Notes on Intuitionistic Fuzzy Sets
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Handling augmented appraisal degrees from multiple perspectives with the universal operator $X_{a,b,c,d,e,f}$

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Abstract: An *augmented appraisal degree* (AAD) is a generalization of a membership grade that indicates not only the *level to which* but also the *reasons why* a proposition is deemed to be true (or false). AADs as such can be used for the augmentation of the membership and non-membership components of each element of an *intuitionistic fuzzy set* (IFS). Such augmented IFSs have been proven to be useful for handling *experience-based evaluations* (XBEs) given by a heterogeneous group of people.

In this paper, a semantic interpretation of the universal operator $X_{a,b,c,d,e,f}$, which is part of the IFS framework, is proposed as a novel option to obtain an approximation of how an AAD characterizing an XBE given by someone is perceived from the perspective of someone else. We illustrate how this interpretation can be applied to handle AADs – and thus XBEs – from multiple perspectives.

Keywords: Augmented computation, Augmented (Atanassov) intuitionistic fuzzy set.
2010 Mathematics Subject Classification: 03E72.

1 Introduction

An *experience-based evaluation*, XBE for short, is a judgment that results from what a person has learned or understood about a given topic by experience. For example, consider the proposition p : '*x is suitable for a Pre-K math class*' where x represents the flash card depicted in Figure 1. Consider also that a Pre-K math teacher and a visual designer, say Alice and Bob respectively, are asked to evaluate the level to which p is true using a unit interval scale, where 0 and 1 denote the lowest and the highest level of truth respectively. After studying the flash card, Alice thinks that x fulfills p with a grade of 0.85 because each number corresponds to the count of items and the depiction of the caterpillar is friendly. In contrast, Bob considers that, although each number corresponds to the count of items, x fulfills p with a grade of 0.35 because of the style of the numbers may be confusing for Pre-K children. In this case, the particular knowledge or understanding that Alice and Bob may have about *flash cards that are suitable for a Pre-K math class*, make them to focus on different characteristics of the flash card to provide their evaluations.

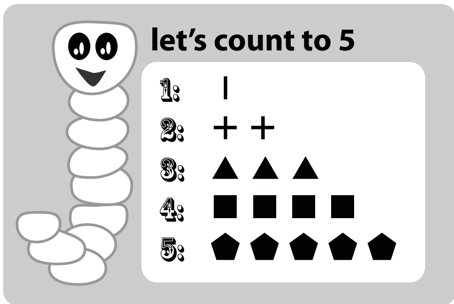


Figure 1. Is this flash card suitable for a Pre-K math class?

Handling XBEs can be a challenging task. One might assume that only the levels are necessary for processing XBEs. In this case, real numbers like 0.85 and 0.35 can be used for the characterization of the XBEs given by Alice and Bob respectively. Opposed to that, one might also assume that the reasons expressed are needed for processing XBEs. If so, structures like $\langle 0.85,$ {*'numbers vs. counts'*,*'caterpillar'* }$\rangle$ and $\langle 0.35,$ {*'style of numbers'*}$\rangle$ can be used for the characterization of the XBEs given by Alice and Bob respectively. An advantage of the latter characterization is that a more reliable comparison can be done: if only the real numbers are used, Alice might think that the evaluation given by Bob is misleading because does not reflect what she noticed on the flash card; however, if the enhanced structure is used, Alice might realize that Bob's evaluation is not misleading, but it has been given from a different perspective.

In [5], Loor and De Tré considered that the reasons given during the evaluation of a proposition are valuable for processing XBEs. Hence, they proposed a ready-for-computation mathematical representation, named *augmented appraisal degree* or AAD for short, for the characterization of an XBE. An AAD is a generalization of a *membership grade* [7] that can be used to record not only the level, but also the reasons expressed in an XBE. Along with the AAD concept, an *augmented framework* for handing XBEs has been proposed. Among the concepts, methods and operators included in that augmented framework is the *'as seen from'* operator.

Using this operator, one can compute an approximation of how an XBE characterized as an AAD looks like from a different perspective. For instance, one can use the *'as seen from'* operator to compute how the XBE given by Bob looks like from the perspective of Alice and, thus, one can perform a more fair comparison between Bob's XBE as seen from Alice's perspective and Alice's XBE.

In this paper, a semantic interpretation of the universal operator $X_{a,b,c,d,e,f}$ [2], which is defined in the framework of *intuitionistic fuzzy sets* (IFSs) [1, 2], is used as a novel option to compute the result of the *'as seen from'* operator. This interpretation demonstrates how the universal operator $X_{a,b,c,d,e,f}$ can be applied to handle an AAD from multiple perspectives. A practical motivation in this regard is to improve the applicability of existing IFS tools to processes like clustering, filtering or querying that involve XBEs given by persons with different knowledge or expertise.

To present the aforementioned interpretation, the reminder of this paper is structured as follows. A brief summary of some of the concepts included in the augmented framework, as well as the formal notation used in this paper are introduced in Section 2. Then, in Section 3 the novel interpretation of the operator $X_{a,b,c,d,e,f}$ is described and it is shown how this interpretation can be used to handle AADs from multiple perspectives. After that, the paper is concluded in Section 4.

2 Preliminaries

As was mentioned above, an *experience-based evaluation* (XBE) is essentially a subjective judgment that results from what a person has experienced about the topic under evaluation. Aiming to model such XBEs, in [5] Loor and De Tré proposed an *augmented framework* that includes, among others, the definitions of *augmented appraisal degrees* (AADs) and *augmented intuitionistic fuzzy sets* (AIFSs). Examples illustrating these concepts are presented below.

2.1 Augmented appraisal degrees

When making a judgment on (the truth value of) a proposition, an evaluator could express his/her appraisal level along with something that hints the reasons of that appraisal. An AAD is a (ready-for-computation) mathematical representation of this kind of judgments. Such a mathematical representation can be described as follows.

Consider an object x consisting of a collection of features $\mathcal{F}$. Consider also a proposition p having the canonical form 'x IS A' meaning 'x *is an instance of* A' [8], where A is a given (fuzzy) set. Finally, consider a person P. With these considerations, an AAD of x, say $\hat{\mu}_{A@P}(x)$, is a pair $\langle \mu_{A@P}(x), F_{\mu_{A@P}}(x) \rangle$ that denotes the level $\mu_{A@P}(x)$ to which x satisfies p, as well as the particular collection of features $F_{\mu_{A@P}}(x) \subseteq \mathcal{F}$ that are taken into account to appraise x from the perspective of P.

By way of illustration, consider that A represents the "*collection of flash cards that are suitable for a Pre-K math class*" in the introductory example (see Section 1). In this context, the XBE given by Alice can be characterized by $\hat{\mu}_{A@Alice}(x) = \langle \mu_{A@Alice}(x), F_{\mu_{A@Alice}}(x) \rangle$, such that $\mu_{A@Alice}(x) = 0.85$ and $F_{\mu_{A@Alice}}(x) = \{$*'numbers vs. counts'*, *'caterpillar'*$\}$. In a similar way, the XBE given by Bob can be characterized by $\hat{\mu}_{A@Bob}(x) = \langle \mu_{A@Bob}(x), F_{\mu_{A@Bob}}(x)\rangle$, such that $\mu_{A@Bob}(x) = 0.35$ and $F_{\mu_{A@Bob}}(x) = \{$*'style of numbers'*$\}$.

2.2 Augmented intuitionistic fuzzy sets

The inclusion of AADs into the definition of an *intuitionistic fuzzy set* (IFS) [1, 2] has been proposed in [5] for handling subjective, imprecise and potentially marked-by-hesitation XBEs given by a heterogeneous group of people. Such an augmented IFS, called *augmented intuitionistic fuzzy set* or AIFS for short, can be described as follows.

Consider a collection of objects $X = \{x_1, \cdots, x_n\}$, where each $x_i \in X$ has a collection of features $\mathcal{F}_i$. Consider also a proposition p having the canonical form 'x_i IS A' [8] meaning '*x_i is an instance of A*', where A is a particular (fuzzy) set. Finally, consider a person P. Assume $I = [0, 1]$ and $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$. Let $\hat{\mu}_{A@P}(x_i) = \langle \mu_{A@P}(x_i), F_{\mu_A@P}(x_i)\rangle$ and $\hat{\nu}_{A@P}(x_i) = \langle \nu_{A@P}(x_i), F_{\nu_A@P}(x_i)\rangle$ in $\langle I, \mathcal{F}\rangle$ be two AADs denoting respectively the evaluation of x_i satisfying and dissatisfying the proposition p according to the standpoint of P. An AIFS is a collection $\hat{A}_{@P}$ that denotes the correspondence between each $x_i \in X$ and both $\hat{\mu}_{A@P}(x_i)$ and $\hat{\nu}_{A@P}(x_i)$ such that

$$\hat{A}_{@P} = \{\langle x_i, \hat{\mu}_{A@P}(x_i), \hat{\nu}_{A@P}(x_i)\rangle \mid (x_i \in X) \wedge (0 \leq \mu_{A@P}(x_i) + \nu_{A@P}(x_i) \leq 1)\}\,. \qquad (1)$$

For instance, let A be a collection of *healthy drinks*, and let X be a collection of drinks constituted by cola, canned orange juice and fresh soursop (a.k.a. *'guanabana'*) juice, i.e., $X = \{$*'cola'*, *'canned orange juice'*, *'fresh soursop juice'*$\}$. Consider a unit interval scale where 1 denotes the highest level of truth and 0 the lowest. Consider also a proposition "x IS A" meaning "the drink x is compatible with the definition of *healthy drinks*." Finally, consider that a person, say P, makes the following judgments:

- A cola dissatisfies the proposition with a grade of 0.9 because it usually contains a lot of added sugar and, also, artificial sweeteners; however, due to the possibility of being refreshing, it satisfies with a grade of 0.1 the proposition as well.
- A canned orange juice satisfies the proposition with a grade of 0.6 because it is usually made from natural oranges and contains vitamin C; however, because it is not fresh, it also dissatisfies with a grade of 0.3 the proposition.
- A fresh soursop juice satisfies the proposition with a grade of 0.9 because it is not preserved and contains magnesium; however, P does not know if it contains zinc or phosphorus.

These judgments can be represented by an AIFS, say $\hat{A}_{@P}$, such that

$$\begin{aligned}\hat{A}_{@P} = \{ & \left\langle \textit{'cola'}, \langle 0.1, \{\textit{'refreshing'}\}\rangle, \langle 0.9, \{\textit{'artificial sweeteners'}, \textit{'added sugar'}\}\rangle \right\rangle, \\ & \left\langle \textit{'canned orange juice'}, \langle 0.6, \{\textit{'natural juice'}, \textit{'vitamin C'}\}\rangle, \langle 0.3, \{\textit{'non-fresh'}\}\rangle \right\rangle, \\ & \left\langle \textit{'fresh soursop juice'}, \langle 0.9, \{\textit{'no preservatives'}, \textit{'magnesium'}\}\rangle, \langle 0, \{\}\rangle \right\rangle \}\,.\end{aligned}$$

Here, the hesitation of P to judge *'fresh soursop juice'* as a member or not of *healthy drinks* could be, e.g., $\hat{h}_{A@P}$(*'fresh soursop juice'*) $= \langle 0.1, \{$*'zinc', 'phosphorus'*$\}\rangle$ – notice that the hesitation level results from $(1 - (0.9 + 0) = 0.1)$.

3 Handling AADs from multiple perspectives

The aim of this section is to provide a novel semantic interpretation of the operator $X_{a,b,c,d,e,f}$ [2] and show how this interpretation can be used to handle AADs from multiple perspectives. To do so, let us first illustrate how an AAD can be seen from a different perspective. Consider a collection $X = \{x_1\}$ and two AIFSs:

$$\hat{A}_{@P} = \{\langle x_1, \langle 0.6, F_{\mu_A @P}\rangle, \langle 0.3, F_{\nu_A @P}\rangle\rangle\} \text{ and}$$
$$\hat{A}_{@Q} = \{\langle x_1, \langle 0.2, F_{\mu_A @Q}\rangle, \langle 0.5, F_{\nu_A @Q}\rangle\rangle\},$$

given by persons P and Q, respectively. Consider also a proposition p having the form 'x IS A', meaning "x is an instance of A." The following cases are possible according to the collection of features that P and Q individually considered for the appraisal of x_1:

1. *The expressions $F_{\mu_A @P} = F_{\mu_A @Q}$ and $F_{\nu_A @P} = F_{\nu_A @Q}$ hold.* In this case, P and Q agree on the features considered for appraising the level to which x_1 satisfies p, as well as on the features considered for appraising the level to which x_1 dissatisfies p. A geometrical interpretation of this case is shown in Figure 2. In this figure, the AIFS elements $\langle x_1, \langle 0.6, F_{\mu_A @P}\rangle, \langle 0.3, F_{\nu_A @P}\rangle\rangle$ and $\langle x_1, \langle 0.2, F_{\mu_A @Q}\rangle, \langle 0.5, F_{\nu_A @Q}\rangle\rangle$ are denoted by ◎ and ⊠, respectively. One can say that this case corresponds to the original definition of an IFS.

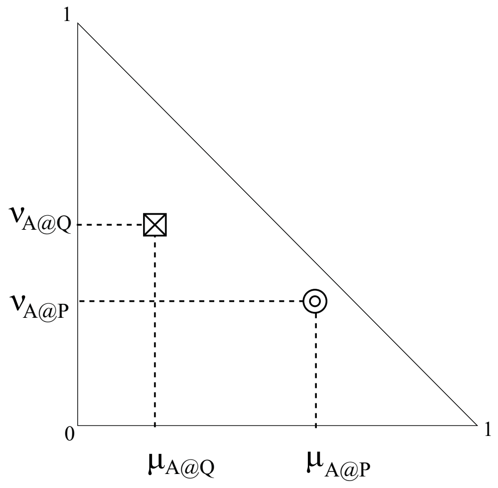


Figure 2. The expressions $F_{\mu_A @P} = F_{\mu_A @Q}$ and $F_{\nu_A @P} = F_{\nu_A @Q}$ hold.

2. *Only the expression $F_{\nu_A @P} = F_{\nu_A @Q}$ holds.* In this case, only the features considered for appraising the level to which x_1 *dissatisfies* p are the same from the perspectives of P and Q. This case is depicted in Figure 3. In a similar way to the previous case, the AIFS

elements $\langle x_1, \langle 0.6, F_{\mu_A@P}\rangle, \langle 0.3, F_{\nu_A@P}\rangle\rangle$ and $\langle x_1, \langle 0.2, F_{\mu_A@Q}\rangle, \langle 0.5, F_{\nu_A@Q}\rangle\rangle$ are denoted by $\circledcirc$ and $\boxtimes$ respectively. As will be explained later, the gray-colored $\boxtimes$ (i.e., $\boxtimes$) denotes how the AAD given by Q (i.e., $\boxtimes$) might be seen from the perspective of P.

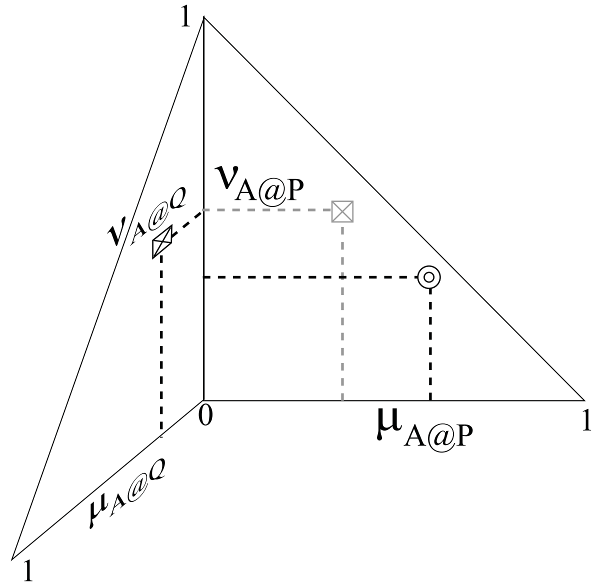


Figure 3. Only the expression $F_{\nu_A@P} = F_{\nu_A@Q}$ holds.

3. *Only the expression* $F_{\mu_A@P} = F_{\mu_A@Q}$ *holds*. In this case, only the features considered for appraising the level to which x_1 *satisfies* p are the same from the perspectives of P and Q. This case is depicted in Figure 4. Analogously to the previous case, while the AIFS elements given by P and Q are denoted by $\circledcirc$ and $\boxtimes$ respectively, the way how $\boxtimes$ might be seen from the perspective of P is denoted by the gray-colored $\boxtimes$ (i.e., $\boxtimes$).

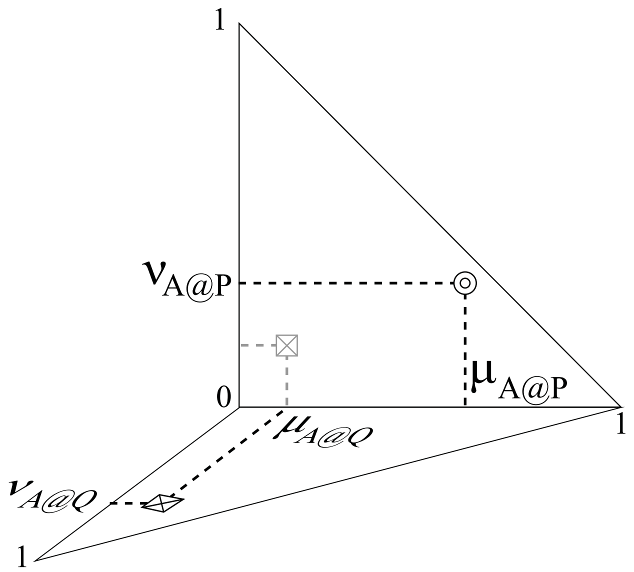


Figure 4. Only the expression $F_{\mu_A@P} = F_{\mu_A@Q}$ holds.

4. *Neither* $F_{\mu_A@P} = F_{\mu_A@Q}$ *nor* $F_{\nu_A@P} = F_{\nu_A@Q}$ *holds.* In this case, there are one or more features considered by P to appraise the level to which x_1 *satisfies* p that differ from those used by Q. The same happens with the features used by P to appraise the level to which x_1 *dissatisfies* p. This case is depicted in Figure 5.

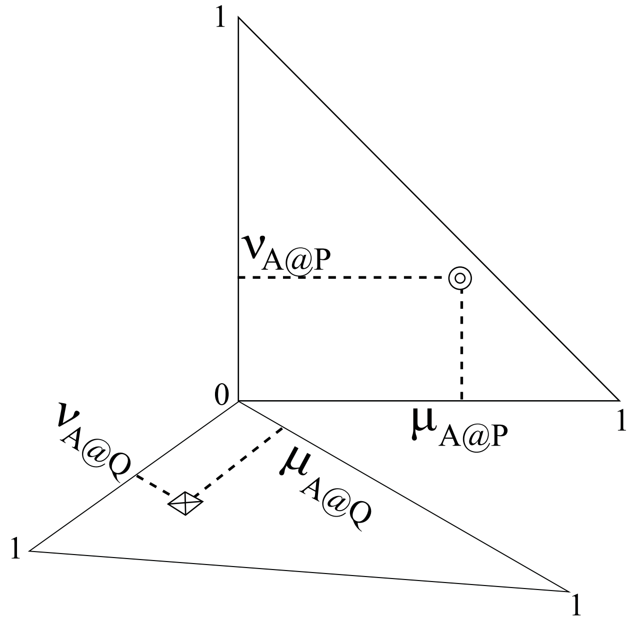


Figure 5. Neither $F_{\mu_A@P} = F_{\mu_A@Q}$ nor $F_{\nu_A@P} = F_{\nu_A@Q}$ holds.

The above-mentioned cases reflect the fact that XBEs given by people with different expertise may have some contextual differences – here, by '*context of an XBE*' is meant the conditions that arise when the evaluation is carried out, which mainly depend on the experience of an evaluator about the concept under analysis. To quantify such contextual differences (or similarities), in [5] Loor and De Tré proposed a number $\Delta_{\mu_A} \in [0, 1]$, named *connotation alikeness factor* (CAF), which indicates the level to which $F_{\mu_{A@P}}$ and $F_{\mu_{A@Q}}$ are perceived as similar. For instance, in the cases depicted in Figures 2 and 4, one can assign the highest value (i.e., 1) to the CAF for the membership component (i.e., Δ_{μ_A}) since $F_{\mu_A@P} = F_{\mu_A@Q}$ holds. In a similar way, in the cases depicted in Figures 2 and 3, one can assign the highest value (i.e., 1) to the CAF for the nonmembership component (i.e., Δ_{ν_A}) since $F_{\nu_A@P} = F_{\nu_A@Q}$ holds. In the case depicted in Figure 5, one can assign a value greater than or equal to 0 but less than 1 to the membership and nonmembership CAFs. In this regard, one can say that $\Delta_{\mu_A} = 1$ means "P and Q focused on the same features to evaluate the level to which x_1 satisfies p" and, by the contrary, $\Delta_{\mu_A} = 0$ means "P and Q focused on totally different features to evaluate the level to which x_1 satisfies p."

A CAF depends on the perspective that is taken as a reference. This means that $\Delta_{\mu_A@P}$, which takes into account the perspective of P, is not necessarily equal to $\Delta_{\mu_A@Q}$, which takes into account the perspective of Q. Because of this, a notation such as $\Delta_{(\mu_A:P,Q)@P}$ meaning "*a CAF between the contexts of the AADs given by P and Q as seen from the perspective of P*," is suggested. For instance, if $F_{\mu_A@P} = \{a, b, c, d, e\}$ and $F_{\mu_A@Q} = \{a, e, f\}$, P can assign $\frac{2}{5}$ to $\Delta_{(\mu_A:P,Q)@P}$ and, at the same time, Q can assign $\frac{2}{3}$ to $\Delta_{(\mu_A:P,Q)@Q}$.

Notice in the previous example that, e.g., P can use $\Delta_{(\mu_A:P,Q)@P}$ to indicate, from his/her perspective, how relevant an AAD given by Q is. Hence, one can use $\Delta_{(\mu_A:P,Q)@P}$ to determine how an AAD given by Q looks like from the perspective of P. This idea has been used in [5] to obtain an approximation of an AAD as seen from a different perspective by means of the *'as seen from'* operator, which can be described as follows:

Consider an object x with a collection of features $\mathcal{F}$. Consider also a proposition p having the form *'x IS A'* meaning "x is an instance of the (fuzzy) set A." Finally, consider two persons, say P and Q, and assume $I_P = I_Q = [0, 1]$. Let $\hat{\mu}_{A@P}(x)$ be an AAD denoting the level to which x satisfies (the proposition) p as seen from the perspective of (person) P. The *'as seen from'* operator is a mapping

$$\begin{aligned} \lfloor\cdot\rfloor_{@Q} : \langle I_P, \mathcal{F}\rangle &\to \langle I_Q, \mathcal{F}\rangle \\ \hat{\mu}_{A@P}(x) &\mapsto \lfloor\hat{\mu}_{A@P}(x)\rfloor_{@Q}, \end{aligned} \tag{2}$$

such that $\lfloor\hat{\mu}_{A@P}(x)\rfloor_{@Q}$ is an AAD that corresponds to $\hat{\mu}_{A@P}(x)$ as seen from the perspective of Q. Here, $\lfloor\hat{\mu}_{A@P}(x)\rfloor_{@Q}$ has the form $\langle\lfloor\mu_{A@P}(x)\rfloor_{@Q}, \lfloor F_{\mu_A@P}(x)\rfloor_{@Q}\rangle$, where $\lfloor\mu_{A@P}(x)\rfloor_{@Q}$ and $\lfloor F_{\mu_A@P(x)}\rfloor_{@Q}$ correspond to $\mu_{A@P}(x)$ and $F_{\mu_A@P}(x)$ respectively as seen from the perspective of Q. Notice that, from the perspective of Q, the features of x considered for the appraisal of p are given by $F_{\mu_A@Q}(x)$. Hence, one can say that the expression

$$\lfloor F_{\mu_A@P}(x)\rfloor_{@Q} = F_{\mu_A@Q}(x) \tag{3}$$

holds. Along with (3), the expression

$$\lfloor\mu_{A@P}(x)\rfloor_{@Q} = \Delta_{(\mu_A:P,Q)@Q} \cdot \mu_{A@P}(x). \tag{4}$$

has been proposed in [5] to obtain $\lfloor\hat{\mu}_{A@P}(x)\rfloor_{@Q}$.

It is worth mentioning that, even though the *'as seen from'* operator has been described above through an AAD denoting the level to which x *satisfies* (the proposition) p, it can also be applied to an AAD denoting the level to which x *dissatisfies* (the proposition) p. Hence, the equations

$$\lfloor F_{\nu_A@P}(x)\rfloor_{@Q} = F_{\nu_A@Q}(x) \tag{5}$$

and

$$\lfloor\nu_{A@P}(x)\rfloor_{@Q} = \Delta_{(\nu_A:P,Q)@Q} \cdot \nu_{A@P}(x) \tag{6}$$

also hold.

As a novel option to obtain $\lfloor\mu_{A@P}(x)\rfloor_{@Q}$ and $\lfloor\nu_{A@P}(x)\rfloor_{@Q}$, in this paper we propose the use of the following semantic interpretation of the universal operator $X_{a,b,c,d,e,f}$, which is included in the IFS framework.

The operator $X_{a,b,c,d,e,f}$ [2] is defined by

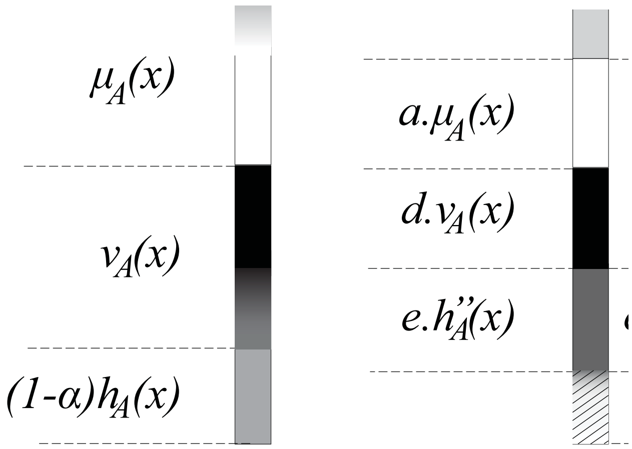
$$\begin{aligned} X_{a,b,c,d,e,f}(A) = \{\langle x, &a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)), \\ &d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle \mid (x \in X)\}, \end{aligned} \tag{7}$$

where A is an IFS, and a, b, c, d, e and f are numbers in $[0, 1]$ such that $(a + e - e.f \leq 1)$ and $(b + d - d.c \leq 1)$ hold. A unit-segment interpretation of this operator is shown in Figure 6.

Figure 6b shows the components of the IFS element depicted in Figure 6a after applying the operator $X_{a,b,c,d,e,f}$. For the sake of illustration, in Figure 6a the hesitation margin, i.e.,

$$h_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

has been split into two components using a value $\alpha \in [0,1]$, which is called *'hesitation splitter'* [4].



(a) (b)

Figure 6. Unit-segment interpretation of the operator $X_{a,b,c,d,e,f}$.

Notice in Figure 6b that two variants of hesitation, namely $h'_A(x) = 1 - \mu_A(x) - c.\nu_A(x)$ and $h''_A(x) = 1 - f.\mu_A(x) - \nu_A(x)$, have been depicted. Notice also that $h'_A(x) \leq h_A(x)$ and $h''_A(x) \leq h_A(x)$. In this regard, one can say that $\tau' = h_A(x) - h'_A(x)$ and $\tau'' = h_A(x) - h''_A(x)$, which are depicted using diagonal lines, might represent any doubt about the transformation performed by the operator $X_{a,b,c,d,e,f}$.

From a semantic point of view, the factors a and d in (7) can be deemed to be the CAFs $\Delta_{(\mu_A:P,Q)@Q}$ and $\Delta_{(\nu_A:P,Q)@Q}$ respectively. This means that, if $\hat{\mu}_{A@P}(x)$ and $\hat{\mu}_{A@Q}(x)$ are two AADs denoting the levels to which x satisfies a proposition p : 'x IS A' according to P and Q respectively, a will indicate the contextual similarities between these AADs as seen from the perspective of Q. Likewise, if $\hat{\nu}_{A@P}(x)$ and $\hat{\nu}_{A@Q}(x)$ are two AADs denoting the levels to which x dissatisfies p according to P and Q respectively, d will indicate the contextual similarities between these AADs as seen from the perspective of Q.

With respect to the factors b and e in (7), since these factors split any hesitation about the level to which x satisfies (or dissatisfies) p, they can be considered as *hesitation splitters* [3]. In this case, b and e will respectively correspond to the *membership hesitation splitter* (i.e., α_A) and to the *nonmembership hesitation splitter* (i.e., β_A). It is worth mentioning that the expression $\alpha_A + \beta_A \leq 1$ must hold for any $\alpha_A, \beta_A \in [0,1]$.

Regarding the factors c and f in (7), they can respectively be considered to be the portions of the nonmembership and membership levels that have been assigned due to specific (or well-known) reasons. For instance, consider that the second judgment of the example presented in Section 2.2 is characterized by the IFS element $\langle$'*canned orange juice*', $0.6, 0.3\rangle$. Assigning the value 0.4 to c in this case suggests that only 40% of ν_A(‘*canned orange juice*’), i.e., $0.4 \cdot 0.3 = 0.12$, has been given after the evaluation of the *‘non-fresh’* feature of a canned orange juice. Analogously, assigning 1 to f indicates that μ_A(‘*canned orange juice*’), i.e., 0.6, has entirely been given due to a canned orange juice is made from natural oranges and contains vitamin C.

Using the aforementioned interpretations, equations (4) and (6) can be rewritten as

$$\lfloor \mu_{A@P}(x) \rfloor_{@Q} = \Delta_{(\mu_A:P,Q)@Q} \cdot \mu_{A@P}(x) + \alpha_{A@Q} \cdot (1 - \mu_{A@P}(x) - c_{@Q} \cdot \nu_{A@P}(x)) \qquad (8)$$

and

$$\lfloor \nu_{A@P}(x) \rfloor_{@Q} = \Delta_{(\nu_A:P,Q)@Q} \cdot \nu_{A@P}(x) + \beta_{A@Q} \cdot (1 - f_{@Q} \cdot \mu_{A@P}(x) - \nu_{A@P}(x)), \qquad (9)$$

respectively. In these equations, the subscript ‘$@Q$’ has been added to the factors c and f to indicate the perspective that is taken as a point of reference during the assignment or computation of these factors. Notice that, if the values of the hesitation splitters $\alpha_{A@Q}$ and $\beta_{A@Q}$ are both fixed to zero, (8) and (9) will be equivalent to (4) and (6), respectively.

To illustrate how (8) and (9) can be applied to handle AADs from multiple perspectives, consider that two additional persons, say R and S, have given the AIFSs

$$\hat{A}_{@R} = \{\langle x_1, \langle 0.6, F_{\mu_A@R}\rangle, \langle 0.3, F_{\nu_A@R}\rangle\rangle\} \text{ and}$$
$$\hat{A}_{@S} = \{\langle x_1, \langle 0.3, F_{\mu_A@S}\rangle, \langle 0.6, F_{\nu_A@S}\rangle\rangle\}\,,$$

respectively for the collection $X = \{x_1\}$ introduced at the beginning of this section. Consider also that the AIFS elements $\langle x_1, \langle 0.6, F_{\mu_A@R}\rangle, \langle 0.3, F_{\nu_A@R}\rangle\rangle$ and $\langle x_1, \langle 0.3, F_{\mu_A@S}\rangle, \langle 0.6, F_{\nu_A@S}\rangle\rangle$ are denoted in that order by $\boxplus$ and $\otimes$.

If P, R and S agree on the features of x_1 that should be considered for appraising the level to which x_1 satisfies (or dissatisfies) the proposition p: ‘x_1 IS A’ (i.e., the expressions $F_{\mu_A@P} = F_{\mu_A@R} = F_{\mu_A@S}$ and $F_{\nu_A@P} = F_{\nu_A@R} = F_{\nu_A@S}$ hold), the values of the CAFs $\Delta_{(\mu_A:P,S)@P}$, $\Delta_{(\mu_A:P,R)@P}$, $\Delta_{(\nu_A:P,S)@P}$ and $\Delta_{(\nu_A:P,R)@P}$ will be equal to 1. Additionally, if P considers that any hesitation expressed in the AADs given by R and S should be discarded, the values of the hesitation splitters $\alpha_{A@P}$ and $\beta_{A@P}$ have to be set to 0. In this case, which is denoted by Case $1a$, one can use the following expressions to compute how the appraisal levels in $\boxplus$ and $\otimes$ are seen from the perspective of P:

$$\begin{aligned}
\lfloor \mu_{A@R}(x_1) \rfloor_{@P} &= 1 \cdot 0.6 + 0 \cdot (1 - 0.6 - c_{@P} \cdot 0.3) = 0.6,\\
\lfloor \nu_{A@R}(x_1) \rfloor_{@P} &= 1 \cdot 0.3 + 0 \cdot (1 - f_{@P} \cdot 0.6 - 0.3) = 0.3,\\
\lfloor \mu_{A@S}(x_1) \rfloor_{@P} &= 1 \cdot 0.3 + 0 \cdot (1 - 0.3 - c_{@P} \cdot 0.6) = 0.3, \text{ and}\\
\lfloor \nu_{A@S}(x_1) \rfloor_{@P} &= 1 \cdot 0.6 + 0 \cdot (1 - f_{@P} \cdot 0.3 - 0.6) = 0.6.
\end{aligned}$$

These results are depicted in Figure 7a. Notice in this case that, even though $F_{\mu_A@P} = F_{\mu_A@S}$ and $F_{\nu_A@P} = F_{\nu_A@S}$ hold, the appraisal levels in $\otimes$ suggest that S has an opposite understanding of A in relation to P. In a different case, if P considers that the hesitation on the AADs given by R and S should be proportionally split, the values of the hesitation splitters $\alpha_{A@P}$ and $\beta_{A@P}$ can be set to 0.5 and, thus, the values of $c_{@P}$ and $f_{@P}$ will be equal to 1. In the latter case, which is denoted by Case $1b$, one can use the following expressions to compute how the appraisal levels in $\boxplus$ and $\otimes$ are seen from the perspective of P:

$$\begin{aligned}
\lfloor \mu_{A@R}(x_1) \rfloor_{@P} &= 1 \cdot 0.6 + 0.5 \cdot (1 - 0.6 - 1 \cdot 0.3) = 0.65, \\
\lfloor \nu_{A@R}(x_1) \rfloor_{@P} &= 1 \cdot 0.3 + 0.5 \cdot (1 - 1 \cdot 0.6 - 0.3) = 0.35, \\
\lfloor \mu_{A@S}(x_1) \rfloor_{@P} &= 1 \cdot 0.3 + 0.5 \cdot (1 - 0.3 - 1 \cdot 0.6) = 0.35, \text{ and} \\
\lfloor \nu_{A@S}(x_1) \rfloor_{@P} &= 1 \cdot 0.6 + 0.5 \cdot (1 - 1 \cdot 0.3 - 0.6) = 0.65.
\end{aligned}$$

These results are depicted in Figure 7b.

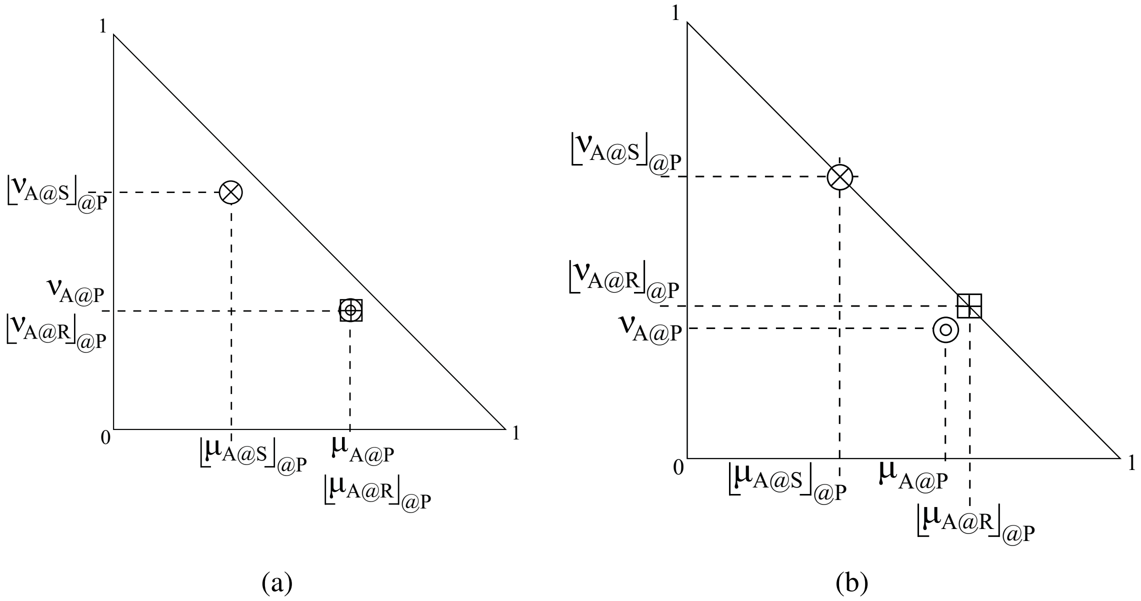


Figure 7. Evaluations given by R and S as seen from the perspective of P: Case $1a$ and Case $1b$.

If only the features considered for appraising the level to which x_1 dissatisfies p are the same from the perspectives of P, R and S (i.e., only $F_{\nu_A@P} = F_{\nu_A@R} = F_{\nu_A@S}$ holds) and assuming that $F_{\mu_A@P} \cap F_{\mu_A@R} = \{\}$ and $F_{\mu_A@P} \cap F_{\mu_A@S} = \{\}$, the values of the CAFs will be as follows: $\Delta_{(\mu_A:P,S)@P} = 0$, $\Delta_{(\mu_A:P,R)@P} = 0$, $\Delta_{(\nu_A:P,S)@P} = 1$ and $\Delta_{(\nu_A:P,R)@P} = 1$. Considering that the hesitation is discarded (i.e., $\alpha_{A@Q} = 0$ and $\beta_{A@Q} = 0$ hold), one can obtain the following results (Case $2a$): $\lfloor \mu_{A@R}(x_1) \rfloor_{@P} = 0$, $\lfloor \nu_{A@R}(x_1) \rfloor_{@P} = 0.3$, $\lfloor \mu_{A@S}(x_1) \rfloor_{@P} = 0$ and $\lfloor \nu_{A@S}(x_1) \rfloor_{@P} = 0.6$ (see Figure 8a). In another case, say Case $2b$, if the values of the hesitation splitters $\alpha_{A@P}$ and $\beta_{A@P}$ are set to 0.5, and the values of $c_{@P}$ and $f_{@P}$ are set to 1, the following results can be obtained: $\lfloor \mu_{A@R}(x_1) \rfloor_{@P} = 0.05$, $\lfloor \nu_{A@R}(x_1) \rfloor_{@P} = 0.35$, $\lfloor \mu_{A@S}(x_1) \rfloor_{@P} = 0.05$ and $\lfloor \nu_{A@S}(x_1) \rfloor_{@P} = 0.65$ (see Figure 8b).

If only the features considered for appraising the level to which x_1 satisfies p are the same from the perspectives of P, R and S (i.e., only $F_{\mu_A@P} = F_{\mu_A@R} = F_{\mu_A@S}$ holds) and assuming that $F_{\nu_A@P} \cap F_{\nu_A@R} = \{\}$ and $F_{\nu_A@P} \cap F_{\nu_A@S} = \{\}$, the values of the CAFs will be as follows: $\Delta_{(\mu_A:P,S)@P} = 1$, $\Delta_{(\mu_A:P,R)@P} = 1$, $\Delta_{(\nu_A:P,S)@P} = 0$ and $\Delta_{(\nu_A:P,R)@P} = 0$. Assuming

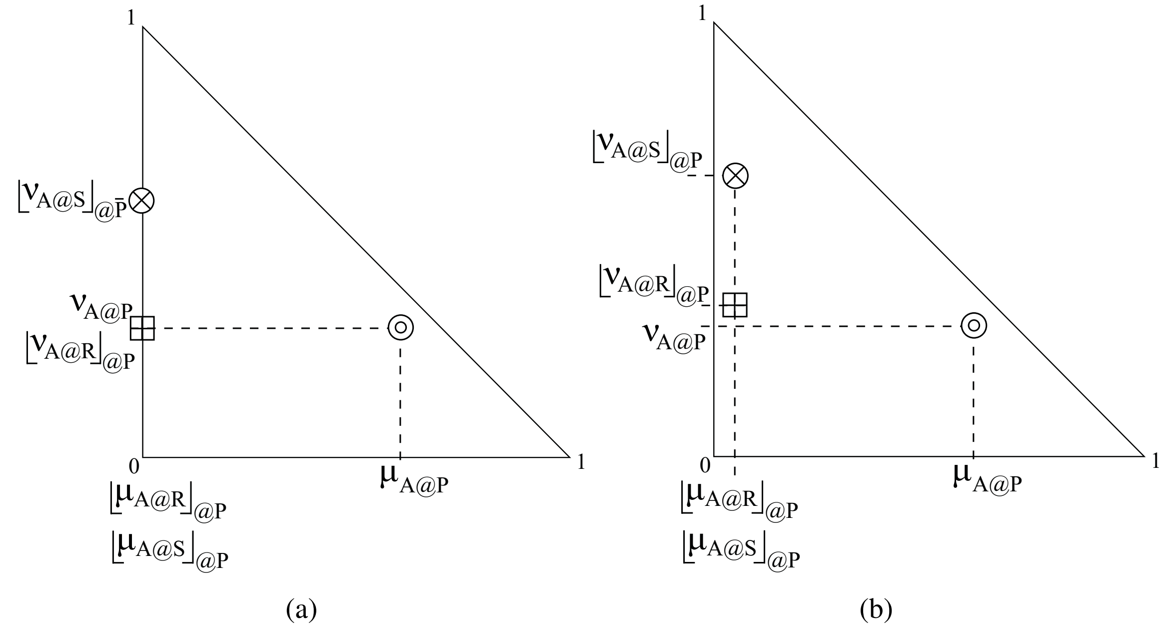


Figure 8. Evaluations given by R and S as seen from the perspective of P: Case $2a$ and Case $2b$.

also that the hesitation is discarded (i.e., $\alpha_{A@Q} = 0$ and $\beta_{A@Q} = 0$ hold), one can obtain the next results (Case $3a$): $\lfloor\mu_{A@R}(x_1)\rfloor_{@P} = 0.6$, $\lfloor\nu_{A@R}(x_1)\rfloor_{@P} = 0$, $\lfloor\mu_{A@S}(x_1)\rfloor_{@P} = 0.3$ and $\lfloor\nu_{A@S}(x_1)\rfloor_{@P} = 0$ (see Figure 9a). In a different case, say Case $3b$, if the values of $\alpha_{A@P}$ and $\beta_{A@P}$ are set to 0.5, and the values of $c_{@P}$ and $f_{@P}$ are set to 1, the following results can be obtained: $\lfloor\mu_{A@R}(x_1)\rfloor_{@P} = 0.65$, $\lfloor\nu_{A@R}(x_1)\rfloor_{@P} = 0.05$, $\lfloor\mu_{A@S}(x_1)\rfloor_{@P} = 0.35$ and $\lfloor\nu_{A@S}(x_1)\rfloor_{@P} = 0.05$ (see Figure 9b).

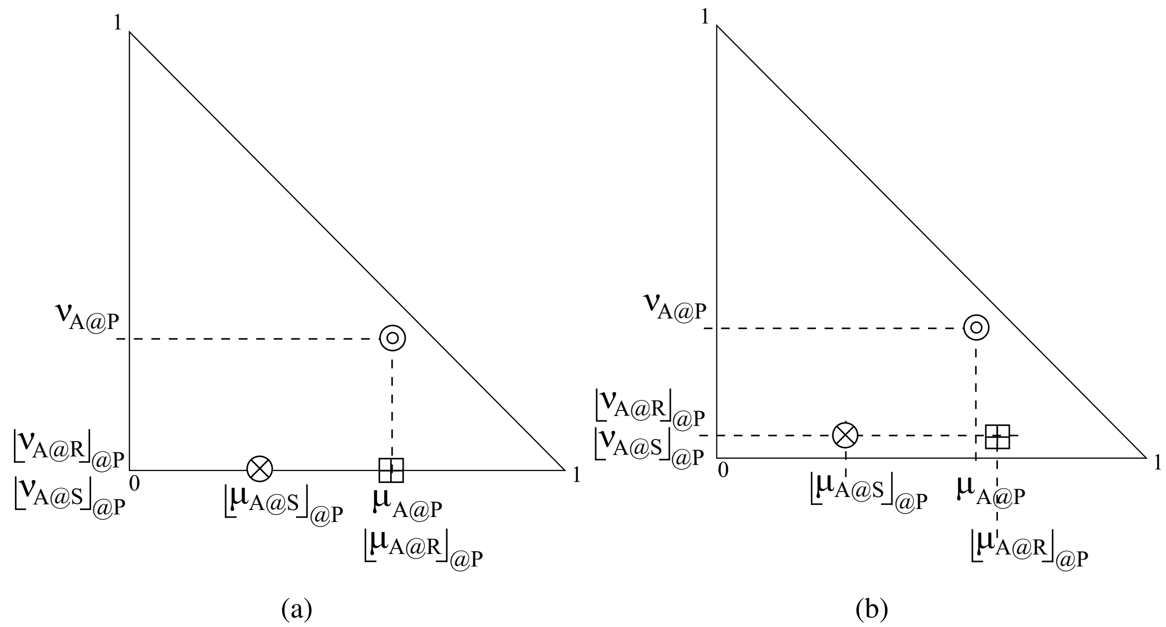


Figure 9. Evaluations given by R and S as seen from the perspective of P: Case $3a$ and Case $3b$.

If $F_{\mu_A@P} \cap F_{\mu_A@R} = \{\}$, $F_{\mu_A@P} \cap F_{\mu_A@S} = \{\}$, $F_{\nu_A@P} \cap F_{\nu_A@R} = \{\}$ and $F_{\nu_A@P} \cap F_{\nu_A@S} = \{\}$ hold, the values of the CAFs $\Delta_{(\mu_A:P,S)@P}$, $\Delta_{(\mu_A:P,R)@P}$, $\Delta_{(\nu_A:P,S)@P}$ and $\Delta_{(\nu_A:P,R)@P}$ will be equal to 0. Assuming that $\alpha_{A@Q} = 0$ and $\beta_{A@Q} = 0$ hold, one can obtain the following results (Case $4a$): $\lfloor \mu_{A@R}(x_1) \rfloor_{@P} = 0$, $\lfloor \nu_{A@R}(x_1) \rfloor_{@P} = 0$, $\lfloor \mu_{A@S}(x_1) \rfloor_{@P} = 0$ and $\lfloor \nu_{A@S}(x_1) \rfloor_{@P} = 0$ (see Figure 10a). Notice in this case that, although P and R both agree on the appraisal levels assigned to x_1, i.e., $\mu_{A@P}(x_1) = \mu_{A@R}(x_1)$ and $\nu_{A@P}(x_1) = \nu_{A@R}(x_1)$, P perceives these levels as $\lfloor \mu_{@R}(x_1) \rfloor_{@P} = 0$ and $\lfloor \nu_{@R}(x_1) \rfloor_{@P} = 0$ respectively because it is considered that P and R have focused on completely different features of x_1 for the appraisal of p. Hence, this case can be deemed to be an example of a problem called *'pseudo-matching'* [3], in which a comparison between two XBEs given by two persons can "match" even though these persons have different understandings of the evaluated concept. In a variant of the previous case, if $\alpha_{A@P} = \beta_{A@P} = 0.5$ and $c_{@P} = f_{@P} = 1$, the following results can be obtained (Case $4b$): $\lfloor \mu_{A@R}(x_1) \rfloor_{@P} = 0.05$, $\lfloor \nu_{A@R}(x_1) \rfloor_{@P} = 0.05$, $\lfloor \mu_{A@S}(x_1) \rfloor_{@P} = 0.05$ and $\lfloor \nu_{A@S}(x_1) \rfloor_{@P} = 0.05$ (see Figure 10b).

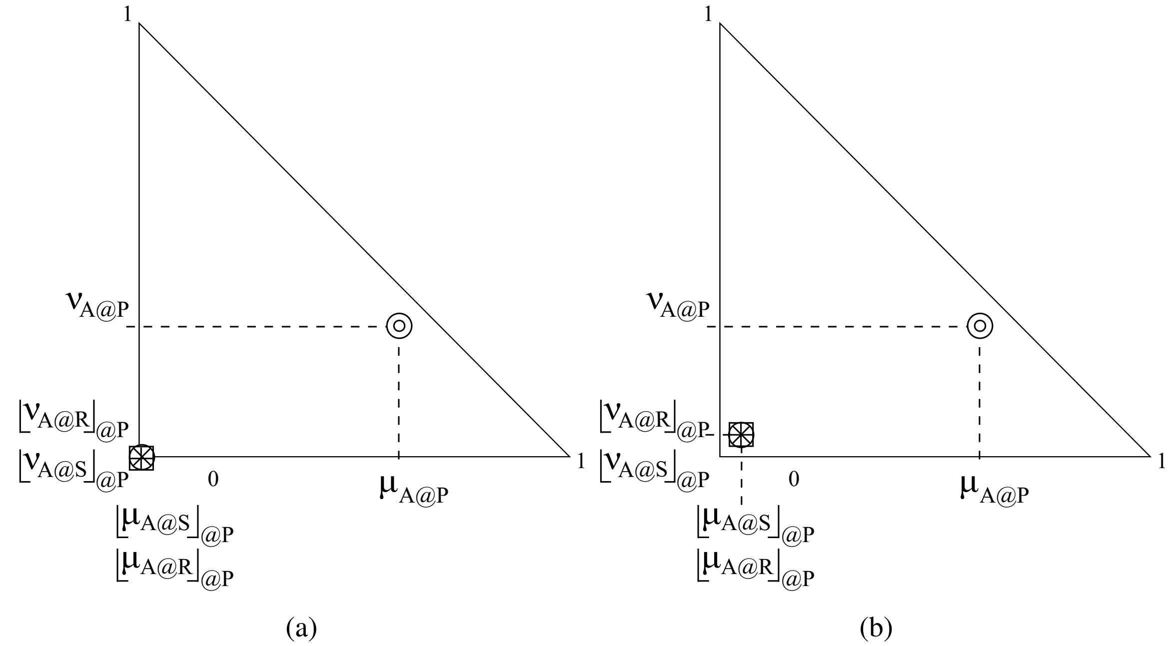


Figure 10. Evaluations given by R and S as seen from the perspective of P: Case $4a$ and Case $4b$.

As could be noticed in the previous cases, the values assigned to the CAFs can have a significant effect on the results computed by (8) and (9). Since such results can be used to compare XBEs in forthcoming processes like filtering or clustering, those values should be carefully assigned. Hence, the study of methods to approximate the level to which the contexts of XBEs are perceived as alike, see e.g., [6], is suggested and subject to further research.

4 Conclusions

In this paper, a semantic interpretation of the universal operator $X_{a,b,c,d,e,f}$ has been presented as a novel way for computing an approximation of how an *augmented appraisal degree* (AAD) is perceived from a different perspective.

It has been shown that some of the parameters of the operator $X_{a,b,c,d,e,f}$ can be related to concepts like *connotation alikeness factor* or *hesitation splitter*, which have been proposed to handle AADs characterizing *experience-based evaluations* (XBEs) given by persons with different knowledge or expertise. The effect of those parameters on the computed approximations has been illustrated through AADs characterizing XBEs given from multiple perspectives.

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