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The Cauchy problem for complex intuitionistic fuzzy differential equations

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Abstract: In this paper, we discuss the existence of a solution to the Cauchy problem for complex intuitionistic fuzzy differential equations. We first propose definitions of complex intuitionistic fuzzy sets and discuss entailed results which parallel those of complex fuzzy sets.
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1 Introduction

The concept of intuitionistic fuzzy sets is introduced by K. Atanasov (1984) ([1, 2]). This concept is a generalization of fuzzy theory introduced by L. Zadeh [3].

The concept of complex fuzzy sets as sets with complex membership functions was first introduced by Ramot et al.,who in [8] demonstrated the increased expressive power gained by endowing a set S with a complex membership function $\mu_S(x) = r_S(x)e^{i\phi_S(x)}$, where $r_S(x)$ and $\phi_S(x)$ are real-valued functions with r_S solely responsible for the fuzzy information and ϕ_S functioning as a phase term containing additional crisp information.

In this work we will be working on the same idea, but this time in the intuitionistic fuzzy theory, we can write the representation of complex membership function μ as $\mu_S(x) = r_S(x)e^{i\phi_S(x)}$ and the non-membership function ν as $\nu_S(x) = r'_S(x)e^{i\phi'_S(x)}$, where $(r_S(x), r'_S(x))$ and $(\phi_S(x), \phi'_S(x))$ are real-valued functions with (r_S, r'_S) solely responsible for the intuitionistic fuzzy information and (ϕ_S, ϕ'_S) functioning. And we study the existence and uniqueness of solutions for complex intuitionistic fuzzy differential equations

$$\begin{cases} U'(t) = \mathcal{H}(t, U(t)), & t \in [0, a], \\ U(0) = U_0 \end{cases} \tag{1}$$

where $\mathcal{H} : [0, a] \times \mathbb{F} \longrightarrow \mathbb{F}$ is continuous. And it will be extensive as an initiation to study other concept, stability, etc. which defined in the fuzzy case [5].

We draw the readers attention to the difference between complex intuitionistic fuzzy sets and intuitionistic fuzzy complex numbers.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

Definition 1. *We denote*

$$IF = \left\{(u, v) : \mathbb{R} \to [0, 1]^2 \;|\forall x \in \mathbb{R} \,/ 0 \leq u(x) + v(x) \leq 1\right\}$$

where

1. (u, v) *is normal i.e there exists* $x_0, x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$.
2. u *is fuzzy convex and* v *is fuzzy concave.*
3. u *is upper semicontinuous and* v *is lower semicontinuous*
4. *supp*$(u, v) =$ *cl*$(\{x \in \mathbb{R} : v(x) < 1\})$ *is bounded.*

For $\alpha \in [0, 1]$ *and* $(u, v) \in IF$*, we define*

$$[(u, v)]^\alpha = \{x \in \mathbb{R} \mid v(x) \leq 1 - \alpha\}$$

and

$$[(u, v)]_\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}.$$

Remark 1. *We can consider* $[(u, v)]_\alpha$ *as* $[u]^\alpha$ *and* $[(u, v)]^\alpha$ *as* $[1 - v]^\alpha$ *in the fuzzy case.*

Definition 2. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0), & x = 0 \\ (0, 1), & x \neq 0 \end{cases}$$

Definition 3. *Let* (u, v) ,$(u', v') \in IF$ *and* $\lambda \in \mathbb{R}$*, we define the addition by:*

$$((u, v) \oplus (u', v'))\,(z) = \left(\sup_{z=x+y} \min(u(x), u'(y)); \inf_{z=x+y} \max(v(x), v'(y))\right)$$

$$\lambda(u, v) = \begin{cases} (\lambda u, \lambda v) & if\ \lambda \neq 0 \\ 0_{(0,1)} & if\ \lambda = 0 \end{cases}$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space IF as follows:

$$[(u,v)\oplus(z,w)]^{\alpha}=[(u,v)]^{\alpha}+[(z,w)]^{\alpha} \tag{2}$$

$$[\lambda(u,v)]^{\alpha}=\lambda[(u,v)]^{\alpha} \tag{3}$$

$$[(u,v)\oplus(z,w)]_{\alpha}=[(u,v)]_{\alpha}+[(z,w)]_{\alpha} \tag{4}$$

$$[\lambda(u,v)]_{\alpha}=\lambda[(u,v)]_{\alpha}, \tag{5}$$

where $(u,v),(z,w)\in IF$ and $\lambda\in\mathbb{R}$.
We denote

$$[(u,v)]_l^+(\alpha)=\inf\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$
$$[(u,v)]_r^+(\alpha)=\sup\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$
$$[(u,v)]_l^-(\alpha)=\inf\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}$$
$$[(u,v)]_r^-(\alpha)=\sup\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}.$$

Remark 2.

$$\big[(u,v)\big]_{\alpha}=\big[[(u,v)]_l^+(\alpha),[(u,v)]_r^+(\alpha)\big]$$
$$\big[(u,v)\big]^{\alpha}=\big[[(u,v)]_l^-(\alpha),[(u,v)]_r^-(\alpha)\big]$$

Theorem 1. *Let* $\mathcal{M}=\{M_\alpha,\ M^\alpha\ :\ \alpha\in[0,1]\}$ *be a family of subsets in* $\mathbb{R}$ *satisfying conditions* $(i)-(iv)$:

(i) $\alpha\leq\beta\Rightarrow M_\beta\subset M_\alpha$ *and* $M^\beta\subset M^\alpha$

(ii) M_α *and* M^α *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha\in[0,1]$.

(iii) *for any nondecreasing sequence* $\alpha_i\to\alpha$ *on* $[0,1]$, *we have* $M_\alpha=\bigcap_i M_{\alpha_i}$ *and* $M^\alpha=\bigcap_i M^{\alpha_i}$.

(iv) *For each* $\alpha\in[0,1]$, $M_\alpha\subset M^\alpha$ *and define* u *and* v, *by*

$$u(x)=\begin{cases}0 & if\ x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & if x\in M_0\end{cases},$$

$$v(x)=\begin{cases}1 & if\ x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & if x\in M^0\end{cases}.$$

Then $(u,v)\in IF$.

Proof. See [7]. □

The space *IF* is metrizable by the distance of the following form:

$$
\begin{aligned}
d_\infty((u,v),(z,w)) = & \frac{1}{4}\sup_{0<\alpha\le 1} |[(u,v)]_r^+(\alpha) - [(z,w)]_r^+(\alpha)| \\
& + \frac{1}{4}\sup_{0<\alpha\le 1} |[(u,v)]_l^+(\alpha) - [(z,w)]_l^+(\alpha)| \\
& + \frac{1}{4}\sup_{0<\alpha\le 1} |[(u,v)]_r^-(\alpha) - [(z,w)]_r^-(\alpha)| \\
& + \frac{1}{4}\sup_{0<\alpha\le 1} |[(u,v)]_l^-(\alpha) - [(z,w)]_l^-(\alpha)|
\end{aligned}
$$

where $|.|$ denotes the usual Euclidean norm in $\mathbb{R}$.

Theorem 2. *(IF, d_∞) is a complete metric space.*

Proof. See [7]. □

We recall the definition of a complex fuzzy set:

Definition 4. *A complex fuzzy set A, defined on a universe of discourse X, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in A. By definition $\mu_A(x)$ a value in the unit circle in the complex plane in the polar case and a value in the unit square in $\mathbb{C}$ in the cartesian case.*

3 Complex intuitionistic fuzzy set

In this section we recall some basic notion on complex intuitionistic fuzzy sets defined in [4]. As the defintion of complex fuzzy set, we give here a defition of complex intuitionistic fuzzy set:

Definition 5. *A Complex intuitionistic fuzzy Set A, defined on a universe of discourse X, is characterized by a membership function $\mu_A(x)$ and non-membership function $\nu_A(x)$ that assigns any element $x \in X$ a complex-valued grade of membership and non-membership in A. By definition, the values $\mu_A(x)$, $\nu_A(x)$ and $\mu_A(x) + \nu_A(x)$ may receive all lie within the unit circle in the complex plane in the polar case. And μ_A, ν_A and $\mu_A(x) + \nu_A(x)$ a value in the unit square in $\mathbb{C}$ in the cartesian case.*

3.1 Cartesian representation of complex grades of membership and non-membership

The complex membership function μ, is defined as

$$\mu(V, z) = \mu_R(V) + i\mu_I(z),$$

likewise, we can define the complex non-membership function as

$$\nu(V, z) = \nu_R(V) + i\nu_I(z),$$

where V is to be interpreted as a set in a intuitionistic fuzzy set of sets and z as an element of V. This definition can be easily extended to $\mathbb{R}$, for $x \in \mathbb{R}$, let

$$f_1(x) = u(x) + iv(x) \qquad \text{and} \qquad f_2(x) = u'(x) + iv'(x),$$

where $f = (u, u') : \mathbb{R} \longrightarrow [0,1]^2$ and $g = (v, v') : \mathbb{R} \longrightarrow [0,1]^2$. For ease of notation, denote $\mathcal{F}$ by (f, g). Thus, $f_1,\ f_2$ assigns to each $x \in \mathbb{R}$ a value in the unit square in $\mathbb{C}$, representing a complex grade of membership and non-membership. Note that $u,\ v,\ u'$ and v' considered individually define non-complex fuzzy sets in $\mathbb{R}$.

Now, for $f = (u, u'),\ g = (v, v') : \mathbb{R} \longrightarrow [0,1]^2$, α-level sets are classically defined as follows:

$$[f]^\alpha = [(u, u')]^\alpha = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha\}\,; \qquad [f]_\alpha = [(u, u')]_\alpha = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha\}$$

and

$$[f]^0 = [(u, u')]^0 = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1\}}; \qquad [f]_0 = [(u, u')]_0 = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0\}}.$$

We use the above to define (α, β)-level sets for $\mathcal{F} = (f, g)$, $0 < \alpha,\ \beta \leq 1$:

$$[\mathcal{F}]^{(\alpha,\beta)} = [(f, g)]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta, \tag{6}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = [(f, g)]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta. \tag{7}$$

Consider the following set of conditions as an alternative definition of $[\mathcal{F}]^{(\alpha,\beta)}$ and $[\mathcal{F}]_{(\alpha,\beta)}$:

$$[\mathcal{F}]^{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) \leq 1 - \beta\}\,, \tag{8}$$

$$[\mathcal{F}]^{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) < 1\}}, \tag{9}$$

$$[\mathcal{F}]^{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) \leq 1 - \beta\}}, \tag{10}$$

$$[\mathcal{F}]^{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) < 1\}}, \tag{11}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) \geq \beta\}\,, \tag{12}$$

$$[\mathcal{F}]_{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) > 0\}}, \tag{13}$$

$$[\mathcal{F}]_{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\ v(x) \geq \beta\}}, \tag{14}$$

$$[\mathcal{F}]_{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\, v(x) > 0\}}. \tag{15}$$

Note that (8) and (11) are equivalent to definition (6), likewise (12) and (15) are equivalent to definition (7) for the corresponding α, β, but (9), (10) and (13), (14) are not: (6) and (7) may not yield closed sets in the case when exactly one of α, β is equal to 0, but (9), (10) and (13), (14) would yield the respective closures of those sets.

For $f, g \in IF$, we have $[f]^\alpha \cap [g]^\beta$, $[f]_\alpha \cap [g]_\beta$ are always compact and $[f]^1 \cap [g]^1 \subset [f]^\alpha \cap [g]^\beta \subset [f]^0 \cap [g]^0$ and $[f]_1 \cap [g]_1 \subset [f]_\alpha \cap [g]_\beta \subset [f]_0 \cap [g]_0$ are nonempty as in order to ensure this, it is sufficient that $[f]^1 \cap [g]^1$ and $[f]_1 \cap [g]_1$ be nonempty, meaning that there should exist some $x_0,\, x_1 \in \mathbb{R}$ such that $x_0 \in [f]^1$, i.e.,$u'(x_0) = 0$, $x_0 \in [g]^1$,i.e. $v'(x_0) = 0$ and $x_1 \in [f]_1$, i.e.,$u(x_1) = 1$, $x_1 \in [g]_1$,i.e. $v(x_1) = 1$. With that in mind, we define the following set:

$$\hat{IF}^2 = \left\{ \Big((u, u'),\, (v, v') \Big) \in IF \times IF \,|\, \exists x_0,\, x_1 \in \mathbb{R},\, s.t\, u(x_1) = v(x_1) = 1,\, u'(x_0) = v'(x_0) = 0 \right\}. \tag{16}$$

Then for $(f, g) \in \hat{IF}^2$, $[\mathcal{F}]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta$, $[\mathcal{F}]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta \in \mathcal{P}_k(\mathbb{R})$ for all $\alpha,\, \beta \in [0, 1]$. And the compactness of the $[\mathcal{F}]^{(\alpha,\beta)}$ sets guarantees the complete equivalence of definition (6) and the set of definitions (8)–(11), and the complete equivalence of definition (7) and the set of definitions (12)–(15).

We recall that IF is closed under addition and scalar multiplication, to establish a similar result for $\hat{IF}^2$. For functions $f = (u, u')$, $g = (v, v') \in IF$, addition and scalar multiplication can be defined via level sets as (2)–(5).

For $\mathcal{F} = (f, g) = \big((u, u'), (v, v')\big)$, $\mathcal{G} = (f', g') = \big((x, x'), (y, y')\big) \in \hat{IF}^2$ and λ is a scalar, let

$$\mathcal{F} + \mathcal{G} = (f, g) + (f', g') = (f + f', g + g'), \tag{17}$$

$$\lambda\mathcal{F} = \lambda(f, g) = (\lambda f, \lambda g). \tag{18}$$

Theorem 3. *([4]) $\hat{IF}^2$ is closed under addition and scalar multiplication.*

Consider the product metric on $IF^2 = IF \times IF$, $\hat{d}_\infty : IF^2 \times IF^2 \longrightarrow \mathbb{R}^+$ by:

$$\hat{d}_\infty(\mathcal{F}, \mathcal{G}) = \max\{d_\infty(f, f')\,,\, d_\infty(g, g')\}, \quad \mathcal{F} = (f, g),\, \mathcal{G} = (f', g') \in \hat{IF}^2. \tag{19}$$

Since $\hat{IF}^2 \subset IF^2$, $\hat{d}_\infty$ is also a metric for $\hat{IF}^2$. Hence, $(\hat{IF}^2, \hat{d}_\infty)$ is a complete metric space.

It will also prove useful to define a zero element in $\hat{IF}^2$. Recall that on IF we define zero element $0_{(1,0)} \in IF$ by

$$0_{(1,0)}(x) = \begin{cases} (1, 0), & x = 0 \\ (0, 1), & x \neq 0 \end{cases}.$$

The zero element on $\hat{IF}^2$ then reads

$$\hat{0} = \big(0_{(1,0)}, 0_{(1,0)}\big) \in IF^2.$$

We have $\hat{0}(0) = \big((1, 0), (1, 0)\big)$, verifying that $\hat{0} \in \hat{IF}^2$.

Theorem 4. *([4]) $\hat{IF}^2 \subset IF \times IF$ is embeddable into a Banach space.*

Remark 3. *In the same manner can be defined $\hat{IF}^n$, $n \geq 3$ and it is shown that is embeddable into a Banach space.*

3.2 Polar representation of complex grades of membership and non-membership

The polar representation of the membership function μ, is defined as

$$\mu(V,z) = r(V)e^{i\sigma\phi(z)},$$

likewise, we can define the polar representation of complex non-membership function as

$$\nu(V,z) = r'(V)e^{i\sigma\phi'(z)},$$

where σ is a scaling factor, does not translate directly to and from the respective Cartesian representation. Therefore the two representations of the corresponding extension to $\mathbb{R}$ are not equivalent as defined, which will be seen below. Thus, depending on the application, one may be more appropriate to use than the other.

For $x \in \mathbb{R}$, the polar form of f_1 and f_2 is defined as follows:

$$f_1(x) = r(x)e^{2\pi\phi(x)i}, \qquad f_2(x) = r'(x)e^{2\pi\phi'(x)i},$$

where $f = (r, r'), g = (\phi, \phi') : \mathbb{R} \longrightarrow [0,1]^2$.

We denote $f_1 = (r, \phi)$ and $f_2 = (r', \phi')$. The scaling factor is taken to be 2π, allowing the range of f_1 and f_2 to be the entire unit circle. Because $e^{2\pi i\phi}$ is periodic, we take the value of ϕ giving the maximum distance from e^0, $\phi = 0.5$, to be the "maximum" membership value.

The level sets for $f = (r, r')$, $[f]^\alpha$ and $[f]_\alpha$ is defined just as

$$[f]^\alpha = [(r,r')]^\alpha = \left\{x \in \mathbb{R} \,|\, r'(x) \leq 1-\alpha\right\},$$

and

$$[f]_\alpha = [(r,r')]_\alpha = \left\{x \in \mathbb{R} \,|\, r(x) \geq \alpha\right\}.$$

And we define the level sets for $g = (\phi, \phi')$, denoted $[g]^{\langle\alpha\rangle}$ and $[g]_{\langle\alpha\rangle}$, must be defined differently to account for the periodicity:

$$[g]^{\langle\alpha\rangle} = \left\{x \in \mathbb{R} \,|\, \phi'(x) \in [\alpha, 1-\alpha],\, \alpha \in (0, 0.5]\right\}, \tag{20}$$

$$[g]_{\langle\alpha\rangle} = \left\{x \in \mathbb{R} \,|\, \phi(x) \in [\alpha, 1-\alpha],\, \alpha \in (0, 0.5]\right\}, \tag{21}$$

$$[g]^{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi'(x) < 1\}}, \tag{22}$$

$$[g]_{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi(x) < 1\}}, \tag{23}$$

$$[g]^{\langle\alpha\rangle} = [(\phi,\phi')]^{\langle 1-\alpha\rangle}, \quad [g]_{\langle\alpha\rangle} = [(\phi,\phi')]_{\langle 1-\alpha\rangle}, \text{ for all } \alpha \in [0,1]. \tag{24}$$

For $\mathcal{F} = (f, g)$, We can then define the level sets $[\mathcal{F}]^{\langle\alpha,\beta\rangle}$ and $[\mathcal{F}]_{\langle\alpha,\beta\rangle}$ as

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [(f,g)]^{\langle\alpha,\beta\rangle} = [f]^{\langle\alpha\rangle} \cap [g]^{\langle\beta\rangle}, \quad \text{and} \quad [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [(f,g)]_{\langle\alpha,\beta\rangle} = [f]_{\langle\alpha\rangle} \cap [g]_{\langle\beta\rangle}, \tag{25}$$

or by the relations:

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r'(x) \leq 1-\alpha,\, \phi'(x) \in [\beta, 1-\beta]\}, \tag{26}$$

$$[\mathcal{F}]_{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\, \phi(x) \in [\beta, 1-\beta]\}, \tag{27}$$

$$[\mathcal{F}]^{<\alpha,0>} = \overline{\{x \in \mathbb{R} \,|\, r'(x) \leq 1-\alpha,\, 0 < \phi'(x) < 1\}}, \tag{28}$$

$$[\mathcal{F}]_{<\alpha,0>} = \overline{\{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\, 0 < \phi(x) < 1\}}, \tag{29}$$

$$[\mathcal{F}]^{<0,\beta>} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\, \phi'(x) \in [\beta, 1-\beta]\}}, \tag{30}$$

$$[\mathcal{F}]_{<0,\beta>} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\, \phi(x) \in [\beta, 1-\beta]\}}, \tag{31}$$

$$[\mathcal{F}]^{<0,0>} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\, 0 < \phi'(x) < 1\}}, \tag{32}$$

$$[\mathcal{F}]_{<0,0>} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\, 0 < \phi(x) < 1\}}, \tag{33}$$

together with

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [\mathcal{F}]^{<\alpha,1-\beta>}, \quad \text{and} \quad [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [\mathcal{F}]_{\langle\alpha,1-\beta\rangle}, \quad \text{for all}\, \alpha, \beta \in [0,1]. \tag{34}$$

It is clear that, for $g = (\phi, \phi') \in IF$, $[g]^{\langle\alpha\rangle} \subset [g]^{\alpha}$ and $[g]_{\langle\alpha\rangle} \subset [g]_{\alpha}$ for all $\alpha \in [0, 0.5]$. However, $[g]^{\langle\alpha\rangle}$, $[g]_{\langle\alpha\rangle}$ need not be compact or convex. In order to address this issue, we define

$$\hat{\mathcal{G}} = \left\{(u,v) : \mathbb{R} \longrightarrow [0,1]^2 \text{ satisfying all of the following conditions}\right\},$$

1. There exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = v(x_1) = 0.5$.
2. u and v are monotone.
3. u is upper semi-continuous on K_1 and lower semi-continuous on K_2, with
$$K_1 = \{x \in \mathbb{R} \,|\, 0 < u(x) \leq 0.5\}, \quad \text{and} \quad K_2 = \{x \in \mathbb{R} \,|\, 0.5 \leq u(x) < 1\}.$$
4. v is lower semi-continuous on K_1' and upper semi-continuous on K_2', with
$$K_1' = \{x \in \mathbb{R} \,|\, 0 < v(x) \leq 0.5\}, \quad \text{and} \quad K_2' = \{x \in \mathbb{R} \,|\, 0.5 \leq v(x) < 1\}.$$
5. $\overline{K_1 \cup K_2}$ and $\overline{K_1' \cup K_2'}$ are compact.

Theorem 5. *([4]) There exists an embedding $l : \hat{\mathcal{G}} \longrightarrow IF \times IF$.*

Now, we define

$$\hat{IF}^2_* = \left\{\Big((r,r'),(\phi,\phi')\Big) \in IF \times \hat{\mathcal{G}} \,|\, \exists\, x_0, x_1 \in \mathbb{R}\, \text{s.t}\; r(x_0) = 1,\, r'(x_1) = 0,\, \phi(x_0) = \phi'(x_1) = 0.5\right\}.$$

Note that, for $\mathcal{F} \in \hat{IF}^2_*$, definition (25) is equivalent to the set of definitions (26)–(34).

Theorem 6. *([4]) $\hat{IF}^2_*$ is embeddable into a Banach space.*

4 Cauchy problem for complex intuitionistic fuzzy differential equations

In this section, we study the existence of an intuitionistic fuzzy mild solution to the Cauchy problem for complex fuzzy differential equations:

$$\begin{cases} U'(t) = \mathcal{H}(t, U(t)), & t \in [0, a]. \\ U(0) = U_0 \in \mathbb{F} \end{cases} \tag{35}$$

For brevity, we shall let $\mathbb{F} = \hat{IF}^2$ when dealing with the Cartesian complex form, and $\mathbb{F} = \hat{IF}^2_*$ when dealing with the polar complex form. We define differentiability as in terms of the Hukuhara difference. For $\mathcal{F}, \mathcal{G} \in \mathbb{F}$, if there exists $\mathcal{K} \in \mathbb{F}$ such that $\mathcal{G} + \mathcal{K} = \mathcal{F}$, we write $\mathcal{F} - \mathcal{G} = \mathcal{K}$ and call $\mathcal{K}$ the difference of $\mathbb{F}$ and $\mathcal{G}$.

Definition 6. *We call a mapping $\mathcal{R} : I = [0, a] \longrightarrow \mathbb{F}$ is differentiable at $t_0 \in I$ if there exists some $\mathcal{R}'(t_0) \in \mathbb{F}$ such that the following limits exist and are equal to $\mathcal{R}'(t_0) \in \mathbb{F}$:*

$$\lim_{h \longrightarrow 0^+} \frac{\mathcal{R}(t_0 + h) - \mathcal{R}(t_0)}{h} \qquad and \lim_{h \longrightarrow 0^+} \frac{\mathcal{R}(t_0) - \mathcal{R}(t_0 - h)}{h}$$

Let $\mathcal{R} : I \longrightarrow \mathbb{F}$ be a continuous mapping. We define $\mathcal{S} : [0, a] \longrightarrow \mathbb{F}$ by

$$\mathcal{S}(t) = \int_0^t \mathcal{R}(s) ds, \quad t \in I.$$

Note that

$$\frac{d}{dt} \mathcal{S}(t) = \mathcal{S}'(t) = \mathcal{R}(t), \quad t \in I.$$

Let $\mathcal{C}(I, \mathbb{F})$ denote the set of all continuous maps from I to $\mathbb{F}$ and let $\hat{d}^{\mathcal{C}}_\infty$ denote a metric on $\mathcal{C}(I, \mathbb{F})$ defined as

$$\hat{d}^{\mathcal{C}}_\infty(\mathcal{F}, \mathcal{G}) = \sup_{t \in I} \hat{d}_\infty(\mathcal{F}(t), \mathcal{G}(t)), \qquad \mathcal{F}, \mathcal{G} \in \mathcal{C}(I, \mathbb{F}).$$

It follows that $(\mathcal{C}(I, \mathbb{F}), \hat{d}^{\mathcal{C}}_\infty)$ is a complete metric space.

Definition 7. *we say that $U : I \longrightarrow \mathbb{F}$ is an intuitionistic fuzzy mild solution to the problem (35) if and only if $U \in \mathcal{C}(I, \mathbb{F})$, and for all $t \geq 0$ and U satisfies the integral equation*

$$U(t) = U_0 + \int_0^t \mathcal{H}(s, U(s)) ds, \quad t \in I.$$

Definition 8. *A mapping $\mathcal{H} : \mathbb{F} \longrightarrow \mathbb{F}$ is Holder continuous if there exists a constant $L > 0$ and a constant $0 < \alpha \leq 1$ such that*

$$\hat{d}_\infty(\mathcal{H}(X), \mathcal{H}(Y)) \leq L(\hat{d}_\infty(X, Y))^\alpha, \qquad \forall X, Y \in \mathbb{F}.$$

Definition 9. *A mapping $\mathcal{H} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ is Lipschitzian with respect to the second argument if there exists a constant $M > 0$ such that*

$$\hat{d}_\infty(\mathcal{H}(t,X),\mathcal{H}(t,Y)) \leq M\hat{d}_\infty(X,Y), \qquad \forall X,Y \in \mathbb{F}, \quad t \geq 0.$$

Theorem 7. *Let $\mathcal{H} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ be continuous and Lipschitzian with respect to the second argument with constant. Then, there exists a unique intuitionistic fuzzy mild solution to the problem (35) on I.*

Proof. Define the operator $\mathcal{O}$ on $\mathcal{C}(I,\mathbb{F})$ by

$$\mathcal{O}U(t) = U_0 + \int_0^t \mathcal{H}(s,U(s))ds, \quad t \in I. \tag{36}$$

It is easy to see that $\mathcal{O}$ is well-defined.
For $X, Y \in \mathcal{C}(I,\mathbb{F})$ and $t \in I$, we have

$$\begin{aligned}
\hat{d}_\infty(\mathcal{O}X(t)\,,\,\mathcal{O}Y(t)) &= \hat{d}_\infty\left(U_0 + \int_0^t \mathcal{H}(s,X(s))ds\,,\,U_0 + \int_0^t \mathcal{H}(s,Y(s)))ds\right)\\
&\leq \hat{d}_\infty\left(\int_0^t \mathcal{H}(s,X(s))ds\,,\,\int_0^t \mathcal{H}(s,Y(s)))ds\right)\\
&\leq \int_0^t \hat{d}_\infty\left(\mathcal{H}(s,X(s))\,,\,\mathcal{H}(s,Y(s)))\right)ds\\
&\leq M\int_0^t \hat{d}_\infty\left(X(s)\,,\,Y(s))\right)ds\\
&\leq tM\hat{d}^{\mathcal{C}}_\infty\left(X\,,\,Y)\right).
\end{aligned}$$

By the some way we have

$$\begin{aligned}
\hat{d}_\infty(\mathcal{O}^2X(t)\,,\,\mathcal{O}^2Y(t)) &\leq M\int_0^t \hat{d}_\infty\left(\mathcal{O}X(s),\,\mathcal{O}Y(s))\right)ds\\
&\leq M^2\hat{d}^{\mathcal{C}}_\infty\left(X\,,\,Y\right)\int_0^t sds\\
&= \frac{M^2t^2}{2}\hat{d}^{\mathcal{C}}_\infty\left(X\,,\,Y\right).
\end{aligned}$$

Then, we can prove that for all $p \in \mathbb{N}^*$, we have

$$\begin{aligned}
\hat{d}^{\mathcal{C}}_\infty(\mathcal{O}^pX\,,\,\mathcal{O}^pY) &\leq \hat{d}_\infty(\mathcal{O}^pX(t)\,,\,\mathcal{O}^pY(t))\\
&\leq \frac{M^pt^p}{p!}\hat{d}^{\mathcal{C}}_\infty\left(X\,,\,Y\right).
\end{aligned}$$

Therefore, there exists $q > 0$ such that $\dfrac{M^qt^q}{q!} < 1$, since

$$\lim_{p \longrightarrow +\infty} \frac{M^pt^p}{p!} = 0.$$

Then, $\mathcal{O}^q$ is a contraction and, by the Banach fixed point theorem, the operator $\mathcal{O}^q$ has a unique fixed point X such that $\mathcal{O}^qX = X$. Thus by the uniqueness of X, X is the unique intuitionistic fuzzy mild solution of (35) (since $\mathcal{O}^q\mathcal{O}X = \mathcal{O}\mathcal{O}^qX = \mathcal{O}X$). □

Corollary 1. *Let $\mathcal{H} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ Holder continuous with constant L, then there exists a unique intuitionistic fuzzy mild solution to the problem (35) on I.*

5 Conclusion

In this article we consider the Cauchy problem for complex intuitionistic fuzzy differential equations. First we establish some basic defintions to build our result. Such we have studied the existence and uniqueness of intuitionistic fuzzy mild solution by the same way for the two spaces with their embedding into a Banach spaces.

The results obtained in this work are fundamental to develop intuitionistic fuzzy differential equations and to be well-to-do.

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