On the intuitionistic fuzzy polynomial ideals of a ring

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Abstract: In this paper we introduce the notion of intuitionistic fuzzy polynomial ideal \(A_x\) of a polynomial ring \(R[x]\) induced by an intuitionistic fuzzy ideal \(A\) of a ring \(R\), and obtain an isomorphism theorem of a ring of intuitionistic fuzzy cosets of \(A_x\). It is shown that an intuitionistic fuzzy ideal \(A\) of a ring is an intuitionistic fuzzy prime if and only if \(A_x\) is an intuitionistic fuzzy prime ideal of \(R[x]\). Moreover, we show that if \(A_x\) is an intuitionistic fuzzy maximal ideal of \(R[x]\), then \(A\) is an intuitionistic fuzzy maximal ideal of \(R\) but converse is not true.

Keywords: Intuitionistic fuzzy polynomial ideal, Intuitionistic fuzzy ideal, \(f\)-invariant, Intuitionistic fuzzy prime (maximal) ideal.

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1 Introduction

One of the remarkable generalizations of the fuzzy sets is the intuitionistic fuzzy sets which was introduced by Atanassov [1, 2]. Biswas was the first one to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of
a group in [5]. Later on, Hur and others in [6] and [7] defined and studied intuitionistic fuzzy subrings and ideals of a ring. With a different approach, Mukerjee and Basnet in [4] also studied intuitionistic fuzzy subrings of a ring. Jun and others in [8] introduced and study the notion of intuitionistic nil radicals of intuitionistic fuzzy ideals and Euclidean intuitionistic fuzzy ideals in rings. Translate of intuitionistic fuzzy subring and ideal was studied by Sharma in [14]. Meena and Thomas in [13] studied the concept of intuitionistic fuzzy subring to lattice setting and introduced the notion of intuitionistic L-fuzzy subring. The concept of characteristic intuitionistic fuzzy subrings of an intuitionistic fuzzy ring was introduced by Meena in [12]. In this paper, we introduce the notion of intuitionistic fuzzy polynomial ideal of a ring and study some of their properties.

2 Preliminaries

In this section, we review some definitions which will be used in the later section. Throughout this paper unless stated otherwise all rings are commutative rings with identity.

**Definition 2.1.** ([3, 4]) Let $R$ be a ring. An IFS $A = (\mu_A, \nu_A)$ of $R$ is said to be an intuitionistic fuzzy ideal (IFI) of $R$ if

(i) $\mu_A(x - y) \geq \mu_A(x) \land \mu_A(y)$ and $\nu_A(x - y) \leq \nu_A(x) \lor \nu_A(y);

(ii) $\mu_A(xy) \geq \mu_A(x) \lor \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \land \nu_A(y)$, $\forall x, y \in R$.

**Definition 2.2.** ([8]) Let $R$ and $S$ be any sets and let $f : R \rightarrow S$ be a function. An IFS $A$ of $R$ is called an $f$-invariant if $f(x) = f(y) \Rightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, where $x, y \in R$.

If $A$ is any $f$-invariant IFS of $R$, then $f^{-1}(f(A)) = A$.

The following results are easy to prove:

**Lemma 2.3.** Let $R$ and $S$ be any sets and $f : R \rightarrow S$ be any function. If $A$ and $B$ are IFS of $R$ and $S$ respectively are $f$-invariant, then $A \cup B$ and $A \cap B$ are $f$-invariant.

**Lemma 2.4.** Let $R$ and $S$ be any sets and $f : R \rightarrow S$ be any function. Let $A$ and $B$ be $f$-invariant IFS of $R$. If $A \subseteq B$, then $f(A) \subseteq f(B)$.

**Theorem 2.5.** Let $f : R \rightarrow R'$ be a homomorphism of rings. If $A$ and $B$ are $f$-invariant IFS of $R$ and $R'$ respectively. Then

(i) $(f(A))_* = f(A_*)$

(ii) $(f^{-1}(B))_* = f^{-1}(B_*)$.

Let $R$ be a commutative ring with identity and let $R[x]$ be the ring of polynomials where $x$ is an indeterminate.

**Definition 2.6.** Let $f : R \rightarrow R'$ be a homomorphism of rings. A map $f_x : R[x] \rightarrow R'[x]$ defined by

$$f_x(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i,$$

is obviously a ring homomorphism, and we call $f_x$ an induced homomorphism by $f$. 49
3 \textbf{Intuitionistic fuzzy polynomial ideals}

In this section, we introduce the notion of intuitionistic fuzzy polynomial ideal of a ring and study their properties. The set of all real numbers is denoted by $\mathbf{R}$.

\textbf{Lemma 3.1.} Let $a_i, b_i \in \mathbf{R} \ (i = 1, 2, \ldots, n)$. Then
\[ \min_i \{\min \{a_i, b_i\}\} = \min_i \{\min_i (a_i), \min_i (b_i)\} \quad \text{and} \quad \max_i \{\max \{a_i, b_i\}\} = \max_i \{\max_i (a_i), \max_i (b_i)\} \]

\textit{Proof.} Straightforward. \hfill \Box

\textbf{Lemma 3.2.} Let $a_i, b_i \in \mathbf{R} \ (i = 1, 2, \ldots, n)$. Then
\[ \min_i \{\max \{a_i, b_i\}\} \geq \max_i \{\min_i (a_i), \min_i (b_i)\} \quad \text{and} \quad \max_i \{\min \{a_i, b_i\}\} \leq \min_i \{\max_i (a_i), \max_i (b_i)\} \]

\textit{Proof.} Straightforward. \hfill \Box

\textbf{Lemma 3.3.} Let $A = (\mu_A, \nu_A)$ be an IFI of a ring $\mathbf{R}$. Then
\[ \mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \geq \max_i \{\min_i(\mu_A(a_i)), \min_i(\mu_A(b_i))\} \quad \text{and} \quad \nu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \leq \min_i \{\max_i(\nu_A(a_i)), \max_i(\nu_A(b_i))\}, \forall a_i, b_i \in \mathbf{R}. \]

\textit{Proof.} Since $A = (\mu_A, \nu_A)$ be an IFI of a ring $\mathbf{R}$. for any $a_i, b_i \in \mathbf{R} \ (i = 1, 2, \ldots, n)$. By Definition (2.1), we have
\[ \mu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \geq \min_i \min_i (\mu_A(a_i), \mu_A(b_i)) \]
\[ \geq \min_i \{\max_i(\mu_A(a_i)), \max_i(\mu_A(b_i))\} \quad \text{and} \quad \nu_A(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) \leq \min_i \min_i (\nu_A(a_i), \nu_A(b_i)) \]
\[ \leq \min_i \{\max_i(\nu_A(a_i)), \max_i(\nu_A(b_i))\}. \]

\textbf{Theorem 3.4.} Let $A = (\mu_A, \nu_A)$ be an IFI of a ring $\mathbf{R}$ and let $f(x) = \sum_{i=0}^{n} a_ix^i \in \mathbf{R}[x]$. Define an IFS $A_x = (\mu_A, \nu_A_x)$ on $\mathbf{R}[x]$ by
\[ \mu_A_x(f(x)) = \min_i \{\mu_A(a_i)\} \quad \text{and} \quad \nu_A_x(f(x)) = \max_i \{\nu_A(a_i)\}. \]

Then $A_x$ is an IFI of $\mathbf{R}[x]$. \hfill \Box

\textit{Proof.} By Definition (2.1), we show that $A_x$ is an IFI of $\mathbf{R}[x]$. Let $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{i=0}^{n} b_ix^i \in \mathbf{R}[x]$. Then by Lemma (3.1), we have
\[ \mu_A_x(f(x) - g(x)) = \min_i \{\mu_A(c_i)\}, \text{ where } c_i = a_i - b_i \]
\[ = \min_i \{\mu_A(a_i - b_i)\} \]
\[ \geq \min_i \{\min_i \{\mu_A(a_i), \mu_A(b_i)\}\} \]
\[ = \min_i \{\min_i \{\mu_A(a_i), \min_i \{\mu_A(b_i)\}\}\} \]
\[ = \min_i \{\mu_A_x(f(x)), \mu_A_x(g(x))\}. \]

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Thus, $\mu_{A_x}(f(x) - g(x)) \geq \min\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}$. Similarly, we can show that $\nu_{A_x}(f(x) - g(x)) \leq \max\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\}$. Also,

\[
\mu_{A_x}(f(x)g(x)) = \min_i \{\mu_A(d_i)\}, \text{ where } d_i = \sum_{i} a_i b_{n+m-i} \\
= \min_i \{\max_i \{\mu_A(a_i), \mu_A(b_i)\}\} \\
\geq \min_i \{\max_i \{\mu_A(a_i), \nu_{A_x}(g(x))\}\} \\
\geq \max_i \{\min_i \{\mu_A(a_i), \min_i \{\nu_{A_x}(g(x))\}\}\} \\
= \max_i \{\nu_{A_x}(f(x)), \mu_{A_x}(g(x))\}.
\]

Thus, $\mu_{A_x}(f(x)g(x)) \geq \max\{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\}$. Similarly, we can show that $\nu_{A_x}(f(x)g(x)) \leq \min\{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\}$. This proves that $A_x$ is an IFI of $R[x]$. \qed

**Definition 3.5.** The intuitionistic fuzzy ideal $A_x$ discussed in Theorem (3.4) is called the intuitionistic fuzzy polynomial ideal (IFPI) of $R[x]$ induced by an intuitionistic fuzzy ideal $A$.

**Proposition 3.6.** Let $f : R \rightarrow R'$ be a homomorphism of rings and let $f_x : R[x] \rightarrow R'[x]$ be an induced homomorphism of $f$. If $A$ is an IFI of the ring $R$ and $A_x$ is its IFPI of $R[x]$, then $A$ is $f$-invariant if and only if $A_x$ is $f_x$-invariant.

**Proof.** Assume that $A$ is $f$-invariant. Let $f_x(r(x)) = f_x(s(x))$, where $r(x) = \sum_{i=0}^{m} a_i x^i$ and $s(x) = \sum_{i=0}^{m} b_i x^i \in R[x]$. Then $\sum_{i=0}^{m} f_x(a_i) x^i = \sum_{i=0}^{m} f_x(b_i) x^i \Rightarrow f(a_i) = f(b_i), \forall i = 1, 2, \ldots, m$. Hence $\mu_{A_x}(r(x)) = \min_i \{\mu_A(a_i)\} = \min_i \{\mu_A(b_i)\} = \mu_{A_x}(s(x))$ and $\nu_{A_x}(r(x)) = \max_i \{\nu_A(a_i)\} = \nu_{A_x}(s(x))$. Thus, $A_x$ is $f_x$-invariant.

Conversely, assume that $A_x$ is an $f_x$-invariant. If $f(a) = f(b)$, then $f_x(a) = f_x(b)$. Since $A_x$ is an $f_x$-invariant. So, we have $\mu_{A_x}(a) = \mu_{A_x}(b)$ and $\nu_{A_x}(a) = \nu_{A_x}(b)$, which implies that $\mu_A(a) = \mu_A(b)$ and $\nu_A(a) = \nu_A(b)$. Thus $A$ is $f$-invariant. \qed

**Proposition 3.7.** Let $A$ be an IFI of the ring $R$. Then the set

\[
S = \{f(x) \in R[x] : \mu_{A_x}(f(x)) = \mu_{A_x}(0) \text{ and } \nu_{A_x}(f(x)) = \nu_{A_x}(0)\}
\]

is a subring of $R[x]$.

**Proof.** Let $f(x), g(x)$ be any two element of $S$, then

\[
\mu_{A_x}(f(x) - g(x)) \geq \min \{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0)
\]

and

\[
\mu_{A_x}(f(x)g(x)) \geq \max \{\mu_{A_x}(f(x)), \mu_{A_x}(g(x))\} = \mu_{A_x}(0).
\]

Similarly, we can show that $\nu_{A_x}(f(x) - g(x)) \leq \max \{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\} = \nu_{A_x}(0)$ and $\nu_{A_x}(f(x)g(x)) \leq \min \{\nu_{A_x}(f(x)), \nu_{A_x}(g(x))\} = \nu_{A_x}(0)$.

On the other hand, $\mu_{A_x}(f(x)) \leq \mu_{A_x}(0) \text{ and } \nu_{A_x}(f(x)) \geq \nu_{A_x}(0), \forall f(x) \in R[x].$

So, $f(x) - g(x), f(x)g(x) \in S$. Thus, $S$ is a subring of $R[x]$. \qed
Remark 3.8. Let $A$ be an IFS of a ring $R$. We denote a level cut set $A_*$ by

$$A_* = \{ x \in R : \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0) \}.$$ 

It is proved in [11] that if $A$ is an IFS of ring $R$, then $A_*$ is an ideal of ring $R$. Note that if $A$ is an IFS of a ring $R$, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in R$.

We denote $A_*[x] = \{ f(x) = \sum_{i=0}^{n} a_i x^i \in R[x] : \text{ where } a_i \in A_*, \forall i = 1, 2, \ldots, n \}$.

**Theorem 3.9.** Let $A$ be an IFS of a ring $R$, then $(A_*[x])_x = A_*[x]$.

**Proof.** It follows that

$$(A_*[x])_x = \{ f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x^i, \mu_{A_*}(f(x)) = \mu_{A_*}(0) \text{ and } \nu_{A_*}(f(x)) = \nu_{A_*}(0) \}$$

$$(A_*[x])_x = \{ f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x^i, \min_i \{ \mu_A(a_i) \} = \mu_A(0) \text{ and } \nu_A(a_i) = \nu_A(0) \}$$

$$(A_*[x])_x = \{ f(x) \in R[x] : f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in A_*, \forall i \} = A_*[x].$$

**Theorem 3.10.** If $A$ and $B$ are two IFIs of a ring $R$, then

(i) $(A \cap B)_x = A_x \cap B_x$.

(ii) $(A \cup B)_x \supseteq A_x \cup B_x$.

(iii) $A_x + B_x \subseteq (A + B)_x$.

(iv) $A_x B_x \subseteq (AB)_x$.

**Proof.** Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be any element of $R[x]$, then

(i) $(A \cap B)_x(f(x)) = (\mu_{(A \cap B)_x}(f(x)), \nu_{(A \cap B)_x}(f(x)))$, where

$$\mu_{(A \cap B)_x} = \min_i \{ \mu_{(A \cap B)}(a_i) \}$$

$$= \min_i \{ \min_i \{ \mu_A(a_i), \mu_B(a_i) \} \}$$

$$= \min_i \{ \min_i \{ \mu_A(a_i), \mu_B(a_i) \} \} \text{ [Using Lemma (3.1)]}$$

$$= \min_i \{ \mu_A(a_i), \min_i \{ \mu_B(a_i) \} \}$$

$$= \min_i \{ \mu_A(a_i), \nu_{B_x}(f(x)) \}$$

$$= \mu_{A \cap B_x}(f(x)).$$

Similarly, we can show that $\nu_{(A \cap B)_x} = \nu_{A \cap B_x}(f(x))$. Hence $(A \cap B)_x = A_x \cap B_x$.

(ii) $(A \cup B)_x(f(x)) = (\mu_{(A \cup B)_x}(f(x)), \nu_{(A \cup B)_x}(f(x)))$, where

$$\mu_{(A \cup B)_x} = \min_i \{ \mu_{(A \cup B)}(a_i) \}$$

$$= \min_i \{ \max_i \{ \mu_A(a_i), \mu_B(a_i) \} \}$$

$$\geq \max_i \{ \min_i \{ \mu_A(a_i), \mu_B(a_i) \} \} \text{ [Using Lemma (3.2)]}$$

$$= \max_i \{ \min_i \{ \mu_A(a_i), \min_i \{ \mu_B(a_i) \} \} \}$$

$$= \max_i \{ \mu_{A_x}(f(x)), \nu_{B_x}(f(x)) \}$$

$$= \mu_{A_x \cup B_x}(f(x)).$$
Similarly, we can show that $\nu_{(A \cup B)_x} \leq \nu_{A_x \cup B_x}(f(x))$. Hence $(A \cup B)_x \supseteq A_x \cup B_x$.

(iii) Now, $(A_x + B_x)(f(x)) = (\mu_{A_x + B_x}(f(x)), \nu_{A_x + B_x}(f(x)))$, where

$$
\mu_{A_x + B_x}(f(x)) = \max_{f(x) = g(x) + h(x)} \{ \min \{ \mu_A(g(x)), \mu_B(h(x)) \} \},
$$

$$
g(x) = \sum_{i=0}^{p} b_i x^i, h(x) = \sum_{i=0}^{p} c_i x^i
$$

$$
= \max_{f(x) = g(x) + h(x)} \{ \min \{ \mu_A(b_i), \min \{ \mu_B(c_i) \} \} \}
$$

$$
= \max_{a_i = b_i + c_i} \{ \min \{ \mu_A(b_i), \min \{ \mu_B(c_i) \} \} \} \text{ [Using Lemma (3.1)]}
$$

$$
\leq \min_{i} \max_{a_i = b_i + c_i} \{ \min \{ \mu_A(b_i), \min \{ \mu_B(c_i) \} \} \} \text{ [Using Lemma (3.1)]}
$$

$$
= \min_{i} \{ \mu_A(b_i) \}
$$

$$
= \mu((A + B)_x)(f(x)).
$$

Thus, we get $\mu_{A_x + B_x}(f(x)) \leq \mu_{(A + B)_x}(f(x))$. Similarly, we can show that $\nu_{A_x + B_x}(f(x)) \geq \nu_{(A + B)_x}(f(x))$. Hence $A_x + B_x \subseteq (A + B)_x$.

(iv) Now, $(A_x B_x)(f(x)) = (\mu_{A_x B_x}(f(x)), \nu_{A_x B_x}(f(x)))$, where

$$
\mu_{A_x B_x}(f(x)) = \sup_{f(x) = g(x)h(x)} \{ \min \{ \mu_A(g(x)), \mu_B(h(x)) \} \},
$$

$$
g(x) = \sum_{i=0}^{n} b_i x^i, h(x) = \sum_{i=0}^{m} c_i x^i, n + m = p
$$

$$
= \sup_{a_i = \sum_{i=0}^{n} b_i x^i c_i x^i} \{ \min \{ \mu_A(b_i), \min \{ \mu_B(c_i) \} \} \}
$$

$$
= \sup_{a_i = \sum_{i=0}^{n} b_i x^i c_i x^i} \{ \min \{ \mu_A(b_i), \min \{ \mu_B(c_i) \} \} \} \text{ [Using Lemma (3.1)]}
$$

$$
\leq \min_{i} \{ \mu_A(b_i) \}
$$

$$
= \mu((A B)_x)(f(x)).
$$

Thus, we get $\mu_{A_x B_x}(f(x)) \leq \mu_{(A B)_x}(f(x))$. Similarly, we can show that $\nu_{A_x B_x}(f(x)) \geq \nu_{(A B)_x}(f(x))$. Hence $A_x B_x \subseteq (A B)_x$. 

\[\square\]

**Theorem 3.11.** Let $f : R \rightarrow R'$ be a homomorphism from $R$ onto $R'$. If $A$ and $B$ are IFIs of $R'$, then

(i) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

**Proof.** Let $x \in R$ be any element.

(i) Now, $f^{-1}(A \cap B)(x) = (\mu_{f^{-1}(A \cap B)(x)}, \nu_{f^{-1}(A \cap B)(x)})$, where

$$
\mu_{f^{-1}(A \cap B)}(x) = \mu_{(A \cap B)}(f(x))
$$

$$
= \min \{ \mu_A(f(x)), \mu_B(f(x)) \}
$$

$$
= \min \{ \mu_{f^{-1}(A)}(x), \mu_{f^{-1}(B)}(x) \}
$$

$$
= \mu_{f^{-1}(A) \cap f^{-1}(B)}(x).
$$

Similarly, we can show that $\nu_{f^{-1}(A \cap B)}(x) = \nu_{f^{-1}(A) \cap f^{-1}(B)}(x)$.

Hence $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. 

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(ii) Now, \( f^{-1}(A \cup B)(x) = (\mu_{f^{-1}(A \cup B)}(x), \nu_{f^{-1}(A \cup B)}(x)) \), where

\[
\begin{align*}
\mu_{f^{-1}(A \cup B)}(x) &= \mu_{(A \cup B)}(f(x)) \\
&= \max \{ \mu_{A}(f(x)), \mu_{B}(f(x)) \} \\
&= \max \{ \mu_{f^{-1}(A)}(x), \mu_{f^{-1}(B)}(x) \} \\
&= \mu_{f^{-1}(A) \cup f^{-1}(B)}(x).
\end{align*}
\]

Similarly, we can show that \( \nu_{f^{-1}(A \cup B)}(x) = \nu_{f^{-1}(A) \cup f^{-1}(B)}(x) \).

Hence \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \). \( \square \)

**Corollary 3.12.** Let \( f : R \to R' \) be a homomorphism from \( R \) onto \( R' \). Let \( f_x \) be an induced homomorphism of \( f \). If \( A \) and \( B \) are IFIs of \( R' \), then

(i) \( f^{-1}_x((A \cap B)_x) = f^{-1}_x(A_x) \cap f^{-1}_x(B_x) \)

(ii) \( f^{-1}_x((A \cup B)_x) = f^{-1}_x(A_x) \cup f^{-1}_x(B_x) \).

**Proof.** (i) It follows from Theorem (3.10)(i) and Theorem (3.11)(ii) that \( f^{-1}_x((A \cap B)_x) = f^{-1}_x(A_x \cap B_x) = f^{-1}_x(A_x) \cap f^{-1}_x(B_x) \).

(ii) By Theorem (3.10)(ii), we have \( A_x \cup B_x \subseteq (A \cup B)_x \Rightarrow f^{-1}_x((A \cap B)_x) \subseteq f^{-1}_x((A \cup B)_x) \).

By applying Theorem (3.10)(ii) and Theorem (3.11)(ii), we obtain \( f^{-1}_x(A_x) \cap f^{-1}_x(B_x) = f^{-1}_x(A_x \cup B_x) \subseteq f^{-1}_x((A \cup B)_x) \), which proves (ii). \( \square \)

**Theorem 3.13.** Let \( f : R \to R' \) be a homomorphism from \( R \) onto \( R' \). Let \( f_x \) be an induced homomorphism of \( f \). If \( A \) is an IFI of \( R' \), then \( (f^{-1}(A))_x = f^{-1}_x(A_x) \).

**Proof.** Let \( r(x) = \sum_{i=0}^{n} a_i x^i \) be any element of \( R[x] \), then we have

Now, \( (f^{-1}(A))_x(r(x)) = (\mu_{(f^{-1}(A))_x}(r(x)), \nu_{(f^{-1}(A))_x}(r(x))) \), where

\[
\mu_{(f^{-1}(A))_x}(r(x)) = \min_i \{ \mu_{f^{-1}(A)}(a_i) \} = \min_i \{ \mu_{A}(f(a_i)) \} = \mu_{A_x}(f_x(r(x))) = \mu_{f^{-1}(A_x)}(r(x)).
\]

Similarly, we can show that \( \nu_{(f^{-1}(A))_x}(r(x)) = \nu_{f^{-1}(A_x)}(r(x)) \).

Hence \( (f^{-1}(A))_x = f^{-1}_x(A_x) \). \( \square \)

**Theorem 3.14.** Let \( f : R \to R' \) be a homomorphism from \( R \) onto \( R' \) and let \( f_x \) be an induced homomorphism of \( f \). If \( A \) is an \( f \)-invariant IFIs of \( R' \), then \( (f(A))_x = f_x(A_x) \).

**Proof.** For any polynomial \( s(x) := \sum_{i=0}^{m} b_i x^i \in R[x] \), we let \( h_j(x) := \sum_{i=0}^{m} a_{ji} x^i \in R[x] \). Then \( A_x(h_j(x)) = \mu_{A_x}(h_j(x)), \nu_{A_x}(h_j(x)) \), where \( \mu_{A_x}(h_j(x)) = \min_i \{ \nu_{A}(a_{ji}) \} \) and \( \nu_{A_x}(h_j(x)) = \min_i \{ \mu_{A}(a_{ji}) \} \).

Assume that \( f_x(h_j(x)) = s(x) \) and \( f_x(h_k(x)) = s(x) \). Then \( \sum_{i=0}^{m} (a_{ji}) x^i = \sum_{i=0}^{m} b_i x^i \) and \( \sum_{i=0}^{m} f(a_{ji}) x^i = \sum_{i=0}^{m} b_i x^i \). It follows that \( f(a_{ji}) = b_i = f(a_{ki}), \forall i = 1, 2, \ldots, m \).

Hence \( \mu_{A_x}(h_j(x)) = \min_i \{ \nu_{A}(a_{ji}) \} = \min_{j=1,2,\ldots} \{ \nu_{A}(a_{ki}) \} = \mu_{A_x}(h_k(x)) \). Similarly, we can show that \( \nu_{A_x}(h_j(x)) = \nu_{A_x}(h_k(x)) \).

Now, \( [f_x(A_x)](s(x)) = (\mu_{f_x(A_x)}(s(x)), \nu_{f_x(A_x)}(s(x))) \), where

\[
\begin{align*}
\mu_{f_x(A_x)}(s(x)) &= \sup \{ \mu_{A_x}(h_j(x)) : h_j(x) = \sum_{i=0}^{m} a_{ji} x^i \text{ such that } f_x(h_j(x)) = s(x) \} \\
&= \sup \{ \min \{ \mu_{A}(a_{ji}) \} \} \\
&= \mu_{A_x}(h_j(x)).
\end{align*}
\]
Similarly, we can show that \( \nu_{f_x(A_x)}(s(x)) = \nu_{A_x}(h_j(x)) \).

Now, for \( i = 1, 2, \ldots, m \). As \( A \) is \( f \)-invariant, we have
\[
(f(A))(b_i) = (\mu_{f(A)}(b_i), \nu_{f(A)}(b_i)), \text{ where }
\]
\[
\mu_{f(A)}(b_i) = \sup \{ \mu_A(a_{ji}), a_{ji} \in R, f(a_{ji}) = b_i \} = \mu_A(a_{0i}) = \mu_A(a_{1i}) = \cdots = \mu_A(a_{ji}).
\]
Similarly, we have \( \nu_{f(A)}(b_i) = \nu_A(a_{0i}) = \nu_A(a_{1i}) = \cdots = \nu_A(a_{ji}) \). It follows from Theorem (3.4) that
\[
\mu_{(f(A))_x}(s(x)) = \min_i \{ \mu_{f(A)}(b_i) \} = \min \{ \mu_{f(A)}(b_0), \mu_{f(A)}(b_1), \ldots \} = \min \{ \mu_A(a_{j0}), \mu_A(a_{j1}), \ldots \} = \mu_A_x \{ \sum_{i=0}^m a_{ji} \} = \mu_{f_x(A_x)}(s(x)).
\]

Similarly, we can show that \( \nu_{(f(A))_x}(s(x)) = \nu_{f_x(A_x)}(s(x)) \). Hence \( (f(A))_x = f_x(A_x) \). \( \Box \)

**Definition 3.15.** Let \( A \) be an IFI of a ring \( R \) and let \( A_x \) be an intuitionistic fuzzy polynomial ideal of \( R[x] \). For any \( f(x) \in R[x] \), define an IFS \( (f(x) + A_x) \) on \( R[x] \) by
\[
(f(x) + A_x)(g(x)) = (\mu_{f(x)+A_x}(g(x)), \nu_{f(x)+A_x}(g(x))),
\]
where \( \mu_{f(x)+A_x}(g(x)) = \mu_{A_x}(f(x) - g(x)) \) and \( \nu_{f(x)+A_x}(g(x)) = \nu_{A_x}(f(x) - g(x)) \), for all \( f(x), g(x) \in R[x] \).

Then \( f(x) + A_x \) is called an intuitionistic fuzzy coset of \( R[x] \) determined by \( f(x) \) and \( A_x \).

**Theorem 3.16.** Let \( A \) be an IFI of a ring \( R \) and let \( A_x \) be an intuitionistic fuzzy polynomial ideal of \( R[x] \). Then \( R[x]/A_x \), the set of all intuitionistic fuzzy cosets of \( A_x \) form a ring under the composition defined by
\[
(f(x) + A_x) + (g(x) + A_x) := (f(x) + g(x)) + A_x \text{ and }
\]
\[
(f(x) + A_x)(g(x) + A_x) := (f(x)g(x)) + A_x, \forall f(x), g(x) \in R[x].
\]

**Proof.** Straightforward result. \( \Box \)

**Lemma 3.17.** Let \( A \) be an IFI of a ring \( R \) and let \( A_x \) be an intuitionistic fuzzy polynomial ideal of \( R[x] \). Then \( f(x) + A_x = g(x) + A_x \) if and only if \( A_x(f(x) - g(x)) = A_x(0) \), for all \( f(x), g(x) \in R[x] \).

**Proof.** Firstly, assume that \( f(x) + A_x = g(x) + A_x \). Then \( (f(x) + A_x)(f(x)) = (g(x) + A_x)(f(x)) \) implies that \( (\mu_{A_x}(f(x) - f(x)), \nu_{A_x}(f(x) - f(x))) = (\mu_{A_x}(g(x) - f(x)), \nu_{A_x}(g(x) - f(x))) \).
i.e., \( (\mu_{A_x}(0), \nu_{A_x}(0)) = (\mu_{A_x}(g(x) - f(x)), \nu_{A_x}(g(x) - f(x))) \)
\[
\Rightarrow \mu_{A_x}(g(x) - f(x)) = \mu_{A_x}(0) \text{ and } \nu_{A_x}(g(x) - f(x)) = \nu_{A_x}(0) \]
\[
\Rightarrow A_x(g(x) - f(x)) = A_x(0).
\]

Conversely, assume that \( A_x(g(x) - f(x)) = A_x(0) \), for all \( f(x), g(x) \in R[x] \).

Consider \( h(x) \in R[x] \) be any element, then we have
\[
(f(x) + A_x)(h(x)) = (\mu_{f(x)+A_x}(h(x)), \nu_{f(x)+A_x}(h(x))),
\]

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where
\[
\mu_{f(x)+A_x}(h(x)) = \mu_{A_x}(h(x) - f(x)) \\
= \mu_{A_x}(h(x) - g(x) + g(x) - f(x)) \\
\geq \min\{\mu_{A_x}(h(x) - g(x)), \mu_{A_x}(g(x) - f(x))\} \\
= \min\{\mu_{A_x}(h(x) - g(x)), \mu_{A_x}(0)\} \\
= \mu_{A_x}(h(x) - g(x)) \\
= \mu_{g(x)+A_x}(h(x)).
\]

Similarly, we can show that \(\nu_{f(x)+A_x}(h(x)) \leq \nu_{g(x)+A_x}(h(x))\). Thus \(g(x) + A_x \subseteq f(x) + A_x\). In a same way, we can show that \(f(x) + A_x \subseteq g(x) + A_x\). Which complete the proof.

\[\square\]

**Theorem 3.18.** Let \(A\) be an IFI of a ring \(R\) and let \(A_x\) be an IFPI of \(R[x]\). Then
\[
R[x]/A_x \cong R[x]/A_x[x].
\]

**Proof.** Define an map \(\gamma : R[x] \rightarrow R[x]/A_x\) by \(\gamma(f(x)) = f(x) + A_x, \forall f(x) \in R[x]\). Then it is easy to see that the map \(\gamma\) is an epimorphism of rings with \(Ker\gamma\), where
\[
Ker\gamma = \{f(x) \in R[x] : \gamma(f(x)) = A_x\} \\
= \{f(x) \in R[x] : f(x) + A_x = A_x\} \\
= \{f(x) \in R[x] : A_x(f(x) - 0) = A_x(0)\} \\
= \{f(x) \in R[x] : A_x(f(x)) = A_x(0)\} \\
= \{f(x) \in R[x] : \mu_{A_x}(f(x)) = \mu_{A_x}(0) \text{ and } \nu_{A_x}(f(x)) = \nu_{A_x}(0)\} \\
= \{f(x) \in R[x] : f(x) \in (A_x)_x\} \\
= \{f(x) \in R[x] : f(x) \in A_x[x]\} \\
= A_x[x].
\]
The result follows by first theorem of homomorphism of rings. \[\square\]

### 4 Prime and maximal intuitionistic fuzzy polynomial ideals

In this section, we study some properties of the prime and maximal intuitionistic fuzzy polynomial ideals.

**Definition 4.1.** An intuitionistic fuzzy ideal \(P\) of a ring \(R\), not necessary constant, is said to be an intuitionistic fuzzy prime ideal, if for any IFIs \(A\) and \(B\) of \(R\) the condition \(AB \subseteq P\) implies that either \(A \subseteq P\) or \(B \subseteq P\).

**Proposition 4.2.** Let \(A\) is an intuitionistic fuzzy prime ideal of a ring \(R\), then \(A_x\) is a prime ideal of \(R\).

**Proposition 4.3.** Let \(J\) be an ideal of a ring \(R\) such that \(J \neq R\). Then \(J\) is a prime ideal of \(R\) if and only if the IFS \(A = (\mu_A, \nu_A)\) on \(R\) defined by
$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ \alpha, & \text{if otherwise} \end{cases}$; $\nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ \beta, & \text{if otherwise} \end{cases}$, $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, is an intuitionistic fuzzy prime ideal of $R$.

**Theorem 4.4.** Let $A$ be an IFI of a ring $R$. Then $A$ is an intuitionistic fuzzy prime ideal of $R$ if and only if $A_x$ is an intuitionistic fuzzy prime ideal of $R[x]$.

**Proof.** Let $A$ be an intuitionistic fuzzy prime ideal of $R$, then $A_x$ is a prime ideal of $R$. By Theorem (3.4), $A_x$ is an intuitionistic fuzzy prime ideal of $R[x]$. To show that $A_x$ is an intuitionistic fuzzy prime ideal of $R[x]$, we have to show that, by Theorem (3.9), $(A_x)_* = A_x[x]$ is a prime ideal of a ring $R[x]$.

Assume that $A_x[x]$ is not a prime ideal of $R[x]$. Then there exists polynomials $f(x) := \Sigma_{i=0}^n a_i x^i$, $g(x) := \Sigma_{i=0}^m b_i x^i \in R[x]$ such that $f(x)g(x) \in A_x[x]$, but $f(x), g(x) \notin A_x[x]$.

Let $i$ be the first smallest non-negative integer such that $\mu_A(a_i) \neq \mu_A(0)$ and $\nu_A(a_i) \neq \nu_A(0)$ and let $j$ be the first smallest non-negative integer such that $\mu_A(b_i) \neq \mu_A(0)$ and $\nu_A(b_i) \neq \nu_A(0)$. Since $f(x)g(x) \in A_x[x]$ implies that $\Sigma_{p=0}^i q_{-i} a_p b_q \in A_x$, since $a_p$ (where $p = 0, 1, \ldots, i-1$) and $b_p$ (where $p = 0, 1, \ldots, j-1$) are all in $A_x$, we have $a_i b_j \in A_x$. Since $A_x$ is prime ideal of $R$, either $\mu_A(a_i) = \mu_A(0)$ and $\nu_A(a_i) = \nu_A(0)$ or $\mu_A(b_i) = \mu_A(0)$ and $\nu_A(b_i) = \nu_A(0)$, a contradiction. Thus $A_x$ is an intuitionistic fuzzy prime ideal of $R[x]$.

Conversely, assume that $A_x$ is an intuitionistic fuzzy prime ideal of $R[x]$. We claim that $A_x$ is a prime ideal of $R$. Let $a, b \in R$ such that $ab \in A_x$. Then $(ax)(bx) = abx^2 \in A_x[x] = (A_x)_*$. Since $(A_x)_*$ is a prime ideal of $R[x]$, either $(ax) \in (A_x)_*$ or $(bx) \in (A_x)_*$, which shows that either $a \in A_x$ or $b \in A_x$. This proves that $A$ is an intuitionistic fuzzy prime ideal of $R$.

**Theorem 4.5.** Let $f : R \rightarrow R'$ be an epimorphism from $R$ onto $R'$ and let $B$ be an intuitionistic fuzzy prime ideal of $R'$ if and only if $f^{-1}(B)$ is an intuitionistic fuzzy prime ideal of $R$.

**Proof.** Firstly, assume that $B$ is an intuitionistic fuzzy prime ideal of $R'$. Then $B_x$ is a prime ideal of $R$. Clearly, $f^{-1}(B)$ is an IFI of $R$. We claim that $(f^{-1}(B))_*$ is a prime ideal of $R$. Let $a, b \in R$ be any element such that $ab \in (f^{-1}(B))_*$. Then $\mu_{f^{-1}(B)}(ab) = \mu_B(a_0)$ and $\nu_{f^{-1}(B)}(ab) = \nu_B(a_0)$, i.e., $\mu_B(f(ab)) = \mu_B(0')$ and $\nu_B(f(ab)) = \nu_B(0') \Rightarrow f(a)f(b) = f(ab) \in B_x$. Since $B_x$ is a prime ideal of $R'$, either $f(a) \in B_x$ or $f(b) \in B_x$. Which means that either $\mu_B(f(a)) = \mu_B(0')$ and $\nu_B(f(a)) = \nu_B(0')$ or $\mu_B(f(b)) = \mu_B(0')$ and $\nu_B(f(b)) = \nu_B(0')$, i.e., either $\mu_{f^{-1}(B)}(a) = \mu_{f^{-1}(B)}(0)$ and $\nu_{f^{-1}(B)}(a) = \nu_{f^{-1}(B)}(0)$ or $\mu_{f^{-1}(B)}(b) = \mu_{f^{-1}(B)}(0)$ and $\nu_{f^{-1}(B)}(b) = \nu_{f^{-1}(B)}(0)$, i.e., either $a \in (f^{-1})(B)_*$ or $b \in (f^{-1})(B)_*$.

**Theorem 4.6.** Let $f : R \rightarrow R'$ be an epimorphism from $R$ onto $R'$ and let $A$ be an $f$-invariant intuitionistic fuzzy ideal of $R$. Then $A$ is an intuitionistic fuzzy prime ideal of $R$ if and only if $f(A_x)$ is an intuitionistic fuzzy prime ideal of $R'$.

**Proof.** Firstly, assume that $A$ is an intuitionistic fuzzy prime ideal of $R$. Then $A_x$ is a prime ideal of $R$. Let $x, y \in R'$ such that $xy \in f(A_x)$. Since $f$ is onto, there exists $c \in A_x$ such that $f(c) = xy$ and there exists $a, b \in R$ such that $f(a) = x, f(b) = y$. Since $f(ab) = f(a)f(b) = xy = f(c)$.
As \( A \) is \( f \)-invariant, therefore, \( \mu_A(ab) = \mu_A(c) = \mu_A(0) \) and \( \nu_A(ab) = \nu_A(c) = \nu_A(0) \). Thus \( ab \in A_* \). Since \( A_* \) is a prime ideal of \( R \), either \( a \in A_* \) or \( b \in A_* \), which shows that either \( x = f(a) \in f(A_*) \) or \( y = f(b) \in f(A_*) \). Hence \( f(A_*) \) is a prime ideal of \( R' \).

Conversely assume that \( f(A_*) \) is a prime ideal of \( R' \) and let \( a, b \in R \) such that \( ab \in A_* \). Thus \( f(a)f(b) = f(ab) \in f(A_*) \). Since \( f(A_*) \) is a prime ideal of \( R' \), either \( f(a) \in f(A_*) \) or \( f(b) \in f(A_*) \), which implies that either there exist \( a' \in A_* \) such that \( f(a) = f(a') \) or there exist \( b' \in A_* \) such that \( f(b) = f(b') \). Since \( A \) is \( f \)-invariant, either \( \mu_A(a) = \mu_A(a') = \mu_A(0) \) and \( \nu_A(a) = \nu_A(a') = \nu_A(0) \) or \( \mu_A(b) = \mu_A(b') = \mu_A(0) \) and \( \nu_A(b) = \nu_A(b') = \nu_A(0) \), i.e., either \( a \in A_* \) or \( b \in A_* \). Hence \( A_* \) is a prime ideal of \( R \) and hence \( A \) is an intuitionistic fuzzy prime ideal of \( R \).

**Corollary 4.7.** Let \( f : R \to R' \) be an epimorphism from \( R \) onto \( R' \) and let \( A \) be an \( f \)-invariant intuitionistic fuzzy ideal of \( R \). Then \( A \) is an intuitionistic fuzzy prime ideal of \( R \) if and only if \( f(A) \) is an intuitionistic fuzzy prime ideal of \( R' \).

**Corollary 4.8.** Let \( f : R \to R' \) be an epimorphism from \( R \) onto \( R' \), \( f_x \) be an induced homomorphism of \( f \). Then an IFI \( B \) of \( R' \) is an intuitionistic fuzzy prime ideal of \( R' \) if and only if \( f_x^{-1}(B_x) \) is an intuitionistic fuzzy prime ideal of \( R[x] \).

**Corollary 4.9.** Let \( f : R \to R' \) be an epimorphism from \( R \) onto \( R' \), \( f_x \) be an induced homomorphism of \( f \). Then an IFI \( A \) of \( R \) is an intuitionistic fuzzy prime ideal of \( R \) if and only if \( f_x(A_x) \) is an intuitionistic fuzzy prime ideal of \( R'[x] \).

**Definition 4.10.** ([10]) A non-constant intuitionistic fuzzy ideal \( A \) of a ring \( R \) is called an intuitionistic fuzzy maximal ideal if for any intuitionistic fuzzy ideal \( B \) of \( R \), if \( A \subseteq B \), then either \( B_* = A_* \) or \( B_* = R \).

**Theorem 4.11.** Let \( A \) be a non-constant intuitionistic fuzzy ideal of a ring \( R \). Then \( A_* \) is maximal intuitionistic fuzzy ideal of \( R[x] \), then \( A \) is an intuitionistic fuzzy maximal ideal of \( R \).

**Proof.** Let \( A \) (non-constant) and \( B \) be IFIs of a ring \( R \) such that \( A \subseteq B \) which implies \( A_x \) and \( B_x \) are IFIs of \( R[x] \) such that \( A_x \subseteq B_x \). Now, \( A_x \) is maximal intuitionistic fuzzy ideal of \( R[x] \) then either \( (B_x)_* = (A_x)_* \) or \( (B_x)_* = R[x] \), i.e., either \( B_*[x] = A_*[x] \) or \( B_*[x] = R[x] \), i.e., either \( B_* = A_* \) or \( B_* = R \). Hence \( A \) is an intuitionistic fuzzy maximal ideal of \( R \).

**Example 4.12.** Let \( Z \) be the set of all integers. Define and IFS \( A \) on \( Z \) by

\[
\mu_A(x) = \begin{cases} 
1, & \text{if } x \in 2\mathbb{Z} \\
0, & \text{if otherwise}
\end{cases} \quad \nu_A(x) = \begin{cases} 
0, & \text{if } x \in 2\mathbb{Z} \\
1, & \text{if otherwise}
\end{cases}
\]

Then \( A \) is an intuitionistic fuzzy maximal ideal of \( Z \), for if \( B \) be any other IFI of \( Z \) such that \( A \subseteq B \), then \( B_* = A_* = 2\mathbb{Z} \) or \( B_* = \mathbb{Z} \).

But \( (A_x)_* = A_*[x] = \{ f(x) : f(x) = \Sigma_{i=0}^{n} a_i x^i, a_i \in A_* \} = \langle 2 \rangle \) is not a maximal ideal of \( Z[x] \), since \( \langle 2 \rangle \subseteq \langle 2, x \rangle \subseteq Z[x] \). Hence \( A_x \) is not an intuitionistic fuzzy maximal ideal of \( Z[x] \).
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References


