ICIFSS, 8–10 January 2018, Erode, Tamil Nadu, India
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 24, 2018, No. 1, 48–59
DOI: 10.7546/nifs.2018.24.1.48-59

On the intuitionistic fuzzy polynomial ideals of a ring

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Received: 19 September 2017 **Revised:** 26 November 2017 **Accepted:** 30 November 2017

Abstract: In this paper we introduce the notion of intuitionistic fuzzy polynomial ideal A_x of a polynomial ring $R[x]$ induced by an intuitionistic fuzzy ideal A of a ring R, and obtain an isomorphism theorem of a ring of intuitionistic fuzzy cosets of A_x. It is shown that an intuitionistic fuzzy ideal A of a ring is an intuitionistic fuzzy prime if and only if A_x is an intuitionistic fuzzy prime ideal of $R[x]$. Moreover, we show that if A_x is an intuitionistic fuzzy maximal ideal of $R[x]$, then A is an intuitionistic fuzzy maximal ideal of R but converse is not true.
Keywords: Intuitionistic fuzzy polynomial ideal, Intuitionistic fuzzy ideal, f-invariant, Intuitionistic fuzzy prime (maximal) ideal.
2010 Mathematics Subject Classification: 03E72, 03F55, 13F20.

1 Introduction

One of the remarkable generalizations of the fuzzy sets is the intuitionistic fuzzy sets which was introduced by Atanassov [1, 2]. Biswas was the first one to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Later on, Hur and others in [6] and [7] defined and studied intuitionistic fuzzy subrings and ideals of a ring. With a different approach, Mukerjee and Basnet in [4] also studied intuitionistic fuzzy subrings of a ring. Jun and others in [8] introduced and study the notion of intuitionistic nil radicals of intuitionistic fuzzy ideals and Euclidean intuitionistic fuzzy ideals in rings. Translate of intuitionistic fuzzy subring and ideal was studied by Sharma in [14]. Meena and Thomas in [13] studied the concept of intuitionistic fuzzy subring to lattice setting and introduced the notion of intuitionistic L-fuzzy subring. The concept of characteristic intuitionistic fuzzy subrings of an intuitionistic fuzzy ring was introduced by Meena in [12]. In this paper, we introduce the notion of intuitionistic fuzzy polynomial ideal of a ring and study some of their properties.

2 Preliminaries

In this section, we review some definitions which will be used in the later section. Throughout this paper unless stated otherwise all rings are commutative rings with identity.

Definition 2.1. ([3, 4]) Let R be a ring. An IFS $A = (\mu_A, \nu_A)$ of R is said to be an intuitionistic fuzzy ideal (IFI) of R if

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y), \forall x, y \in R$.

Definition 2.2. ([8]) Let R and S be any sets and let $f : R \to S$ be a function. An IFS A of R is called an f-invariant if $f(x) = f(y) \Rightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, where $x, y \in R$.

If A is any f-invariant IFS of R, then $f^{-1}(f(A)) = A$.

The following results are easy to prove:

Lemma 2.3. *Let R and S be any sets and $f : R \to S$ be any function. If A and B are IFS of R and S respectively are f-invariant, then $A \cup B$ and $A \cap B$ are f-invariant.*

Lemma 2.4. *Let R and S be any sets and $f : R \to S$ be any function. Let A and B be f-invariant IFS of R. If $A \subseteq B$, then $f(A) \subseteq f(B)$.*

Theorem 2.5. *Let $f : R \to R'$ be a homomorphism of rings. If A and B are f-invariant IFS of R and R' respectively. Then*

(i) $(f(A))_* = f(A_*)$

(ii) $(f^{-1}(B))_* = f^{-1}(B_*)$.

Let R be a commutative ring with identity and let $R[x]$ be the ring of polynomials where x is an indeterminate.

Definition 2.6. Let $f : R \to R'$ be a homomorphism of rings. A map $f_x : R[x] \to R'[x]$ defined by

$$f_x(\Sigma_{i=0}^{n} a_i x^i) = \Sigma_{i=0}^{n} f(a_i) x^i,$$

is obviously a ring homomorphism, and we call f_x an induced homomorphism by f.

3 Intuitionistic fuzzy polynomial ideals

In this section, we introduce the notion of intuitionistic fuzzy polynomial ideal of a ring and study their properties. The set of all real numbers is denoted by **R**.

Lemma 3.1. *Let* $a_i, b_i \in \boldsymbol{R}(i = 1, 2, \ldots, n)$. *Then*
$\min_i[\min\{a_i, b_i\}] = \min\{\min_i(a_i), \min_i(b_i)\}$ *and* $\max_i[\max\{a_i, b_i\}] = \max\{\max_i(a_i), \max_i(b_i)\}$

Proof. Straightforward. □

Lemma 3.2. *Let* $a_i, b_i \in \boldsymbol{R}(i = 1, 2, \ldots, n)$. *Then*
$\min_i[\max\{a_i, b_i\}] \geq \max\{\min_i(a_i), \min_i(b_i)\}$ *and* $\max_i[\min\{a_i, b_i\}] \leq \min\{\max_i(a_i), \max_i(b_i)\}$

Proof. Straightforward. □

Lemma 3.3. *Let* $A = (\mu_A, \nu_A)$ *be an IFI of a ring* $\boldsymbol{R}$. *Then*
$\mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \geq \max\{\min_i(\mu_A(a_i)), \min_i\{\mu_A(b_i)\}\}$ *and*
$\nu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \leq \min\{\max_i(\nu_A(a_i)), \max_i\{\nu_A(b_i)\}\}, \forall a_i, b_i \in \boldsymbol{R}$.

Proof. Since $A = (\mu_A, \nu_A)$ be an IFI of a ring **R**. for any $a_i, b_i \in \mathbf{R}(i = 1, 2, \ldots, n)$.
By Definition (2.1), we have

$$\begin{aligned}
\mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) &\geq \min_i\{\mu_A\{a_ib_{n+1-i}\}\} \\
&\geq \min_i\{\max\{\mu_A(a_i), \mu_A(b_{n+1-i})\}\} \\
&\geq \max\{\min_i\{\mu_A(a_i), \mu_A(b_i)\}\} \text{ and} \\
\nu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) &\leq \max_i\{\nu_A\{a_ib_{n+1-i}\}\} \\
&\leq \max_i\{\min\{\nu_A(a_i), \nu_A(b_{n+1-i})\}\} \\
&\leq \min\{\max_i\{\nu_A(a_i), \nu_A(b_i)\}\}.
\end{aligned}$$

□

Theorem 3.4. *Let* $A = (\mu_A, \nu_A)$ *be an IFI of a ring* R *and let* $f(x) = \Sigma_{i=0}^n a_ix^i \in R[x]$.
Define an IFS $A_x = (\mu_{A_x}, \nu_{A_x})$ *on* $R[x]$ *by*

$$\mu_{A_x}(f(x)) = \min_i\{\mu_A(a_i)\} \text{ and } \nu_{A_x}(f(x)) = \max_i\{\nu_A(a_i)\}.$$

Then A_x *is an IFI of* $R[x]$.

Proof. By Definition (2.1), we show that A_x is an IFI of $R[x]$.
Let $f(x) = \Sigma_{i=0}^n a_ix^i$ and $g(x) = \Sigma_{i=0}^n b_ix^i \in R[x]$. Then by Lemma (3.1), we have

$$\begin{aligned}
\mu_{A_x}(f(x) - g(x)) &= \min_i\{\mu_A(c_i)\}, \text{ where } c_i = a_i - b_i \\
&= \min_i\{\mu_A(a_i - b_i)\} \\
&\geq \min_i\{\min\{\mu_A(a_i), \mu_A(b_i)\}\} \\
&= \min\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_A(b_i)\}\} \\
&= \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}.
\end{aligned}$$

Thus, $\mu_{A_x}(f(x) - g(x)) \geq \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}$. Similarly, we can show that $\nu_{A_x}(f(x) - g(x)) \leq \max\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\}$. Also,

$$\begin{aligned}
\mu_{A_x}(f(x)g(x)) &= \min_i\{\mu_A(d_i)\}, \text{ where } d_i = \sum_i^{n+m} a_i b_{n+m-i} \\
&= \min_i\{\max\{\mu_A(a_i)\}, \{\mu_A(b_{n+m-i})\}\} \\
&\geq \min_i\{\max\{\mu_A(a_i), \mu_A(b_i)\}\} \\
&\geq \max\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_A(b_i)\}\} \\
&= \max\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}.
\end{aligned}$$

Thus, $\mu_{A_x}(f(x)g(x)) \geq \max\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}$. Similarly, we can show that $\nu_{A_x}(f(x)g(x)) \leq \min\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\}$. This proves that A_x is an IFI of $R[x]$. $\square$

Definition 3.5. The intuitionistic fuzzy ideal A_x discussed in Theorem (3.4) is called the intuitionistic fuzzy polynomial ideal (IFPI) of $R[x]$ induced by an intuitionistic fuzzy ideal A.

Proposition 3.6. *Let $f : R \to R'$ be a homomorphism of rings and let $f_x : R[x] \to R'[x]$ be an induced homomorphism of f. If A is an IFI of the ring R and A_x be its IFPI of $R[x]$, then A is f-invariant if and only if A_x is f_x-invariant.*

Proof. Assume that A is f-invariant. Let $f_x(r(x)) = f_x(s(x))$, where $r(x) = \Sigma_{i=0}^m a_i x^i$ and $s(x) = \Sigma_{i=0}^m b_i x^i \in R[x]$. Then $\Sigma_{i=0}^m f(a_i)x^i = \Sigma_{i=0}^m f(b_i)x^i \Rightarrow f(a_i) = f(b_i), \forall i = 1, 2, \ldots, m$. Hence $\mu_{A_x}(r(x)) = \min_i\{\mu_A(a_i)\} = \min_i\{\mu_A(b_i)\} = \mu_{A_x}(s(x))$ and $\nu_{A_x}(r(x)) = \max_i\{\nu_A(a_i)\}$ $= \max_i\{\nu_A(b_i)\} = \nu_{A_x}(s(x))$. Thus, A_x is f_x-invariant.

Conversely, assume that A_x is an f_x-invariant. If $f(a) = f(b)$, then $f_x(a) = f_x(b)$. Since A_x is an f_x-invariant. So, we have $\mu_{A_x}(a) = \mu_{A_x}(b)$ and $\nu_{A_x}(a) = \nu_{A_x}(b)$, which implies that $\mu_A(a) = \mu_A(b)$ and $\nu_A(a) = \nu_A(b)$. Thus A is f-invariant. $\square$

Proposition 3.7. *Let A be an IFI of the ring R. Then the set*

$$S = \{f(x) \in R[x] : \mu_{A_x}(f(x)) = \mu_{A_x}(0) \textit{ and } \nu_{A_x}(f(x)) = \nu_{A_x}(0)\}$$

is a subring of $R[x]$.

Proof. Let $f(x), g(x)$ be any two element of S, then

$$\mu_{A_x}(f(x) - g(x)) \geq \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0)$$

and

$$\mu_{A_x}(f(x)g(x)) \geq \max\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0).$$

Similarly, we can show that $\nu_{A_x}(f(x) - g(x)) \leq \max\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\} = \nu_{A_x}(0)$ and $\nu_{A_x}(f(x)g(x)) \leq \min\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\} = \nu_{A_x}(0)$.

On the other hand, $\mu_{A_x}(f(x)) \leq \mu_{A_x}(0)$ and $\nu_{A_x}(f(x)) \geq \nu_{A_x}(0), \forall f(x) \in R[x]$.

So, $f(x) - g(x), f(x)g(x) \in S$. Thus, S is a subring of $R[x]$. $\square$

Remark 3.8. Let A be an IFS of a ring R. We denote a level cut set A_* by

$$A_* = \{x \in R : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}.$$

It is proved in [11] that if A is an IFI of ring R, then A_* is an ideal of ring R. Note that if A is an IFI of a ring R, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in R$.

We denote $A_*[x] = \{f(x) = \Sigma_{i=0}^{n} a_i x^i \in R[x] : \text{ where } a_i \in A_*, \forall i = 1, 2, \ldots, n\}$.

Theorem 3.9. *Let A be an IFI of a ring R, then $(A_x)_* = A_*[x]$.*

Proof. It follows that

$$\begin{aligned}(A_x)_* &= \{f(x) \in R[x] : f(x) = \Sigma_{i=0}^{n} a_i x^i, \mu_{A_x}(f(x)) = \mu_{A_x}(0) \text{ and } \nu_{A_x}(f(x)) = \nu_{A_x}(0)\} \\ &= \{f(x) \in R[x] : f(x) = \Sigma_{i=0}^{n} a_i x^i, \min_i\{\mu_A(a_i)\} = \mu_A(0) \text{ and } \nu_A(a_i) = \nu_A(0)\} \\ &= \{f(x) \in R[x] : f(x) = \Sigma_{i=0}^{n} a_i x^i, \mu_A(a_i) = \mu_A(0) \text{ and } \nu_A(a_i) = \nu_A(0), \forall i\} \\ &= \{f(x) \in R[x] : f(x) = \Sigma_{i=0}^{n} a_i x^i, a_i \in A_*, \forall i\} = A_*[x].\end{aligned}$$

□

Theorem 3.10. *If A and B are two IFIs of a ring R, then*

(i) $(A \cap B)_x = A_x \cap B_x$.

(ii) $(A \cup B)_x \supseteq A_x \cup B_x$.

(iii) $A_x + B_x \subseteq (A + B)_x$.

(iv) $A_x B_x \subseteq (AB)_x$.

Proof. Let $f(x) = \Sigma_{i=0}^{n} a_i x^i$ be any element of $R[x]$, then
(i) $(A \cap B)_x(f(x)) = (\mu_{(A\cap B)_x}(f(x)), \nu_{(A\cap B)_x}(f(x)))$, where

$$\begin{aligned}\mu_{(A\cap B)_x} &= \min_i\{\mu_{(A\cap B)}(a_i)\} \\ &= \min_i\{\min\{\mu_A(a_i), \mu_B(a_i)\}\} \\ &= \min\{\min_i\{\mu_A(a_i), \mu_B(a_i)\}\} [\text{ Using Lemma (3.1)}] \\ &= \min\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_B(a_i)\}\} \\ &= \min\{\mu_{A_x}(f(x)), \mu_{B_x}(f(x))\} \\ &= \mu_{A_x \cap B_x}(f(x)).\end{aligned}$$

Similarly, we can show that $\nu_{(A\cap B)_x} = \nu_{A_x \cap B_x}(f(x))$. Hence $(A \cap B)_x = A_x \cap B_x$.
(ii) $(A \cup B)_x(f(x)) = (\mu_{(A\cup B)_x}(f(x)), \nu_{(A\cup B)_x}(f(x)))$, where

$$\begin{aligned}\mu_{(A\cup B)_x} &= \min_i\{\mu_{(A\cup B)}(a_i)\} \\ &= \min_i\{\max\{\mu_A(a_i), \mu_B(a_i)\}\} \\ &\geq \max\{\min_i\{\mu_A(a_i), \mu_B(a_i)\}\} [\text{ Using Lemma (3.2)}] \\ &= \max\{\min_i\{\mu_A(a_i)\}, \min_i\{\mu_B(a_i)\}\} \\ &= \max\{\mu_{A_x}(f(x)), \mu_{B_x}(f(x))\} \\ &= \mu_{A_x \cup B_x}(f(x)).\end{aligned}$$

Similarly, we can show that $\nu_{(A\cup B)_x} \leq \nu_{A_x\cup B_x}(f(x))$. Hence $(A\cup B)_x \supseteq A_x \cup B_x$.

(iii) Now, $(A_x + B_x)(f(x)) = (\mu_{A_x+B_x}(f(x)), \nu_{A_x+B_x}(f(x)))$, where

$$\begin{aligned}
\mu_{A_x+B_x}(f(x)) &= \max_{f(x)=g(x)+h(x)} \{\min\{\mu_{A_x}(g(x)), \mu_{B_x}(h(x))\}\}, g(x) = \sum_{i=0}^{p} b_i x^i, h(x) = \Sigma_{i=0}^{p} c_i x^i \\
&= \max_{f(x)=g(x)+h(x)} \{\min\{\min_i\{\mu_A(b_i)\}, \min_i\{\mu_B(c_i)\}\}\} \\
&= \max_{a_i=b_i+c_i} \{\min_i\{\min\{\mu_A(b_i), \mu_B(c_i)\}\}\} \text{ [Using Lemma (3.1)]} \\
&\leq \min_i\{\max_{a_i=b_i+c_i} \{\min\{\mu_A(b_i), \mu_B(c_i)\}\}\} \text{ [Using Lemma (3.1)]} \\
&= \min_i\{\mu_{A+B}(a_i)\} \\
&= \mu_{(A+B)_x}(f(x)).
\end{aligned}$$

Thus, we get $\mu_{A_x+B_x}(f(x)) \leq \mu_{(A+B)_x}(f(x))$. Similarly, we can show that $\nu_{A_x+B_x}(f(x)) \geq \nu_{(A+B)_x}(f(x))$. Hence $A_x + B_x \subseteq (A+B)_x$.

(iv) Now, $(A_xB_x)(f(x)) = (\mu_{A_xB_x}(f(x)), \nu_{A_xB_x}(f(x)))$, where

$$\begin{aligned}
\mu_{A_xB_x}(f(x)) &= Sup_{f(x)=g(x)h(x)}\{\min\{\mu_{A_x}(g(x)), \mu_{B_x}(h(x))\}\}, g(x) = \sum_{i=0}^{n} b_i x^i, h(x) = \sum_{i=0}^{m} c_i x^i, n+m=p \\
&= Sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{min\{\min_i\{\mu_A(b_i)\}, \min_i\{\mu_B(c_{n+m-i})\}\}\} \\
&= Sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{\min_i\{\min\{\mu_A(b_i), \mu_B(c_{n+m-i})\}\}\} \text{ [Using Lemma (3.1)]} \\
&\leq \min_i\{Sup_{a_i=\sum_{i=0}^{n+m=p}(b_i c_{n+m-i})}\{\min\{\mu_A(b_i), \mu_B(c_{n+m-i})\}\}\} \text{ [Using Lemma (3.2)]} \\
&= \min_i\{\mu_{AB}(a_i)\} \\
&= \mu_{(AB)_x}(f(x)).
\end{aligned}$$

Thus, we get $\mu_{A_xB_x}(f(x)) \leq \mu_{(AB)_x}(f(x))$. Similarly, we can show that $\nu_{A_xB_x}(f(x)) \geq \nu_{(AB)_x}(f(x))$. Hence $A_xB_x \subseteq (AB)_x$. □

Theorem 3.11. *Let $f : R \to R'$ be a homomorphism from R onto R'. If A and B are IFIs of R', then*

(i) $f^{-1}(A\cap B) = f^{-1}(A) \cap f^{-1}(B)$

(ii) $f^{-1}(A\cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof. Let $x \in R$ be any element.
(i) Now, $f^{-1}(A\cap B)(x) = (\mu_{f^{-1}(A\cap B)}(x), \nu_{f^{-1}(A\cap B)}(x))$, where

$$\begin{aligned}
\mu_{f^{-1}(A\cap B)}(x) &= \mu_{(A\cap B)}(f(x)) \\
&= \min\{\mu_A(f(x)), \mu_B(f(x))\} \\
&= \min\{\mu_{f^{-1}(A)}(x), \mu_{f^{-1}(B)}(x)\} \\
&= \mu_{f^{-1}(A)\cap f^{-1}(B)}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{f^{-1}(A\cap B)}(x) = \nu_{f^{-1}(A)\cap f^{-1}(B)}(x)$.
Hence $f^{-1}(A\cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) Now, $f^{-1}(A\cup B)(x)=(\mu_{f^{-1}(A\cup B)}(x),\nu_{f^{-1}(A\cup B)}(x))$, where

$$\begin{aligned}\mu_{f^{-1}(A\cup B)}(x) &= \mu_{(A\cup B)}(f(x))\\ &= \max\{\mu_A(f(x)),\mu_B(f(x))\}\\ &= \max\{\mu_{f^{-1}(A)}(x),\mu_{f^{-1}(B)}(x)\}\\ &= \mu_{f^{-1}(A)\cup f^{-1}(B)}(x).\end{aligned}$$

Similarly, we can show that $\nu_{f^{-1}(A\cup B)}(x)=\nu_{f^{-1}(A)\cup f^{-1}(B)}(x)$.
Hence $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$. □

Corollary 3.12. *Let $f:R\to R'$ be a homomorphism from R onto R'. Let f_x be an induced homomorphism of f. If A and B are IFIs of R', then*

(i) $f_x^{-1}((A\cap B)_x)=f_x^{-1}(A_x)\cap f_x^{-1}(B_x)$

(ii) $f_x^{-1}((A\cup B)_x)=f_x^{-1}(A_x)\cup f_x^{-1}(B_x)$.

Proof. *(i)* It follows from Theorem (3.10)*(i)* and Theorem (3.11)*(ii)* that
$f_x^{-1}((A\cap B)_x)=f_x^{-1}(A_x\cap B_x)=f_x^{-1}(A_x)\cap f_x^{-1}(B_x)$.
(ii) By Theorem (3.10)*(ii)*, we have $A_x\cup B_x\subseteq(A\cup B)_x\Rightarrow f_x^{-1}((A\cap B)_x)\subseteq f_x^{-1}((A\cup B)_x)$.
By applying Theorem (3.10)*(ii)* and Theorem (3.11)*(ii)*, we obtain
$f_x^{-1}(A_x)\cap f_x^{-1}(B_x)=f_x^{-1}(A_x\cup B_x)\subseteq f_x^{-1}((A\cup B)_x)$, which proves (ii). □

Theorem 3.13. *Let $f:R\to R'$ be a homomorphism from R onto R'. Let f_x be an induced homomorphism of f. If A is an IFI of R', then $(f^{-1}(A))_x=f_x^{-1}(A_x)$.*

Proof. Let $r(x)=\Sigma_{i=0}^{n}a_ix^i$ be any element of $R[x]$, then we have
Now, $(f^{-1}(A))_x(r(x))=(\mu_{(f^{-1}(A))_x}(r(x)),\nu_{(f^{-1}(A))_x}(r(x)))$, where

$$\mu_{(f^{-1}(A))_x}(r(x))=\min_i\{\mu_{f^{-1}(A)}(a_i)\}=\min_i\{\mu_A(f(a_i))\}=\mu_{A_x}(f_x(r(x)))=\mu_{f_x^{-1}(A_x)}(r(x)).$$

Similarly, we can show that $\nu_{(f^{-1}(A))_x}(r(x))=\nu_{f_x^{-1}(A_x)}(r(x))$.
Hence $(f^{-1}(A))_x=f_x^{-1}(A_x)$. □

Theorem 3.14. *Let $f:R\to R'$ be a homomorphism from R onto R' and let f_x be an induced homomorphism of f. If A is an f-invariant IFIs of R', then $(f(A))_x=f_x(A_x)$.*

Proof. For any polynomial $s(x):=\Sigma_{i=0}^{m}b_ix^i\in R[x]$, we let $h_j(x):=\Sigma_{i=0}^{m}a_{ji}x^i\in R[x]$. Then $A_x(h_j(x))=(\mu_{A_x}(h_j(x)),\nu_{A_x}(h_j(x)))$, where $\mu_{A_x}(h_j(x))=\min_i\{\mu_A(a_{ji})\}$ and $\nu_{A_x}(h_j(x))=\min_i\{\nu_A(a_{ji})\}$. Assume that $f_x(h_j(x))=s(x)$ and $f_x(h_k(x))=s(x)$. Then $\Sigma_{i=0}^{m}(f(a_{ji})x^i=\Sigma_{i=0}^{m}b_ix^i$ and $\Sigma_{i=0}^{m}(f(a_{ki})x^i=\Sigma_{i=0}^{m}b_ix^i$. It follows that $f(a_{ji})=b_i=f(a_{ki}),\forall i=1,2,\ldots,m$.

Hence $\mu_{A_x}(h_j(x))=\min_i\{\mu_A(a_{ji})\}=\min_i\{\mu_A(a_{ki})\}=\mu_{A_x}(h_k(x))$. Similarly, we can show that $\nu_{A_x}(h_j(x))=\nu_{A_x}(h_k(x))$.

Now, $[f_x(A_x)](s(x))=(\mu_{f_x(A_x)}(s(x)),\nu_{f_x(A_x)}(s(x)))$, where

$$\begin{aligned}\mu_{f_x(A_x)}(s(x)) &= Sup\{\mu_{A_x}(h_j(x)):h_j(x)=\Sigma_{i=0}^{m}a_{ji}x^i \text{ such that } f_x(h_j(x))=s(x)\}\\ &= Sup\{\min_i\{\mu_A(a_{ji})\},j=1,2,\ldots\}\\ &= \mu_{A_x}(h_j(x)).\end{aligned}$$

Similarly, we can show that $\nu_{f_x(A_x)}(s(x)) = \nu_{A_x}(h_j(x))$.
Now, for $i = 1, 2, \dots, m$. As A is f-invariant, we have

$$(f(A))(b_i) = (\mu_{f(A)}(b_i), \nu_{f(A)}(b_i)), \text{ where}$$

$\mu_{f(A)}(b_i) = Sup\{\mu_A(a_{ji}), a_{ji} \in R, f(a_{ji}) = b_i\} = \mu_A(a_{0i}) = \mu_A(a_{1i}) = \cdots = \mu_A(a_{ji})$.
Similarly, we have $\nu_{f(A)}(b_i) = \nu_A(a_{0i}) = \nu_A(a_{1i}) = \cdots = \nu_A(a_{ji})$. It follows from Theorem (3.4) that

$$\begin{aligned}
\mu_{(f(A))_x}(s(x)) &= \min_i\{\mu_{f(A)}(b_i)\} \\
&= \min_i\{\mu_{f(A)}(b_0), \mu_{f(A)}(b_1), \dots\} \\
&= \min_i\{\mu_A(a_{j0}), \mu_A(a_{j1}), \dots\} \\
&= \mu_{A_x}\{\Sigma_{i=0}^m a_{ji}\} \\
&= \mu_{f_x(A_x)}(s(x)).
\end{aligned}$$

Similarly, we can show that $\nu_{(f(A))_x}(s(x)) = \nu_{f_x(A_x)}(s(x))$. Hence $(f(A))_x = f_x(A_x)$. □

Definition 3.15. Let A be an IFI of a ring R and let A_x be an intuitionistic fuzzy polynomial ideal of $R[x]$. For any $f(x) \in R[x]$, define an IFS $(f(x) + A_x)$ on $R[x]$ by

$$(f(x) + A_x)(g(x)) = (\mu_{f(x)+A_x}(g(x)), \nu_{f(x)+A_x}(g(x))) \text{ , where}$$

$\mu_{f(x)+A_x}(g(x)) = \mu_{A_x}(f(x)-g(x))$ and $\nu_{f(x)+A_x}(g(x)) = \nu_{A_x}(f(x)-g(x)), \forall f(x), g(x) \in R[x]$.
Then $f(x) + A_x$ is called an intuitionistic fuzzy coset of $R[x]$ determined by $f(x)$ and A_x.

Theorem 3.16. *Let A be an IFI of a ring R and let A_x be an intuitionistic fuzzy polynomial ideal of $R[x]$. Then $R[x]/A_x$, the set of all intuitionistic fuzzy cosets of A_x form a ring under the composition defined by*

$$(f(x) + A_x) + (g(x) + A_x) := (f(x) + g(x)) + A_x \textit{ and}$$

$$(f(x) + A_x)(g(x) + A_x) := (f(x)g(x)) + A_x, \forall f(x), g(x) \in R[x].$$

Proof. Straightforward result. □

Lemma 3.17. *Let A be an IFI of a ring R and let A_x be an intuitionistic fuzzy polynomial ideal of $R[x]$. Then $f(x) + A_x = g(x) + A_x$ if and only if $A_x(f(x) - g(x)) = A_x(0)$, for all $f(x), g(x) \in R[x]$.*

Proof. Firstly, assume that $f(x)+A_x = g(x)+A_x$. Then $(f(x)+A_x)(f(x)) = (g(x)+A_x)(f(x))$ implies that $(\mu_{A_x}(f(x) - f(x)), \nu_{A_x}(f(x) - f(x))) = (\mu_{A_x}(g(x) - f(x)), \nu_{A_x}(g(x) - f(x)))$
i.e., $(\mu_{A_x}(0), \nu_{A_x}(0)) = (\mu_{A_x}(g(x) - f(x)), \nu_{A_x}(g(x) - f(x)))$
$\Rightarrow \mu_{A_x}(g(x) - f(x)) = \mu_{A_x}(0)$ and $\nu_{A_x}(g(x) - f(x))) = \nu_{A_x}(0))$
$\Rightarrow A_x(g(x) - f(x)) = A_x(0)$.

Conversely, assume that $A_x(g(x) - f(x)) = A_x(0)$, for all $f(x), g(x) \in R[x]$.

Consider $h(x) \in R[x]$ be any element, then we have

$$(f(x) + A_x)(h(x)) = (\mu_{f(x)+A_x}(h(x)), \nu_{f(x)+A_x}(h(x))),$$

where

$$\begin{aligned}\mu_{f(x)+A_x}(h(x)) &= \mu_{A_x}(h(x)-f(x))\\ &= \mu_{A_x}(h(x)-g(x)+g(x)-f(x))\\ &\geq \min\{\mu_{A_x}(h(x)-g(x)),\mu_{A_x}(g(x)-f(x))\}\\ &= \min\{\mu_{A_x}(h(x)-g(x)),\mu_{A_x}(0)\}\\ &= \mu_{A_x}(h(x)-g(x))\\ &= \mu_{g(x)+A_x}(h(x)).\end{aligned}$$

Similarly, we can show that $\nu_{f(x)+A_x}(h(x)) \leq \nu_{g(x)+A_x}(h(x))$. Thus $g(x)+A_x \subseteq f(x)+A_x$. In a same way, we can show that $f(x)+A_x \subseteq g(x)+A_x$. Which complete the proof. □

Theorem 3.18. *Let A be an IFI of a ring R and let A_x be an IFPI of $R[x]$. Then*

$$R[x]/A_x \cong R[x]/A_*[x].$$

Proof. Define an map $\gamma : R[x] \to R[x]/A_x$ by $\gamma(f(x)) = f(x)+A_x, \forall f(x) \in R[x]$. Then it is easy to see that the map γ is an epimorphism of rings with $Ker\gamma$, where

$$\begin{aligned}Ker\gamma &= \{f(x)\in R[x] : \gamma(f(x)) = A_x\}\\ &= \{f(x)\in R[x] : f(x)+A_x = A_x\}\\ &= \{f(x)\in R[x] : A_x(f(x)-0) = A_x(0)\}\\ &= \{f(x)\in R[x] : A_x(f(x)) = A_x(0)\}\\ &= \{f(x)\in R[x] : \mu_{A_x}(f(x)) = \mu_{A_x}(0) \text{ and } \nu_{A_x}(f(x)) = \nu_{A_x}(0)\}\\ &= \{f(x)\in R[x] : f(x)\in (A_x)_*\}\\ &= \{f(x)\in R[x] : f(x)\in A_*[x]\}\\ &= A_*[x].\end{aligned}$$

The result follows by first theorem of homomorphism of rings. □

4 Prime and maximal intuitionistic fuzzy polynomial ideals

In this section, we study some properties of the prime and maximal intuitionistic fuzzy polynomial ideals.

Definition 4.1. An intuitionistic fuzzy ideal P of a ring R, not necessary constant, is said to be an intuitionistic fuzzy prime ideal, if for any IFIs A and B of R the condition $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

Proposition 4.2. *Let A is an intuitionistic fuzzy prime ideal of a ring R, then A_* is a prime ideal of R.*

Proposition 4.3. *Let J be an ideal of a ring R such that $J \neq R$. Then J is a prime ideal of R if and only if the IFS $A = (\mu_A, \nu_A)$ on R defined by*

$$\mu_A(x) = \begin{cases} 1, & \textit{if } x \in J \\ \alpha, & \textit{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \textit{if } x \in J \\ \beta, & \textit{if otherwise} \end{cases}, \alpha, \beta \in [0,1] \textit{ such that } \alpha + \beta \leq 1,$$

is an intuitionistic fuzzy prime ideal of R.

Theorem 4.4. *Let A be an IFI of a ring R. Then A is an intuitionistic fuzzy prime ideal of R if and only if A_x is an intuitionistic fuzzy prime ideal of $R[x]$.*

Proof. Let A be an intuitionistic fuzzy prime ideal of R, then A_* is a prime ideal of R. By Theorem (3.4), A_x is an intuitionistic fuzzy ideal of $R[x]$. To show that A_x is an intuitionistic fuzzy prime ideal of $R[x]$, we have to show that, by Theorem (3.9), $(A_x)_* = A_*[x]$ is a prime ideal of a ring $R[x]$.

Assume that $A_*[x]$ is not a prime ideal of $R[x]$. Then there exists polynomials $f(x) := \Sigma_{i=0}^{n} a_i x^i, g(x) := \Sigma_{i=0}^{m} b_i x^i \in R[x]$ such that $f(x)g(x) \in A_*[x]$, but $f(x), g(x) \notin A_*[x]$.

Let i be the first smallest non-negative integer such that $\mu_A(a_i) \neq \mu_A(0)$ and $\nu_A(a_i) \neq \nu_A(0)$ and let j be the first smallest non-negative integer such that $\mu_A(b_i) \neq \mu_A(0)$ and $\nu_A(b_i) \neq \nu_A(0)$. Since $f(x)g(x) \in A_*[x]$ implies that $\Sigma_{p,q=0,p+q=i+j}^{i+j} a_p b_q \in A_*$, since a_p (where $p = 0, 1, \ldots, i-1$) and b_p (where $p = 0, 1, \ldots, j-1$) are all in A_*, we have $a_i b_j \in A_*$. Since A_* is prime ideal of R, either $\mu_A(a_i) = \mu_A(0)$ and $\nu_A(a_i) = \nu_A(0)$ or $\mu_A(b_i) = \mu_A(0)$ and $\nu_A(b_i) = \nu_A(0)$, a contradiction. Thus A_x is an intuitionistic fuzzy prime ideal of $R[x]$.

Conversely, assume that A_x is an intuitionistic fuzzy prime ideal of $R[x]$. We claim that A_x is a prime ideal of R. Let $a, b \in R$ such that $ab \in A_*$. Then $(ax)(bx) = abx^2 \in A_*[x] = (A_x)_*$. Since $(A_x)_*$ is a prime ideal of $R[x]$, either $(ax) \in (A_x)_*$ or $(bx) \in (A_x)_*$, which shows that either $a \in A_*$ or $b \in A_*$. This proves that A is an intuitionistic fuzzy prime ideal of R. □

Theorem 4.5. *Let $f : R \to R'$ be an epimorphism from R onto R' and let B be an intuitionistic fuzzy prime ideal of R' if and only if $f^{-1}(B)$ is an intuitionistic fuzzy prime ideal of R.*

Proof. Firstly, assume that B is an intuitionistic fuzzy prime ideal of R'. Then B_* is a prime ideal of R'. Clearly, $f^{-1}(B)$ is an IFI of R. We claim that $(f^{-1}(B))_*$ is a prime ideal of R. Let $a, b \in R$ be any element such that $ab \in (f^{-1}(B))_*$. Then $\mu_{f^{-1}(B)}(ab) = \mu_{f^{-1}(B)}(0)$ and $\nu_{f^{-1}(B)}(ab) = \nu_{f^{-1}(B)}(0)$, i.e., $\mu_B(f(ab)) = \mu_B(0')$ and $\nu_B(f(ab)) = \nu_B(0') \Rightarrow f(a)f(b) = f(ab) \in B_*$. Since B_* is a prime ideal of R', either $f(a) \in B_*$ or $f(b) \in B_*$. Which means that either $\mu_B(f(a)) = \mu_B(0')$ and $\nu_B(f(a)) = \nu_B(0')$ or $\mu_B(f(b)) = \mu_B(0')$ and $\nu_B(f(b)) = \nu_B(0')$, i.e., either $\mu_{f^{-1}B}(a) = \mu_{f^{-1}B}(0)$ and $\nu_{f^{-1}B}(a) = \nu_{f^{-1}B}(0)$ or $\mu_{f^{-1}B}(b) = \mu_{f^{-1}B}(0)$ and $\nu_{f^{-1}B}(b) = \nu_{f^{-1}B}(0)$, i.e., either $a \in (f^{-1})(B)_*$ or $b \in (f^{-1})(B)_*$. □

Theorem 4.6. *Let $f : R \to R'$ be an epimorphism from R onto R' and let A be an f-invariant intuitionistic fuzzy ideal of R. Then A is an intuitionistic fuzzy prime ideal of R if and only if $f(A_*)$ is an intuitionistic fuzzy prime ideal of R'.*

Proof. Firstly, assume that A is an intuitionistic fuzzy prime ideal of R. Then A_* is a prime ideal of R. Let $x, y \in R'$ such that $xy \in f(A_*)$. Since f is onto, there exists $c \in A_*$ such that $f(c) = xy$ and there exists $a, b \in R$ such that $f(a) = x, f(b) = y$. Since $f(ab) = f(a)f(b) = xy = f(c)$.

As A is f-invariant, therefore, $\mu_A(ab) = \mu_A(c) = \mu_A(0)$ and $\nu_A(ab) = \nu_A(c) = \nu_A(0)$. Thus $ab \in A_*$. Since A_* is a prime ideal of R, either $a \in A_*$ or $b \in A_*$, which shows that either $x = f(a) \in f(A_*)$ or $y = f(b) \in f(A_*)$. Hence $f(A_*)$ is a prime ideal of $R^{'}$.

Conversely assume that $f(A_*)$ is a prime ideal of $R^{'}$ and let $a, b \in R$ such that $ab \in A_*$. Thus $f(a)f(b) = f(ab) \in f(A_*)$. Since $f(A_*)$ is a prime ideal of $R^{'}$, either $f(a) \in f(A_*)$ or $f(b) \in f(A_*)$, which implies that either there exist $a^{'} \in A_*$ such that $f(a) = f(a^{'})$ or there exist $b^{'} \in A_*$ such that $f(b) = f(b^{'})$. Since A is f-invariant, either $\mu_A(a) = \mu_A(a^{'}) = \mu_A(0)$ and $\nu_A(a) = \nu_A(a^{'}) = \nu_A(0)$ or $\mu_A(b) = \mu_A(b^{'}) = \mu_A(0)$ and $\nu_A(b) = \nu_A(b^{'}) = \nu_A(0)$, i.e., either $a \in A_*$ or $b \in A_*$. Hence A_* is a prime ideal of R and hence A is an intuitionistic fuzzy prime ideal of R. □

Corollary 4.7. *Let $f : R \to R^{'}$ be an epimorphism from R onto $R^{'}$ and let A be an f-invariant intuitionistic fuzzy ideal of R. Then A is an intuitionistic fuzzy prime ideal of R if and only if $f(A)$ is an intuitionistic fuzzy prime ideal of $R^{'}$*

Corollary 4.8. *Let $f : R \to R^{'}$ be an epimorphism from R onto $R^{'}$, f_x be an induced homomorphism of f. Then an IFI B of $R^{'}$ is an intuitionistic fuzzy prime ideal of $R^{'}$ if and only if $f_x^{-1}(B_x)$ is an intuitionistic fuzzy prime ideal of $R[x]$.*

Corollary 4.9. *Let $f : R \to R^{'}$ be an epimorphism from R onto $R^{'}$, f_x be an induced homomorphism of f. Then an IFI A of R is an intuitionistic fuzzy prime ideal of R if and only if $f_x(A_x)$ is an intuitionistic fuzzy prime ideal of $R^{'}[x]$.*

Definition 4.10. ([10]) A non-constant intuitionistic fuzzy ideal A of a ring R is called an intuitionistic fuzzy maximal ideal if for any intuitionistic fuzzy ideal B of R, if $A \subseteq B$, then either $B_* = A_*$ or $B_* = R$.

Theorem 4.11. *Let A be a non-constant intuitionistic fuzzy ideal of a ring R. Then A_x is maximal intuitionistic fuzzy ideal of $R[x]$, then A is an intuitionistic fuzzy maximal ideal of R.*

Proof. Let A (non-constant) and B be IFIs of a ring R such that $A \subseteq B$ which implies A_x and B_x are IFIs of $R[x]$ such that $Ax \subseteq B_x$. Now, A_x is maximal intuitionistic fuzzy ideal of $R[x]$ then either $(B_x)_* = (A_x)_*$ or $(B_x)_* = R[x]$, i.e., either $B_*[x] = A_*[x]$ or $B_*[x] = R[x]$, i.e., either $B_* = A_*$ or $B_* = R$. Hence A is an intuitionistic fuzzy maximal ideal of R. □

Example 4.12. Let **Z** be the set of all integers. Define and IFS A on **Z** by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 2\mathbf{Z} \\ 0, & \text{if otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 2\mathbf{Z} \\ 1, & \text{if otherwise} \end{cases}.$$

Then A is an intuitionistic fuzzy maximal ideal of **Z**, for if B be any other IFI of **Z** such that $A \subseteq B$, then $B_* = A_* = 2\mathbf{Z}$ or $B_* = \mathbf{Z}$.

But $(A_x)_* = A_*[x] = \{f(x) : f(x) = \Sigma_{i=0}^{n} a_i x^i, a_i \in A_*\}$ = $\langle 2 \rangle$ is not a maximal ideal of $\mathbf{Z}[x]$, since $\langle 2 \rangle \subseteq \langle 2, x \rangle \subseteq \mathbf{Z}[x]$. Hence A_x is not an intuitionistic fuzzy maximal ideal of $\mathbf{Z}[x]$.

Acknowledgements

The second author would like to thank IKG PT University, Jalandhar for providing the opportunity to do research work.

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