p = \frac{1}{2}(d(y,p) - d(z,p) + d(x,y) - d(x,z)) \leq \frac{1}{2}(d(y,z) + d(y,z)) = d(y,z)$$

In this way we proved that

$$(x+y)_p - (x+z)_p \leq d(y,z)$$

Similarly, we are able to prove that

$$(x+z)_p - (x+y)_p \leq d(y,z)$$

From previous we obtain that

$$|(x+y)_p - (x+z)_p| \leq d(y,z)$$

(6) Analogously to (5)

(7)

$$(y+z)_x = \frac{1}{2}(d(y,x) + d(z,x) + d(y,z)) \geq \frac{1}{2}(d(y,z) + d(y,z)) = d(y,z)$$

(8)

$$(y+z)_x = \frac{1}{2}(d(y,x) + d(z,x) + d(y,z)) = (x+y)_z$$
$$(y+z)_x = \frac{1}{2}(d(y,x) + d(z,x) + d(y,z)) = (x+z)_y$$

(9)

$$(y+z)_y = \frac{1}{2}(d(y,y) + d(z,y) + d(y,z)) = d(z,y)$$
$$(y+z)_z = \frac{1}{2}(d(y,z) + d(z,z) + d(y,z)) = d(z,y)$$

(10)

$$(x.y)_z + (x.z)_y + (y.z)_x$$
$$= \frac{1}{2}(d(x,z) + d(y,z) - d(x,y) + d(x,y) + d(z,y) - d(x,z) + d(y,x) + d(z,x) - d(y,z))$$
$$= \frac{1}{2}(d(x,z) + d(z,y) + d(y,x)) = (x+y)_z$$

□

4 Gromov products and sums on intuitionistic fuzzy sets

Now we consider Gromov products and sums on intuitionistic fuzzy sets.

Let A, B and C are intuitionistic fuzzy sets [1] in $X = \{x_1, x_2, \ldots, x_n\}$. Using [3], we obtain the following Gromov products and sums between intuitionistic fuzzy sets:

- the Hamming–Gromov product and sum

$$(A.B)_C{}^{d^1_{IFS}} = \frac{1}{2}(d^1_{IFS}(A,C) + d^1_{IFS}(B,C) - d^1_{IFS}(A,B))$$

$$(A+B)_C{}^{d^1_{IFS}} = \frac{1}{2}(d^1_{IFS}(A,C) + d^1_{IFS}(B,C) + d^1_{IFS}(A,B))$$

- the Euclidean–Gromov product and sum

$$(A.B)_C{}^{e^1_{IFS}} = \frac{1}{2}(e^1_{IFS}(A,C) + e^1_{IFS}(B,C) - e^1_{IFS}(A,B))$$

$$(A+B)_C{}^{e^1_{IFS}} = \frac{1}{2}(e^1_{IFS}(A,C) + e^1_{IFS}(B,C) + e^1_{IFS}(A,B))$$

- the normalized Hamming–Gromov product and sum

$$(A.B)_C{}^{l^1_{IFS}} = \frac{1}{2}(l^1_{IFS}(A,C) + l^1_{IFS}(B,C) - l^1_{IFS}(A,B))$$

$$(A+B)_C{}^{l^1_{IFS}} = \frac{1}{2}(l^1_{IFS}(A,C) + l^1_{IFS}(B,C) + l^1_{IFS}(A,B))$$

- the normalized Euclidean–Gromov product and sum

$$(A.B)_C{}^{q^1_{IFS}} = \frac{1}{2}(q^1_{IFS}(A,C) + q^1_{IFS}(B,C) - q^1_{IFS}(A,B))$$

$$(A+B)_C{}^{q^1_{IFS}} = \frac{1}{2}(q^1_{IFS}(A,C) + q^1_{IFS}(B,C) + q^1_{IFS}(A,B))$$

5 Conclusion

In the context of connection between Gromov sums and Gromov products, the paper suggests an opportunity for a future profound study in their meaning and their intuitionistic fuzzy analogies.

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