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Examples of intuitionistic fuzzy algebras and intuitionistic fuzzy measures

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Abstract: We present some examples of remarkable intuitionistic fuzzy σ-algebras and σ-additive intuitionistic fuzzy measures.
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1 Preliminaries

We consider the usual definitions and notations (see [1] or [2]) of intuitionistic fuzzy set, union, intersection, complementary of intuitionistic fuzzy sets (IFSs, for short) and we denote $\widetilde{X} = \{\langle x, 1, 0\rangle ; x \in X\}, \widetilde{\emptyset} = \{\langle x, 0, 1\rangle ; x \in X\}$. We recall the main definitions used in this paper.

Definition 1.1 *([5], see also [4]) An intuitionistic fuzzy σ-algebra on $X \neq \emptyset$ is a family $\widetilde{\mathcal{A}}$ of IFSs in X satisfying the following properties:*
(i) $\widetilde{X} \in \widetilde{\mathcal{A}}$;
(ii) $\widetilde{A} \in \widetilde{\mathcal{A}}$ *implies* $\widetilde{A}^c \in \widetilde{\mathcal{A}}$;
(iii) $\bigcup_{n\in\mathbb{N}} \widetilde{A}_n \in \widetilde{\mathcal{A}}$ *for every sequence* $\left(\widetilde{A}_n\right)_{n\in\mathbb{N}}$ *of IFSs in* X.

Definition 1.2 *(see [5]) Let $\widetilde{\mathcal{A}}$ be an intuitionistic fuzzy σ-algebra in X. A function $\widetilde{m} : \widetilde{\mathcal{A}} \to [0, \infty]$ is said to be an intuitionistic fuzzy measure if it satisfies the following conditions:*
(i) $\widetilde{m}\left(\widetilde{\emptyset}\right) = 0$;
(ii) For any $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}}, \widetilde{A} \subseteq \widetilde{B}$ *implies* $\widetilde{m}\left(\widetilde{A}\right) \leq \widetilde{m}\left(\widetilde{B}\right)$.

The intuitionistic fuzzy measure $\widetilde{m}$ is called σ-additive if $\widetilde{m}\left(\bigcup_{n\in\mathbb{N}} \widetilde{A}_n\right) = \sum_{n\in\mathbb{N}} \widetilde{m}\left(\widetilde{A}_n\right)$ *for every sequence* $\left(\widetilde{A}_n\right)_{n\in\mathbb{N}}$ *of pairwise disjoint IFSs in* $\widetilde{\mathcal{A}}$.

2 Examples of intuitionistic fuzzy σ-algebras

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle; x \in X\right\} \in IFS(X)$ is called $\mathcal{A}$-measurable if the functions $\mu_{\widetilde{A}}$ and $\nu_{\widetilde{A}}$ are $\mathcal{A}$-measurable.

Proposition 2.1 *The family $\widetilde{\mathcal{I}}_{\mathcal{A}}$ of $\mathcal{A}$-measurable IFSs is an intuitionistic fuzzy σ-algebra.*

Proof. It is obvious because the constant functions are $\mathcal{A}$-measurable and the $\mathcal{A}$-measurability of f_n, for every $n \in \mathbb{N}$, implies the $\mathcal{A}$-measurability of $\sup\limits_{n\in\mathbb{N}} f_n$ and $\inf\limits_{n\in\mathbb{N}} f_n$. ■

Let $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle; x \in X\right\} \in IFS(X)$. Let us denote $\Omega_{\widetilde{A}} = \left\{x \in X; \mu_{\widetilde{A}}(x) > 0\right\}, \Lambda_{\widetilde{A}} = \left\{x \in X; \nu_{\widetilde{A}}(x) > 0\right\}$ and

$$\mathcal{N} = \left\{\widetilde{A} \in IFS(X); \Omega_{\widetilde{A}} \text{ or } \Lambda_{\widetilde{A}} \text{ is finite or countable}\right\}.$$

Proposition 2.2 *The family $\mathcal{N}$ of IFSs is an intuitionistic fuzzy σ-algebra.*

Proof. $\widetilde{X} = \{\langle x, 1, 0\rangle; x \in X\} \in \mathcal{N}$ because $\Lambda_{\widetilde{X}} = \emptyset$. If $A \in \mathcal{N}$ then $\Omega_{\widetilde{A}}$ or $\Lambda_{\widetilde{A}}$ is finite or countable, that is $\widetilde{A}^c \in \mathcal{N}$.
Let $\widetilde{A}_n = \left\{\left\langle x, \mu_{\widetilde{A}_n}(x), \nu_{\widetilde{A}_n}(x)\right\rangle; x \in X\right\} \in \mathcal{N}, \forall n \in \mathbb{N}$. If $\Omega_{\widetilde{A}_n}$ is finite or countable for every $n \in \mathbb{N}$ then $\Omega_{\underset{n\in\mathbb{N}}{\cup}\widetilde{A}_n}$ is finite or countable because $\Omega_{\underset{n\in\mathbb{N}}{\cup}\widetilde{A}_n} \subseteq \underset{n\in\mathbb{N}}{\cup}\Omega_{\widetilde{A}_n}$. If there exists $n_0 \in \mathbb{N}$ such that $\Omega_{\widetilde{A}_{n_0}}$ is uncountable then $\Lambda_{\widetilde{A}_{n_0}}$ is finite or countable and

$$\Lambda_{\underset{n\in\mathbb{N}}{\cup}\widetilde{A}_n} = \left\{x \in X; \inf_{n\in\mathbb{N}} \nu_{\widetilde{A}_n}(x) > 0\right\} \subseteq \left\{x \in X; \nu_{\widetilde{A}_{n_0}}(x) > 0\right\} = \Lambda_{\widetilde{A}_{n_0}}$$

implies $\Lambda_{\underset{n\in\mathbb{N}}{\cup}\widetilde{A}_n}$ finite or countable, therefore $\underset{n\in\mathbb{N}}{\cup}\widetilde{A}_n \in \mathcal{N}$. ■

In [3] and [6] the image and the preimage of an intuitionistic fuzzy set by a crisp function $f : X \to Y$ are introduced.

Definition 2.3 *If $\widetilde{B} = \left\{\left\langle y, \mu_{\widetilde{B}}(y), \nu_{\widetilde{B}}(y)\right\rangle; y \in Y\right\} \in IFS(Y)$ then the preimage of $\widetilde{B}$ under f, denoted by $f^{-1}\left(\widetilde{B}\right)$, is the IFS in X defined by $f^{-1}\left(\widetilde{B}\right) = \left\{\left\langle x, f^{-1}\left(\mu_{\widetilde{B}}\right)(x), f^{-1}\left(\nu_{\widetilde{B}}\right)(x)\right\rangle; x \in X\right.$ where $f^{-1}\left(\mu_{\widetilde{B}}\right)(x) = \mu_{\widetilde{B}}(f(x))$ and $f^{-1}\left(\nu_{\widetilde{B}}\right)(x) = \nu_{\widetilde{B}}(f(x))$.*

In the paper [6], Corollary 2.10, some properties of image and preimage are proved. Among these, $f^{-1}\left(\bigcup\limits_{j\in J}\widetilde{B}_j\right) = \bigcup\limits_{j\in J} f^{-1}\left(\widetilde{B}_j\right)$ for every index set J and $\widetilde{B}_j \in IFS(Y)$ and $f^{-1}\left(\widetilde{B}^c\right) = \left(f^{-1}\left(\widetilde{B}\right)\right)^c$ for every $\widetilde{B} \in IFS(Y)$. In addition, $f^{-1}\left(\widetilde{Y}\right) = \widetilde{X}$, therefore the following example is immediate.

Proposition 2.4 *Let X, Y be two sets, $\widetilde{\mathcal{A}}$ an intuitionistic fuzzy σ-algebra on Y and let $f : X \to Y$ be a mapping. Then the system*

$$\widetilde{\mathcal{A}}_f = \left\{ f^{-1}\left(\widetilde{B}\right); \widetilde{B} \in \widetilde{\mathcal{A}} \right\}$$

is an intuitionistic fuzzy σ-algebra on X.

In [7] the so-called (k_1, k_2)-intuitionistic fuzzy sets are introduced and studied.

Definition 2.5 *Let X be a nonempty set. A (k_1, k_2)-intuitionistic fuzzy sets is an IFS on X, $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle; x \in X\right\}$, satisfying the property*

$$k_1 \leq \min\left\{\mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\} \leq k_2, \forall x \in X,$$

where $k_1, k_2 \in [0, 0.5]$ are two constants.

We denote by $IFS_{k_1,k_2}(X)$ the collection of all (k_1, k_2)-intuitionistic fuzzy sets on X.

Proposition 2.6 $IFS_{k_1,k_2}(X) \cup \left\{\widetilde{\emptyset}, \widetilde{X}\right\}$ *is an intuitionistic fuzzy σ-algebra.*

Proof. It is obvious that $\widetilde{A} \in IFS_{k_1,k_2}(X)$ implies $\widetilde{A}^c \in IFS_{k_1,k_2}(X)$. Let $\widetilde{A}_n \in IFS_{k_1,k_2}(X), \forall n \in \mathbb{N}$, that is

$$k_1 \leq \min\left\{\mu_{\widetilde{A}_n}(x), \nu_{\widetilde{A}_n}(x)\right\} \leq k_2, \forall x \in X, \forall n \in \mathbb{N}.$$

We must prove

$$k_1 \leq \min\left\{\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x), \inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x)\right\} \leq k_2, \forall x \in X.$$

To prove the left inequality, let us assume

$$k_1 > \min\left\{\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0), \inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x_0)\right\},$$

where $x_0 \in X$. Then $\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0) < k_1$ or $\inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x_0) < k_1$. If $\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0) < k_1$ then $\mu_{\widetilde{A}_n}(x_0) < k_1, \forall n \in \mathbb{N}$, a contradiction with $\widetilde{A}_n \in IFS_{k_1,k_2}(X)$. If $\inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x_0) < k_1$ then there exists $n_0 \in \mathbb{N}$ such that $\nu_{\widetilde{A}_{n_0}}(x_0) < k_1$, a contradiction with $\widetilde{A}_{n_0} \in IFS_{k_1,k_2}(X)$.
To prove the right inequality, let us assume

$$k_2 < \min\left\{\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0), \inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x_0)\right\},$$

where $x_0 \in X$. Then $\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0) > k_2$ and $\inf_{n\in\mathbb{N}}\nu_{\widetilde{A}_n}(x_0) > k_2$. We get $\nu_{\widetilde{A}_n}(x_0) > k_2, \forall n \in \mathbb{N}$ therefore $\mu_{\widetilde{A}_n}(x_0) \leq k_2, \forall n \in \mathbb{N}$ because $\widetilde{A}_n \in IFS_{k_1,k_2}(X)$. This implies $\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0) \leq k_2$, a contradiction with
$\sup_{n\in\mathbb{N}}\mu_{\widetilde{A}_n}(x_0) > k_2$. As a conclusion, $\bigcup_{n\in\mathbb{N}} \widetilde{A}_n \in IFS_{k_1,k_2}(X)$. ■

3 Examples of intuitionistic fuzzy measures

Let $X \neq \emptyset, A \subseteq X$ and $\alpha, \beta \in [0,1], 0 \leq \alpha + \beta \leq 1$. The intuitionistic fuzzy set $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle; x \in X\right\}$, where $\mu_{\widetilde{A}}(x) = \alpha$, if $x \in A, \mu_{\widetilde{A}}(x) = 0$ if $x \notin A$ and $\nu_{\widetilde{A}}(x) = \beta$, if $x \in A, \nu_{\widetilde{A}}(x) = 1$ if $x \notin A$,is denoted by $\langle x, \alpha, \beta\rangle_A$ (see [5]). For example, $\widetilde{\emptyset} = \langle x, 0, 1\rangle_X$ and $\widetilde{X} = \langle x, 1, 0\rangle_X$.

Proposition 3.1 *Let $p \in X, \alpha > 0, \beta \geq 0, \alpha + \beta \leq 1$ and $\widetilde{\mathcal{A}}$ be an intuitionistic fuzzy σ-algebra. The function $\widetilde{m}_\circ : \widetilde{\mathcal{A}} \to [0, \infty]$ defined by*

$$\widetilde{m}_\circ\left(\widetilde{A}\right) = \begin{cases} 1, & \text{if } \langle x, \alpha, \beta\rangle_{\{p\}} \subseteq \widetilde{A} \\ 0, & \text{if } \langle x, \alpha, \beta\rangle_{\{p\}} \subsetneq \widetilde{A} \end{cases}$$

is a σ-additive intuitionistic fuzzy measure.

Proof. We recall that $\langle x, \alpha, \beta\rangle_{\{p\}} \subseteq \widetilde{A}$ means $\alpha \leq \mu_{\widetilde{A}}(p)$ and $\beta \geq \nu_{\widetilde{A}}(p)$.

$\widetilde{m}_\circ\left(\widetilde{\emptyset}\right) = 0$ holds clearly.

Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}}$ and $\widetilde{A} \subseteq \widetilde{B}$. If $\langle x, \alpha, \beta\rangle_{\{p\}} \subseteq \widetilde{A}$ then $\langle x, \alpha, \beta\rangle_{\{p\}} \subseteq \widetilde{B}$ therefore $\widetilde{m}_\circ\left(\widetilde{A}\right) = \widetilde{m}_\circ\left(\widetilde{B}\right) = 1$. If $\langle x, \alpha, \beta\rangle_{\{p\}} \subsetneq \widetilde{A}$ then $\widetilde{m}_\circ\left(\widetilde{A}\right) = 0$ and the inequality $\widetilde{m}_\circ\left(\widetilde{A}\right) \leq \widetilde{m}_\circ\left(\widetilde{B}\right)$ is obvious.

Let $\left(\widetilde{A}_n\right)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint IFSs in $\widetilde{\mathcal{A}}$ and $\widetilde{A} = \bigcup_{n\in\mathbb{N}} \widetilde{A}_n$.

If $\langle x, \alpha, \beta\rangle_{\{p\}} \subseteq \widetilde{A}$ then there exists an unique $n_0 \in \mathbb{N}$ such that $\nu_{\widetilde{A}_{n_0}}(p) < 1$. Indeed, if $\nu_{\widetilde{A}_n}(p) = 1, \forall n \in \mathbb{N}$ then $\beta \geq \nu_{\widetilde{A}}(p) \geq \inf_{n\in\mathbb{N}} \nu_{\widetilde{A}_n}(p) = 1$, a contradiction, because $\beta < 1$. If there exist $n_1, n_2 \in \mathbb{N}$ such that $\nu_{\widetilde{A}_{n_1}}(p) < 1$ and $\nu_{\widetilde{A}_{n_2}}(p) < 1$ then $\nu_{\widetilde{A}_{n_1} \cap \widetilde{A}_{n_2}}(p) = \max\left\{\nu_{\widetilde{A}_{n_1}}(p), \nu_{\widetilde{A}_{n_2}}(p)\right\} < 1$, a contradiction with $\widetilde{A}_{n_1} \cap \widetilde{A}_{n_2} = \widetilde{\emptyset}$. Because $\nu_{\widetilde{A}_n}(p) = 1, \forall n \in \mathbb{N}, n \neq n_0$ we obtain $\mu_{\widetilde{A}_n}(p) = 0, \forall n \in \mathbb{N}, n \neq n_0$, therefore $\langle x, \alpha, \beta\rangle_{\{p\}} \subsetneq \widetilde{A}, \forall n \in \mathbb{N}, n \neq n_0$. We have $\mu_{\widetilde{A}}(p) = \mu_{\widetilde{A}_{n_0}}(p)$ and $\nu_{\widetilde{A}}(p) = \nu_{\widetilde{A}_{n_0}}(p)$, therefore

$$\widetilde{m}_\circ\left(\widetilde{A}\right) = \sum_{n\in\mathbb{N}} \widetilde{m}_\circ\left(\widetilde{A}_n\right) = \widetilde{m}_\circ\left(\widetilde{A}_{n_0}\right) = 1.$$

If $\langle x, \alpha, \beta\rangle_{\{p\}} \subsetneq \widetilde{A}$ then $\alpha > \sup_{n\in\mathbb{N}} \mu_{\widetilde{A}_n}(p)$ or $\beta < \inf_{n\in\mathbb{N}} \nu_{\widetilde{A}_n}(p)$ therefore $\alpha > \mu_{\widetilde{A}_n}(p)$ or $\beta < \nu_{\widetilde{A}_n}(p)$ for every $n \in \mathbb{N}$. In this case

$$\widetilde{m}_\circ\left(\widetilde{A}\right) = 0 = \sum_{n\in\mathbb{N}} \widetilde{m}_\circ\left(\widetilde{A}_n\right).$$

■

Let X be a finite set and $\widetilde{\mathcal{A}} \subseteq IFS(X)$ be an intuitionistic fuzzy σ-algebra.

Proposition 3.2 *The function* $\widetilde{m} : \widetilde{\mathcal{A}} \to [0, +\infty]$ *defined by*

$$\widetilde{m}\left(\widetilde{A}\right) = \frac{1}{2}\sum_{x \in X}\left(\mu_{\widetilde{A}}(x) + 1 - \nu_{\widetilde{A}}(x)\right),$$

for any $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle ; x \in X\right\} \in \widetilde{\mathcal{A}}$, *is a* σ*-additive intuitionistic fuzzy measure.*

Proof. $\mu_{\widetilde{\emptyset}}(x) = 0, \forall x \in X$ and $\nu_{\widetilde{\emptyset}}(x) = 1, \forall x \in X$ therefore $\widetilde{m}\left(\widetilde{\emptyset}\right) = 0$. If $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}}, \widetilde{A} \subseteq \widetilde{B}$ then $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x), \forall x \in X$ and $\nu_{\widetilde{A}}(x) \geq \nu_{\widetilde{B}}(x), \forall x \in X$. We have $\mu_{\widetilde{A}}(x) + 1 - \nu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x) + 1 - \nu_{\widetilde{B}}(x), \forall x \in X$ which implies $\widetilde{m}\left(\widetilde{A}\right) \leq \widetilde{m}\left(\widetilde{B}\right)$. Let $\left(\widetilde{A}_n\right)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint IFSs in $\widetilde{\mathcal{A}}$. For a fixed $x \in X$ there exists at most an $m_x \in \mathbb{N}$ such that $\mu_{\widetilde{A}_{m_x}}(x) > 0$ and at most an $n_x \in \mathbb{N}$ such that $\nu_{\widetilde{A}_{n_x}}(x) < 1$. Contrariwise the condition $\widetilde{A}_i \cap \widetilde{A}_j = \widetilde{\emptyset}, \forall i, j \in \mathbb{N}, i \neq j$ is violated. We have

$$\begin{aligned}
\widetilde{m}\left(\bigcup_{n \in \mathbb{N}} \widetilde{A}_n\right) &= \frac{1}{2}\sum_{x \in X}\left(\mu_{\bigcup_{n \in \mathbb{N}} \widetilde{A}_n}(x) + 1 - \nu_{\bigcup_{n \in \mathbb{N}} \widetilde{A}_n}(x)\right) \\
&= \frac{1}{2}\left(\sum_{x \in X}\sup_{n \in \mathbb{N}}\mu_{\widetilde{A}_n}(x) + cardX - \sum_{x \in X}\inf_{n \in \mathbb{N}}\nu_{\widetilde{A}_n}(x)\right) \\
&= \frac{1}{2}\left(\sum_{x \in X}\mu_{\widetilde{A}_{m_x}}(x) + cardX - \sum_{x \in X}\nu_{\widetilde{A}_{n_x}}(x)\right) \\
&= \frac{1}{2}\sum_{x \in X}\left(\sum_{n \in \mathbb{N}}\mu_{\widetilde{A}_n}(x) + cardX - \sum_{n \in \mathbb{N}}\nu_{\widetilde{A}_n}(x)\right) \\
&= \sum_{n \in \mathbb{N}}\left(\frac{1}{2}\sum_{x \in X}\left(\mu_{\widetilde{A}_n}(x) + 1 - \nu_{\widetilde{A}_n}(x)\right)\right) \\
&= \sum_{n \in \mathbb{N}}\widetilde{m}\left(\widetilde{A}_n\right).
\end{aligned}$$

■

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra and $m : \mathcal{A} \to [0, +\infty]$ be a measure on $\mathcal{A}$. We denote

$$L_{\alpha,\beta}\left(\widetilde{A}\right) = \left\{x \in X; \mu_{\widetilde{A}}(x) \geq \alpha \text{ and } \nu_{\widetilde{A}}(x) \leq \beta\right\},$$

where $\widetilde{A} = \left\{\left\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right\rangle ; x \in X\right\} \in IFS(X)$ and $\alpha, \beta \in [0, 1], \alpha \neq 0$ or $\beta \neq 1$. It is obvious that $\widetilde{A} \in \widetilde{\mathcal{I}}_{\mathcal{A}}$ (Proposition 2.1) implies $L_{\alpha,\beta}\left(\widetilde{A}\right) \in \mathcal{A}$.

Proposition 3.3 *For fixed* α, β *as above, the function* $\widetilde{m}_{\mathcal{A}} : \widetilde{\mathcal{I}}_{\mathcal{A}} \to [0, +\infty]$ *defined by*

$$\widetilde{m}_{\mathcal{A}}\left(\widetilde{A}\right) = m\left(L_{\alpha,\beta}\left(\widetilde{A}\right)\right)$$

is a σ*-additive intuitionistic fuzzy measure.*

Proof. We have $L_{\alpha,\beta}\left(\widetilde{\emptyset}\right) = \{x \in X; 0 \geq \alpha \text{ and } 1 \leq \beta\} = \emptyset$, therefore $\widetilde{m}_{\mathcal{A}}\left(\widetilde{\emptyset}\right) = m\left(\emptyset\right) = 0$.
If $\widetilde{A} \subseteq \widetilde{B}$ then $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x), \forall x \in X$ and $\nu_{\widetilde{A}}(x) \geq \nu_{\widetilde{B}}(x), \forall x \in X$ therefore $L_{\alpha,\beta}\left(\widetilde{A}\right) \subseteq L_{\alpha,\beta}\left(\widetilde{B}\right)$. We obtain

$$\widetilde{m}_{\mathcal{A}}\left(\widetilde{A}\right) = m\left(L_{\alpha,\beta}\left(\widetilde{A}\right)\right) \leq m\left(L_{\alpha,\beta}\left(\widetilde{B}\right)\right) = \widetilde{m}_{\mathcal{A}}\left(\widetilde{B}\right).$$

Let us prove $\widetilde{A} \cap \widetilde{B} = \widetilde{\emptyset}$ implies $L_{\alpha,\beta}\left(\widetilde{A}\right) \cap L_{\alpha,\beta}\left(\widetilde{B}\right) = \emptyset$. Let $x \in X$. Then $\mu_{\widetilde{A}}(x) = 0$ or $\mu_{\widetilde{B}}(x) = 0$ and $\nu_{\widetilde{A}}(x) = 1$ or $\nu_{\widetilde{B}}(x) = 1$. If $\mu_{\widetilde{A}}(x) = 0$ and $\nu_{\widetilde{A}}(x) = 1$ then $x \notin L_{\alpha,\beta}\left(\widetilde{A}\right)$. If $\mu_{\widetilde{A}}(x) = 0$ and $\nu_{\widetilde{B}}(x) = 1$ or $\mu_{\widetilde{B}}(x) = 0$ and $\nu_{\widetilde{A}}(x) = 1$ then $x \notin L_{\alpha,\beta}\left(\widetilde{A}\right)$ or $x \notin L_{\alpha,\beta}\left(\widetilde{B}\right)$. If $\mu_{\widetilde{B}}(x) = 0$ and $\nu_{\widetilde{B}}(x) = 1$ then $x \notin L_{\alpha,\beta}\left(\widetilde{B}\right)$.
The inclusion $\bigcup\limits_{n\in\mathbb{N}} L_{\alpha,\beta}\left(\widetilde{A}_n\right) \subseteq L_{\alpha,\beta}\left(\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n\right)$ is immediate for every sequence $\left(\widetilde{A}_n\right)_{n\in\mathbb{N}}$ of IFSs. If $\left(\widetilde{A}_n\right)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint IFSs in $\widetilde{\mathcal{I}}_{\mathcal{A}}$ then, for a fixed $x \in X$, there exists at most an index $m_x \in \mathbb{N}$ such that $\mu_{\widetilde{A}_{m_x}}(x) > 0$ and at most an $n_x \in \mathbb{N}$ such that $\nu_{\widetilde{A}_{n_x}}(x) < 1$ (see the proof of Proposition 3.2). But $m_x = n_x$ because contrariwise $\mu_{\widetilde{A}_{m_x}}(x) + \nu_{\widetilde{A}_{m_x}}(x) > 1$, a contradiction. These imply $\mu_{\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n}(x) = \mu_{\widetilde{A}_{m_x}}(x)$ and $\nu_{\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n}(x) = \nu_{\widetilde{A}_{m_x}}(x)$. If $x \in L_{\alpha,\beta}\left(\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n\right)$ then $\mu_{\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n}(x) = \mu_{\widetilde{A}_{m_x}}(x) \geq \alpha$ and $\nu_{\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n}(x) = \nu_{\widetilde{A}_{m_x}}(x) \leq \beta$ that is $x \in L_{\alpha,\beta}\left(\widetilde{A}_{m_x}\right)$. We obtain $x \in \bigcup\limits_{n\in\mathbb{N}} L_{\alpha,\beta}\left(\widetilde{A}_n\right)$, therefore $L_{\alpha,\beta}\left(\bigcup\limits_{n\in\mathbb{N}} \widetilde{A}_n\right) \subseteq \bigcup\limits_{n\in\mathbb{N}} L_{\alpha,\beta}\left(\widetilde{A}_n\right)$. We have

$$\begin{aligned}
\widetilde{m}_{\mathcal{A}}\left(\bigcup_{n\in\mathbb{N}} \widetilde{A}_n\right) &= m\left(L_{\alpha,\beta}\left(\bigcup_{n\in\mathbb{N}} \widetilde{A}_n\right)\right) = m\left(\bigcup_{n\in\mathbb{N}} L_{\alpha,\beta}\left(\widetilde{A}_n\right)\right) \\
&= \sum_{n\in\mathbb{N}} m\left(L_{\alpha,\beta}\left(\widetilde{A}_n\right)\right) = \sum_{n\in\mathbb{N}} \widetilde{m}_{\mathcal{A}}\left(\widetilde{A}_n\right).
\end{aligned}$$

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