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Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space

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Abstract: In the present paper the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space has been developed with the help of intuitionistic fuzzy basis.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy vector space, Intuitionistic fuzzy dimension.

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1 Introduction

The notion of intuitionistic fuzzy set (IFS) was introduced by Atanassov [1, 2, 3, 4] as a generalization of Zadeh's fuzzy set [22]. There are situations where IFS theory is more appropriate to deal with [7]. IFS theory have successfully been applied in knowledge engineering, medical diagnosis, decision making, career determination, etc., [11, 21, 12]. Several researchers have extended various mathematical aspects such as groups, rings, topological spaces, metric spaces, topological groups, topological vector spaces etc. in IFS [6, 10, 13, 16, 17, 18, 19]. The notion of fuzzy vector subspaces has been introduced by Katsaras [14] and a notion of fuzzy bases and fuzzy dimension was studied by Shi *et al.* [20]. We have introduced a notion of intuitionistic fuzzy vector space and intuitionistic fuzzy basis in [9]. As a continuation of our paper [9], here we introduced the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space with the help of intuitionistic fuzzy basis and studied some of its basic results.

2 Preliminaries

Definition 2.1. [1] Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X is defined as an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A , respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\}$.

In this paper, we use the symbols $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Definition 2.2. [1] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a set X . Then

- (1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
- (2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
- (3) $A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$
- (4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X\}$.
- (5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X\}$.
- (6) $\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in X\}$, $\diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in X\}$.

Definition 2.3. [4] Let A be an IFS in a set X . Then for $\lambda, \xi \in [0, 1]$ with $\lambda + \xi \leq 1$, the set $A^{[\lambda, \xi]} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \xi\}$ is called (λ, ξ) -level subset of A .

Proposition 2.4. [4] Let A be an IFS in a set X and $(\lambda_1, \xi_1), (\lambda_2, \xi_2) \in \text{Im}(A)$. If $\lambda_1 \leq \lambda_2$ and $\xi_1 \geq \xi_2$, then $A^{[\lambda_1, \xi_1]} \supseteq A^{[\lambda_2, \xi_2]}$.

Definition 2.5. [15, 5] Let X be a vector space over the field K , the field of real and complex numbers, $\alpha \in K$, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of X . Then

- (1) the sum of A and B is defined to be the intuitionistic fuzzy set $A + B = (\mu_{A+B}, \nu_{A+B})$ of X given by

$$\mu_{A+B}(x) = \begin{cases} \sup_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} & \text{if } x = a + b \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \inf_{x=a+b} \{\nu_A(a) \vee \nu_B(b)\} & \text{if } x = a + b \\ 1 & \text{otherwise.} \end{cases}$$

- (2) αA is defined to be the IFS $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ of X , where

$$\mu_{\alpha A}(x) = \begin{cases} \mu_A(\alpha^{-1}x) & \text{if } \alpha \neq 0 \\ \sup_{y \in X} \mu_A(y) & \text{if } \alpha = 0, x = \theta \\ 0 & \text{if } \alpha = 0, x \neq \theta, \end{cases}$$

$$v_{\alpha A}(x) = \begin{cases} v_A(\alpha^{-1}x) & \text{if } \alpha \neq 0 \\ \inf_{y \in X} v_A(y) & \text{if } \alpha = 0, x = \theta \\ 1 & \text{if } \alpha = 0, x \neq \theta. \end{cases}$$

Proposition 2.6. [9] Let A, A_1, \dots, A_n be intuitionistic fuzzy sets in a vector space X and $\lambda_1, \dots, \lambda_n$ be scalars. Then the following assertions are equivalent:

(1) $\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_n A_n \subseteq A$.

(2) For all x_1, x_2, \dots, x_n in X , we have

$$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n) \geq \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)\} \text{ and } v_A(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n) \leq \max\{v_{A_1}(x_1), v_{A_2}(x_2), \dots, v_{A_n}(x_n)\}.$$

Definition 2.7. [9] An IFS $V = (\mu_V, v_V)$ of a vector space X over the field K is said to be intuitionistic fuzzy vector space over X if

(i) $V + V \subseteq V$

(ii) $\alpha V \subseteq V$, for every scalar α .

We denote the set of all intuitionistic fuzzy vector spaces over a vector space X by $IFVS(X)$.

Remark 2.8. [9] Let X be a vector space.

(1) If μ_V is a fuzzy subspace of X , then $V = (\mu_V, \mu_V^c) \in IFVS(X)$.

(2) If $V \in IFVS(X)$, then μ_V and v_V^c are fuzzy vector subspace of X .

(3) If $V \in IFVS(X)$, then $\square V, \diamond V \in IFVS(X)$.

Lemma 2.9. [9] Let V be an intuitionistic fuzzy set in a vector space X . Then, the following are equivalent:

(1) V is an intuitionistic fuzzy vector space over X .

(2) For all scalars α, β , we have $\alpha V + \beta V \subseteq V$.

(3) For all scalars α, β and for all $x, y \in X$, we have

$$\mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y) \text{ and } v_V(\alpha x + \beta y) \leq v_V(x) \vee v_V(y).$$

Remark 2.10. [9] Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [19] and [8].

Proposition 2.11. [8] If $V, W \in IFVS(X)$, then $V + W \in IFVS(X)$.

Proposition 2.12. [9] If $V \in IFVS(X)$ $\alpha \in K$, then $\alpha V \in IFVS(X)$.

Proposition 2.13. [8] If $\{V_i\}_{i \in I} \in IFVS(X)$, then $\bigcap_{i \in I} V_i \in IFVS(X)$.

Proposition 2.14. [9] Let $V \in IFVS(X)$. Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x)$, $\forall x \in X$.

Proposition 2.15. [9] Let $V \in IFVS(X)$. Then for each $(\lambda, \xi) \in [0, 1] \times [0, 1]$ with $\lambda + \xi \leq 1$, $\lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$, $V^{[\lambda, \xi]}$ is a subspace of the vector space X ,

Definition 2.16. [9] For any $(a, b), (c, d) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $c + d \leq 1$, we say that:

- (1) $(a, b) \geq (c, d)$ if $a \geq b$ and $c \leq d$.
- (2) $(a, b) \leq (c, d)$ if $a \leq b$ and $c \geq d$.
- (3) $(a, b) > (c, d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.
- (4) $(a, b) < (c, d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.
- (5) $(a, b) = (c, d)$ if $a = b$ and $c = d$.

Proposition 2.17. [9] Let $V \in IFVS(X)$ with $\dim X = m$. Then $Im(V)$ contains at most $m + 1$ points of $[0, 1] \times [0, 1]$.

Definition 2.18. [9] Let $V = (\mu_V, \nu_V) \in IFVS(X)$. Then for any $\lambda \in \mu_V(X)$, $\xi \in \nu_V(X)$ we define

$$\mu_V^{[\lambda]} = \{x \in X : \mu_V(x) \geq \lambda\} \text{ and } \nu_V^{[\xi]} = \{x \in X : \nu_V(x) \leq \xi\}, [\lambda 1_{\mu_V^{[\lambda]}}](x) = \begin{cases} \lambda, & \text{if } x \in \mu_V^{[\lambda]} \\ 0, & \text{otherwise} \end{cases},$$

$$[\xi 1_{\nu_V^{[\xi]}}](x) = \begin{cases} \xi, & \text{if } x \in \nu_V^{[\xi]} \\ 1, & \text{otherwise} \end{cases}.$$

Theorem 2.19. [9] (Representation Theorem) Let $V \in IFVS(X)$ with $\dim X = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \dots, (\lambda_k, \xi_k)\}$, $k \leq m$ such that $(1, 0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \dots > (\lambda_k, \xi_k) \geq (0, 1)$. Then there exists nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0, \xi_0]} \subsetneq V^{[\lambda_1, \xi_1]} \subsetneq \dots \subsetneq V^{[\lambda_k, \xi_k]} = X$ such that $\mu_V = \lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee \dots \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}$ and $\nu_V = \xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \dots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}$. Also,

- (1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1})$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta, \rho]} = V^{[\eta, \sigma]} = V^{[\lambda_i, \xi_i]}$.
- (2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1}), (\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta, \rho]} \supsetneq V^{[\eta, \sigma]}$.

Definition 2.20. [9] Let $V \in IFVS(X)$ with $\dim X = m$. Consider Theorem 2.19. Let B_{V_i} be the basis of $V^{[\lambda_i, \xi_i]}$, $i = 0, 1, \dots, k$ such that

$$B_{V_0} \subsetneq B_{V_1} \subsetneq \dots \subsetneq B_{V_k}. \quad (*)$$

If $V^{(\lambda_0, \xi_0)} = \{\theta\}$, we start with $V^{(\lambda_1, \xi_1)}$.

Define a map \mathbb{B} from X to $[0, 1] \times [0, 1]$ by

$$\mu_{\mathbb{B}}(x) = \begin{cases} \vee \{\lambda_i : x \in B_{V_i}\} \\ 0, \text{ otherwise} \end{cases} \text{ and } \nu_{\mathbb{B}}(x) = \begin{cases} \wedge \{\xi_i : x \in B_{V_i}\} \\ 1, \text{ otherwise} \end{cases}.$$

Let $\mu_{\mathbb{B}}(x) = \lambda_j$. Then $x \in B_{V_j}$ and $x \notin B_{V_{j-1}}$ i.e. $x \in V^{[\lambda_j, \xi_j]}$ and $x \notin V^{[\lambda_{j-1}, \xi_{j-1}]}$. Thus $\mu_V(x) \geq \lambda_j$ and $\nu_V(x) \leq \xi_j$. If $\mu_V(x) > \lambda_j$, then $\mu_V(x) = \lambda_l$ for some $l < j$. Then $x \in V^{[\lambda_l, \xi_l]}$ and $\mu_{(\mathbb{B})}(x) = \lambda_l$, which is a contradiction. Therefore $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$ i.e. $\nu_{\mathbb{B}}(x) = \xi_j$. Therefore \mathbb{B} is an intuitionistic fuzzy set and it is called intuitionistic fuzzy basis of V corresponding to $(*)$.

Proposition 2.21. [9] Let \mathbb{B} be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition 2.20. Then

- (1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1})$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $\mathbb{B}^{[\zeta, \rho]} = \mathbb{B}^{[\eta, \sigma]} = B_{V_i}$.
- (2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1}), (\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $\mathbb{B}^{[\zeta, \rho]} \supsetneq \mathbb{B}^{[\eta, \sigma]}$.
- (3) $\mathbb{B}^{[\lambda, \xi]}$ is a basis of $V^{[\lambda, \xi]}$ for $\lambda \in (0, 1], \xi \in [0, 1)$ with $\lambda + \xi \leq 1$.

Proposition 2.22. Let \mathbb{B} be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition 2.20. Then $\mu_{\mathbb{B}}^{[\lambda_i]} = B_{V_i} = \nu_{\mathbb{B}}^{[\xi_i]}$, for $i = 0, 1, 2, \dots, k$.

Proof. Let $x \in \mu_{\mathbb{B}}^{[\lambda_i]} \Rightarrow \mu_{\mathbb{B}}(x) \geq \lambda_i$. Let $\mu_{\mathbb{B}}(x) = \lambda_j \Rightarrow x \in B_{V_j} \subset B_{V_i}$.

Thus $\mu_{\mathbb{B}}^{[\lambda_i]} \subseteq B_{V_i}$. Conversely, let $x \in B_{V_i} \Rightarrow \mu_V(x) \geq \lambda_i$.

Let $\mu_V(x) = \lambda_j$. If $\lambda_j > \lambda_i$, then $\mu_{\mathbb{B}}(x) = \lambda_j$.

If $\lambda_j = \lambda_i$, then $\mu_{\mathbb{B}}(x) \geq \lambda_i$. Therefore, in any case $x \in \mu_{\mathbb{B}}^{[\lambda_i]}$.

Thus $B_{V_i} \subseteq \mu_{\mathbb{B}}^{[\lambda_i]}$. Hence $\mu_{\mathbb{B}}^{[\lambda_i]} = B_{V_i}$.

Similarly, it can be proved that $B_{V_i} = \nu_{\mathbb{B}}^{[\xi_i]}$. □

Proposition 2.23. Let $V \in IFVS(X)$ with $\dim X = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \dots, (\lambda_k, \xi_k)\}, k \leq m$ such that $(1, 0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \dots > (\lambda_k, \xi_k) \geq (0, 1)$. Then for $i = 0, 1, \dots, k$, $V^{[\lambda_i, \xi_i]} = \mu_V^{[\lambda_i]} = \nu_V^{[\xi_i]}$.

Proof. Obviously, $V^{[\lambda_i, \xi_i]} \subseteq \mu_V^{[\lambda_i]}$.

Let $x \in \mu_V^{[\lambda_i]}$.

$\Rightarrow \mu_V(x) \geq \lambda_i$.

Let $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$.

$\Rightarrow x \in V^{[\lambda_j, \xi_j]}$

$\Rightarrow x \in V^{[\lambda_i, \xi_i]}$ [as either $(\lambda_j, \xi_j) = (\lambda_i, \xi_i)$ or $(\lambda_j, \xi_j) > (\lambda_i, \xi_i)$].

Thus $\mu_{\mathbb{B}}^{[\lambda_i]} \subseteq V^{[\lambda_i, \xi_i]}$. Therefore $V^{[\lambda_i, \xi_i]} = \mu_V^{[\lambda_i]}$.

Similarly we have $V^{[\lambda_i, \xi_i]} = \nu_V^{[\xi_i]}$. □

Proposition 2.24. Let \mathbb{B} be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition 2.20. Then $|\mu_{\mathbb{B}}^{[\lambda_i]}| = \dim(\mu_V^{[\lambda_i]})$ and $|\nu_{\mathbb{B}}^{[\xi_i]}| = \dim(\nu_V^{[\xi_i]})$, for $i = 0, 1, 2, \dots, k$.

Proof. $|\mu_{\mathbb{B}}^{[\lambda_i]}| = |B_{V_i}| = \dim(V^{[\lambda_i, \xi_i]}) = \dim(\mu_V^{[\lambda_i]})$ [By Proposition 2.22 and 2.23].

The rest part is similar. □

3 Intuitionistic fuzzy dimension

Definition 3.1. Let A be an intuitionistic fuzzy set over X . Define a map $|A| : \mathbb{N} \rightarrow [0, 1] \times [0, 1]$ such that $\forall n \in \mathbb{N}$, $\mu_{|A|}(n) = \vee\{a : (a, b) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\} \text{ with } a + b \leq 1 \text{ and } |A^{[a,b]}| \geq n\}$ and $\nu_{|A|}(n) = \wedge\{b : (a, b) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\} \text{ with } a + b \leq 1 \text{ and } |A^{[a,b]}| \geq n\}$. Then $|A|$ is an intuitionistic fuzzy set over \mathbb{N} , which is called the cardinality of A .

Definition 3.2. For two IFS A, B over X , the addition $|A| + |B|$ of $|A|$ and $|B|$ is defined as follows: for any $n \in \mathbb{N}$, $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l))$ and $\nu_{(|A|+|B|)}(n) = \wedge_{k+l=n}(\nu_{|A|}(k) \vee \nu_{|B|}(l))$.

Proposition 3.3. For two IFS $|A|, |B|$ over \mathbb{N} and for any $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $\mu_{(|A|+|B|)}^{[a]} = \mu_{|A|}^{[a]} + \mu_{|B|}^{[a]}$ and $\nu_{(|A|+|B|)}^{[b]} = \nu_{|A|}^{[b]} + \nu_{|B|}^{[b]}$.

Proof. First we prove that $\mu_{(|A|+|B|)}^{[a]} \subseteq \mu_{|A|}^{[a]} + \mu_{|B|}^{[a]}$.

Let $n \in \mu_{(|A|+|B|)}^{[a]}$. Then $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$.

Hence there exist k, l such that $n = k + l$ and $\mu_{|A|}(k) \wedge \mu_{|B|}(l) \geq a$. Then $k \in \mu_{|A|}^{[a]}$ and $l \in \mu_{|B|}^{[a]}$, i.e., $n = k + l \in \mu_{|A|}^{[a]} + \mu_{|B|}^{[a]}$. Similarly, it can be proved that $\nu_{(|A|+|B|)}^{[b]} \subseteq \nu_{|A|}^{[b]} + \nu_{|B|}^{[b]}$.

Conversely suppose that $n \in \mu_{|A|}^{[a]} + \mu_{|B|}^{[a]}$.

Then there exist k, l such that $n = k + l$ with $k \in \mu_{|A|}^{[a]}$, $l \in \mu_{|B|}^{[a]}$. Then $(\mu_{|A|})(k) \geq a$, $(\mu_{|B|})(l) \geq a$.

Therefore $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$. Thus $n \in \mu_{(|A|+|B|)}^{[a]}$.

Hence $\mu_{|A|}^{[a]} + \mu_{|B|}^{[a]} \subseteq \mu_{(|A|+|B|)}^{[a]}$.

Similarly, we have $\nu_{|A|}^{[b]} + \nu_{|B|}^{[b]} \subseteq \nu_{(|A|+|B|)}^{[b]}$. Hence proved. \square

Definition 3.4. Let $V \in IFVS(X)$ with an intuitionistic fuzzy basis \mathbb{B} . Define $\dim(V) = |\mathbb{B}|$. Then $\dim(V)$ is called intuitionistic fuzzy dimension of V .

Proposition 3.5. Let \mathbb{B} and \mathbb{B}' be two intuitionistic fuzzy bases of an intuitionistic fuzzy vector space $V \in IFVS(X)$. Then $|\mathbb{B}| = |\mathbb{B}'|$.

Proof. By Proposition 2.21, $\mathbb{B}^{[a,b]}$ and $\mathbb{B}'^{[a,b]}$ are bases of $V^{[a,b]}$ for $a \in (0, 1], b \in [0, 1]$ with $a + b \leq 1$. Then $|\mathbb{B}^{[a,b]}| = |\mathbb{B}'^{[a,b]}|$.

Hence for any $n \in \mathbb{N}$,

$$\begin{aligned} \mu_{|\mathbb{B}|}(n) &= \vee\{a : (a, b) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\} \text{ with } a + b \leq 1 \text{ and } |\mathbb{B}^{[a,b]}| \geq n\} \\ &= \vee\{a : (a, b) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\} \text{ with } a + b \leq 1 \text{ and } |\mathbb{B}'^{[a,b]}| \geq n\} \\ &= \mu_{|\mathbb{B}'|}(n). \end{aligned}$$

Similarly, for any $n \in \mathbb{N}$, $\nu_{|\mathbb{B}|}(n) = \nu_{|\mathbb{B}'|}(n)$. Hence proved. \square

Remark 3.6. Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space is independent of intuitionistic fuzzy basis.

Proposition 3.7. Let X be a vector space with $\dim X = m$ and $V \in IFVS(X)$. Then for any $(a, b) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\}$ with $a + b \leq 1$ and $n \in \mathbb{N}$, $n \in \mu_{\dim(V)}^{[a]} \Leftrightarrow n \leq \dim(\mu_V^{[a]})$ and $n \in \nu_{\dim(V)}^{[b]} \Leftrightarrow n \leq \dim(\nu_V^{[b]})$.

Proof. Suppose that $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \dots, (\lambda_k, \xi_k)\}, k \leq m$ such that $(1, 0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \dots > (\lambda_k, \xi_k) \geq (0, 1)$. Then there exists a nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0, \xi_0]} \subsetneq V^{[\lambda_1, \xi_1]} \subsetneq \dots \subsetneq V^{[\lambda_k, \xi_k]} = X$.

Let B_{V_i} be the basis of $V^{[\lambda_i, \xi_i]}, i = 0, 1, \dots, k$ such that $B_{V_0} \subsetneq B_{V_1} \subsetneq \dots \subsetneq B_{V_k} \dots \dots (*)$.

Let \mathbb{B} be an intuitionistic fuzzy basis corresponding to $(*)$ defined as in Definition 2.20. Let $n \in \mu_{dim(V)}^{[a]} \Rightarrow \mu_{dim(V)}(n) \geq a \Rightarrow \vee \{c : (c, d) \in (0, 1] \times [0, 1] \text{ with } c + d \leq 1 \text{ and } |\mathbb{B}^{[c, d]}| \geq n\} \geq a$.

Then there exists $(c, d) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\}$ with $c + d \leq 1$ such that $c \geq a$ and $|\mathbb{B}^{[c, d]}| \geq n$. Now $dim(\mu_V^{[a]}) = |\mu_{\mathbb{B}}^{[a]}| \geq |\mu_{\mathbb{B}}^{[c]}| \geq |\mathbb{B}^{[c, d]}| \geq n$.

Conversely suppose that $n \leq dim(\mu_V^{[a]}) = |\mu_{\mathbb{B}}^{[a]}|$. Now $a \in (\lambda_{i+1}, \lambda_i]$, for some i . Hence $|\mu_{\mathbb{B}}^{[a]}| = |\mu_{\mathbb{B}}^{[\lambda_i]}| = |B_{V_i}| = |\mathbb{B}^{[\lambda_i, \xi_i]}|$. Then $\mu_{dim(V)}(n) = \vee \{c : (c, d) \in [0, 1] \times [0, 1] \setminus \{(0, 1)\} \text{ with } c + d \leq 1 \text{ and } |\mathbb{B}^{[c, d]}| \geq n\} \geq \lambda_i \geq a \Rightarrow n \in \mu_{dim(V)}^{[a]}$. Hence $n \in \mu_{dim(V)}^{[a]} \Leftrightarrow n \leq dim(\mu_V^{[a]})$.

Similarly it can be proved that $n \in \nu_{dim(V)}^{[b]} \Leftrightarrow n \leq dim(\nu_V^{[b]})$. \square

Proposition 3.8. *Let X be a vector space with $dimX = m$ and $V_1, V_2 \in IFVS(X)$. Then we have the following results:*

(1) For all $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $\mu_{V_1 \cap V_2}^{[a]} = \mu_{V_1}^{[a]} \cap \mu_{V_2}^{[a]}$ and $\nu_{V_1 \cap V_2}^{[b]} = \nu_{V_1}^{[b]} \cap \nu_{V_2}^{[b]}$.

(2) For all $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $\mu_{(V_1+V_2)}^{[a]} = \mu_{V_1}^{[a]} + \mu_{V_2}^{[a]}$ and $\nu_{(V_1+V_2)}^{[b]} = \nu_{V_1}^{[b]} + \nu_{V_2}^{[b]}$.

Proof. We only give the proof of (2). For any $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, we have

$$x \in \mu_{(V_1+V_2)}^{[a]} \Leftrightarrow \sup_{x=x_1+x_2} \{\mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2)\} \geq a$$

$$\Leftrightarrow \text{there exist } x_1, x_2 \text{ such that } x_1 + x_2 = x \text{ and } \mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2) \geq a$$

$$\Leftrightarrow \text{there exist } x_1, x_2 \text{ such that } x_1 + x_2 = x \text{ and } x_1 \in \mu_{V_1}^{[a]} \text{ and } x_2 \in \mu_{V_2}^{[a]}.$$

Similarly it can be proved that $\nu_{(V_1+V_2)}^{[b]} = \nu_{V_1}^{[b]} + \nu_{V_2}^{[b]}$. \square

Proposition 3.9. *Let X be a vector space with $dimX = m$ and $V_1, V_2 \in IFVS(X)$. Then $dim(V_1 + V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$.*

Proof. For any $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, let $n \in \mu_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}^{[a]}$. Then there exist k, l such that $n = k + l$ and $k \in \mu_{dim(V_1+V_2)}^{[a]}$ and $l \in \mu_{dim(V_1 \cap V_2)}^{[a]}$. Then by Proposition 3.7, $k \leq dim(\mu_{(V_1+V_2)}^{[a]}) = dim(\mu_{V_1}^{[a]} + \mu_{V_2}^{[a]})$ and $l \leq dim(\mu_{(V_1 \cap V_2)}^{[a]}) = dim(\mu_{V_1}^{[a]} \cap \mu_{V_2}^{[a]})$. Thus $n \leq dim(\mu_{V_1}^{[a]} + \mu_{V_2}^{[a]}) + dim(\mu_{V_1}^{[a]} \cap \mu_{V_2}^{[a]}) = dim(\mu_{V_1}^{[a]}) + dim(\mu_{V_2}^{[a]})$.

Then there exist k' and l' such that $n = k' + l'$ and $k' \leq dim(\mu_{V_1}^{[a]})$ and $l' \leq dim(\mu_{V_2}^{[a]})$. Now by Proposition 3.7, $k' \in \mu_{dim(V_1)}^{[a]}$ and $l' \in \mu_{dim(V_2)}^{[a]}$. Therefore $n = k' + l' \in \mu_{dim(V_1)}^{[a]} + \mu_{dim(V_2)}^{[a]} = \mu_{dim(V_1)+dim(V_2)}^{[a]}$. Hence $\mu_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}^{[a]} \subseteq \mu_{dim(V_1)+dim(V_2)}^{[a]}$.

Similarly, $\nu_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}^{[b]} \subseteq \nu_{dim(V_1)+dim(V_2)}^{[b]}$.

Also, it can be proved that for any $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $\mu_{dim(V_1)+dim(V_2)}^{[a]} \subseteq \mu_{dim(V_1+V_2)+dim(V_1 \cap V_2)}^{[a]}$ and $\nu_{dim(V_1)+dim(V_2)}^{[b]} \subseteq \nu_{dim(V_1+V_2)+dim(V_1 \cap V_2)}^{[b]}$. Thus for any $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $\mu_{dim(V_1)+dim(V_2)}^{[a]} = \mu_{dim(V_1+V_2)+dim(V_1 \cap V_2)}^{[a]}$ and $\nu_{dim(V_1)+dim(V_2)}^{[b]} = \nu_{dim(V_1+V_2)+dim(V_1 \cap V_2)}^{[b]}$. Hence $dim(V_1 + V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$. \square

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