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Properties of the intuitionistic fuzzy implication $\rightarrow_{187}$

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Abstract: In [4], the new intuitionistic fuzzy implication $\rightarrow_{187}$ is defined and some of its properties are studied. Here, new propertirs of the new implication are studied.
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To Janusz!

1 Introduction

In [4], the intuitionistic fuzzy implication $\rightarrow_{187}$ was introduced and some of its properties were studied. Here, we continue the preceding research.

Initially, we remind the reader that in intuitionistic fuzzy logic (see [1, 2]), each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity" $\mu(p)$ and "falsity degree" or "degree of non-validity" $\nu(p)$. Thus, to each one of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

Let

$$\pi(p) = 1 - \mu(p) - \nu(p).$$

The above function determines the degree of uncertainty (indeterminacy).

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$.

We assume that the evaluation function V assigns to the logical truth T, $V(T) = \langle 1, 0 \rangle$, and to the logical falsity F, $V(F) = \langle 0, 1 \rangle$.

Here, we define only the operations "negation", "disjunction" and "conjunction", originally introduced in [1, 2], that have classical logic analogues, as follows:

$$V(\neg_1 p) = \langle \nu(p), \mu(p) \rangle,$$

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle.$$

Below, for simplicity, we write $\neg$ instead of $\neg_1$.

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 2]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a \geq b,$$

while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a = 1,\ b = 0.$$

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" instead of the intuitionistic fuzzy evaluation of A.

In [3], we called the object $\langle \mu(p), \nu(p) \rangle$ an Intuitionistic Fuzzy Pair (IFP).

For brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A) \rangle$ we will use the IFP $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b \rangle$ and $\langle c, d \rangle$ are IFPs, to have

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \geq d$$

and

$$\langle a,b\rangle \geq \langle c,d\rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

If an IFP is an IFT, we call it Intuitionistic Fuzzy Tautological Pair (IFTP) and if it is a tautology – Tautological Intuitionistic Fuzzy Pair (TIFP).

In [4], the intuitionistic fuzzy implication $\rightarrow_{187}$ is defined by:

$$x \rightarrow_{187} y = \neg x \vee y = \langle \max(b,c), ad\rangle.$$

Were, there was shown that this implication generates the standard negation $\neg_1$, because

$$\langle a,b\rangle \rightarrow_{187} \langle 0,1\rangle = \langle b,a\rangle.$$

2 Main results

For brevity, below we will write $\rightarrow$ instead of $\rightarrow_{187}$.

First, we check which intuitionistic fuzzy implications and negations satisfy C. A. Meredith's axiom (see, e.g., [9]).

Theorem 1. For every five formulas A, B, C, D and E, C. A. Meredith's axiom

$$((((A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A))$$

is an IFT.

Proof: Let $V(A) = \langle a,b\rangle$, $V(B) = \langle c,d\rangle$, $V(C) = \langle e,f\rangle$, $V(D) = \langle g,h\rangle$, $V(E) = \langle i,j\rangle$, where $a, b, \ldots, j \in [0,1]$ and $a+b \leq 1$, $c+d \leq 1$, $e+f \leq 1$, $g+h \leq 1$ and $i+j \leq 1$. Then

$$V(((((A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A)))$$

$$= (((((\langle a,b\rangle \rightarrow \langle c,d\rangle) \rightarrow (\langle f,e\rangle \rightarrow \langle h,g\rangle)) \rightarrow \langle e,f\rangle) \rightarrow \langle i,j\rangle)$$
$$\rightarrow ((\langle i,j\rangle \rightarrow \langle a,b\rangle) \rightarrow (\langle g,h\rangle \rightarrow \langle a,b\rangle))$$
$$= ((((\langle \max(b,c), ad\rangle \rightarrow \langle \max(e,h), fg\rangle) \rightarrow \langle e,f\rangle) \rightarrow \langle i,j\rangle)$$
$$\rightarrow (\langle \max(a,j), bi\rangle \rightarrow \langle \max(a,h), bg\rangle)$$
$$= (((\langle \max(ad,e,h), \max(b,c)fg\rangle \rightarrow \langle e,f\rangle) \rightarrow \langle i,j\rangle) \rightarrow \langle \max(bi,a,h), \max(a,j)bg\rangle$$
$$= ((\langle \max(\max(b,c)fg, e), \max(ad,e,h)f\rangle \rightarrow \langle i,j\rangle) \rightarrow \langle \max(bi,a,h), \max(a,j)bg\rangle$$
$$= ((\langle \max(bfg, cfg, e), \max(adf, ef, fh)\rangle \rightarrow \langle i,j\rangle) \rightarrow \langle \max(bi,a,h), \max(a,j)bg\rangle$$
$$= \langle \max(adf, ef, fh, i), \max(bfg, cfg, e)j\rangle \rightarrow \langle \max(bi,a,h), \max(a,j)bg\rangle$$
$$= \langle \max(adf, ef, fh, i), \max(bfgj, cfgj, ej)\rangle \rightarrow \langle \max(bi,a,h), \max(a,j)bg\rangle$$
$$= \langle \max(bfgj, cfgj, ej, bi, a, h), \max(adf, ef, fh, i)\max(a,j)bg\rangle.$$

Let

$$X \equiv \max(bfgj, cfgj, ej, bi, a, h) - \max(adf, ef, fh, i)\max(a,j)bg.$$

Let $a \geq j$. Then

$$X \geq a - \max(adf, ef, fh, i)abg \geq 0.$$

Let $a < j$. Then

$$X = \max(bfgj, cfgj, ej, bi, a, h) - \max(adf, ef, fh, i)bgj$$

$$= \max(bfgj, cfgj, ej, bi, a, h) - \max(abdfgh, befgh, bfghj, bgij).$$

If $abdfgh = \max(abdfgh, befgh, bfghj, bgij)$, then

$$X = \max(bfgj, cfgj, ej, bi, a, h) - abdfgh \geq a - abdfgh \geq 0.$$

If $befgh = \max(abdfgh, befgh, bfghj, bgij)$, then

$$X = \max(bfgj, cfgj, ej, bi, a, h) - befgh \geq h - befgh \geq 0.$$

If $bfghj = \max(abdfgh, befgh, bfghj, bgij)$, then

$$X = \max(bfgj, cfgj, ej, bi, a, h) - bfghj \geq h - bfghj \geq 0.$$

If $bgij = \max(abdfgh, befgh, bfghj, bgij)$, then

$$X = \max(bfgj, cfgj, ej, bi, a, h) - bgij \geq bi - bgij \geq 0.$$

Therefore, in all cases, $X \geq 0$, i.e., Meredith's axiom is an IFT. □

The next assertions are proved by the same manner so we will omit their proofs.
The axioms of the intuitionistic logic (see, e.g., [10]) are the following.

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \wedge B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \wedge \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \wedge \neg B)$,

(IL10) $(\neg A \wedge \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \wedge B)$, (IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 2. All axioms of the intuitionistic logic are IFTs.

The axioms of A. Kolmogorov (see, e.g., [11]) are the following.

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 3. All axioms of A. Kolmogorov are IFTs.

The axioms of J. Łukasiewicz and A. Tarski (see, e.g., [11]) are the following.

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 4. All axioms of J. Łukasiewicz and A. Tarski are IFTs.

3 Conclusion

In [5] another intuitionstc fuzzy implication ($\to_{188}$) was introduced and some of its properties were studied. In next research other properties of this implication will be studied. All the properties show that intuitionstc fuzzy sets and logics in the sense, described in [2] correspond to the ideas of Brouwer's intuitionism (see [6, 7, 8]).

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