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New properties of the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$

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Abstract: Some new properties of the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$ are formulated and proved. Some open problems are formulated.
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1 Introduction

In the previous papers of the authors [4, 5], a new type of intuitionistic fuzzy modal operator, marked by $\otimes_{\alpha,\beta,\gamma,\delta}$ is introduced and some of its properties are studied. Now, new properties of this operator are studied.

Let a set E be fixed. The Intuitionistic Fuzzy Set (IFS) A in E is defined by (see, e.g., [1]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Different relations, operations and operators are introduced over the IFSs. One of them is the classical neegation, defined by

$$\neg A = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}.$$

For the needs of the present research, we introduce the extended modal operators, defined over IFSs (see, e.g. [1, 2]). Let $\alpha, \beta \in [0,1]$ and let:

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1,$$
$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$
$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$
$$H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$
$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$
$$J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}.$$

These operators are partial cases of the followingg operator (see [2, 3])

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)),$$
$$d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$

where $a, b, c, d, e, f \in [0,1]$ and there, the following two conditions are given:

$$a + e - e.f \leq 1, \tag{1}$$

$$b + d - b.c \leq 1, \tag{2}$$

$$b + e \leq 1. \tag{3}$$

In [4], we introduced the following new operator from modal type:

$$\otimes_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0,1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.

According to this definition, on one hand, the operator reduces by α the degree of membership $\mu_A(x)$ original IFS A's and sums it up with a part of the degree of non-membership $(\gamma.\nu_A(x))$, and in the same time it reduces the original A's degree of non-membership $(\nu_A(x))$ by δ and sums it up with a part of the degree of membership $(\beta.\mu_A(x))$.

As it is mentioned in [4], it is easy to see that

$$\otimes_{1,0,0,1}A = A,$$

$$\otimes_{0,1,1,0}A = \neg A,$$

and the set $\otimes_{\alpha,\beta,\gamma,\delta}A$ is an IFS. There, they are proved also that for every IFS A and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$

$$\neg \otimes_{\alpha,\beta,\gamma,\delta} \neg A = \otimes_{\delta,\gamma,\beta,\alpha}A.$$

In [5] it is proved that for the IFS A and for $a, b, c, d, e, f, g, h \in [0, 1]$, so that $a+b, c+d, e+f, g+h \in [0, 1]$. Then

$$\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A) = \otimes_{ae+bg,af+bh,ce+dg,cf+dh}(A).$$

2 Main results

Here, we formulate and prove some new assertions, related to the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$.

First, we mention that in [1, 2] the following five operations are defined for the IFSs

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

and

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\} :$$

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x \in E\},$$

$$A.B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle \mid x \in E\},$$

$$A@B = \{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\rangle | x \in E\}.$$

Theorem 1. Let A be an IFS and let $\alpha, \beta, \gamma, \delta \in [0, 0.25]$, $\varepsilon, \zeta, \eta, \theta \in [0, 1]$. Then

$$J_{1,0}(H_{4\alpha,\varepsilon}A@H_{4\gamma,\zeta}\neg A)@H_{0,1}(J_{\eta,4\beta}\neg A@J_{\theta,4\delta}A) = \otimes_{\alpha,\beta,\gamma,\delta}A. \tag{4}$$

Proof. Let the IFS A and the real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$ satisfy the conditions from Theorem 1. Then

$$J_{1,0}(H_{4\alpha,\varepsilon}A@H_{4\gamma,\zeta}\neg A)@H_{0,1}(J_{\eta,4\beta}\neg A@J_{\theta,4\delta})$$

$$= J_{1,0}(H_{4\alpha,\varepsilon}\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}@H_{4\gamma,\zeta}\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\})$$

$$@H_{0,1}(J_{\eta,4\beta}\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}@J_{\theta,4\delta}\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\})$$

$$= J_{1,0}(\{\langle x, 4\alpha\mu_A(x), \nu_A(x) + \varepsilon\pi_A(x)\rangle | x \in E\}@\{\langle x, 4\gamma\nu_A(x), \mu_A(x) + \zeta\pi_A(x)\rangle | x \in E\})$$

$$@H_{0,1}(\{\langle x, \nu_A(x) + \eta\pi_A(x), 4\beta\mu_A(x)\rangle | x \in E\} @ \{\langle x, \mu_A(x) + \theta\pi_A(x), 4\delta\nu_A(x)\rangle | x \in E\})$$

$$= J_{1,0}\{\langle x, 2\alpha\mu_A(x) + 2\gamma\nu_A(x), \frac{\mu_A(x) + \nu_A(x) + (\varepsilon + \zeta)\pi_A(x)}{2}\rangle | x \in E\}$$

$$@H_{0,1}\{\langle x, \frac{\mu_A(x) + \nu_A(x) + (\eta + \theta)\pi_A(x)}{2}, 2\beta\mu_A(x) + 2\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, 2\alpha\mu_A(x) + 2\gamma\nu_A(x), 0 | x \in E\} @ \{\langle x, 0, 2\beta\mu_A(x) + 2\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \alpha\mu_A(x) + \gamma\nu_A(x), \beta\mu_A(x) + \delta\nu_A(x)\rangle | x \in E\}$$

$$= \otimes_{\alpha,\beta,\gamma,\delta} A.$$

Therefore, (4) is valid. So, the Theorem is proved. □

Theorem 2. Let A be an IFS and let $\alpha, \beta, \gamma, \delta \in [0, 0.25]$, $\varepsilon, \zeta, \eta, \theta \in [0, 1]$. Then

$$J^*_{1,0}(H_{4\alpha,\varepsilon} A @ H_{4\gamma,\zeta} \neg A) @ H^*_{0,1}(J_{\eta,4\beta} \neg A @ J_{\theta,4\delta} A) = \otimes_{\alpha,\beta,\gamma,\delta} A.$$

The proof is similar to the above one. The same is valid for Theorem 3.

Theorem 3. Let A be an IFS and let $\alpha, \beta, \gamma, \delta \in [0, 0.5]$. Then

$$\otimes_{\alpha,\beta,\gamma,\delta} A = G_{2\alpha,2\gamma} A @ G_{2\beta,2\delta} A.$$

These three theorems give answer to the **Open Problem 1** from [5]: *Can operator* $\otimes_{\alpha,\beta,\gamma,\delta}$ *be represented by the extended modal operators?*

On the other hand, we can check directly that for each IFS A and for $\alpha, \beta \in [0, 1]$,

$$G_{\alpha,\beta} A = \otimes_{\alpha,0,0,\beta} A,$$

$$\neg G_{\alpha,\beta} A = \otimes_{0,\beta,\alpha,0} A.$$

In [4] it is proved that for every two IFSs A and B and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$, it holds that

(a) $\otimes_{\alpha,\beta,\gamma,\delta} (A \cup B) = \otimes_{\alpha,\beta,\gamma,\delta} A \cup \otimes_{\alpha,\beta,\gamma,\delta} B$,
(b) $\otimes_{\alpha,\beta,\gamma,\delta} (A \cap B) = \otimes_{\alpha,\beta,\gamma,\delta} A \cap \otimes_{\alpha,\beta,\gamma,\delta} B$,
(c) $\otimes_{\alpha,\beta,\gamma,\delta} (A + B) = \otimes_{\alpha,\beta,\gamma,\delta} A + \otimes_{\alpha,\beta,\gamma,\delta} B$,
(d) $\otimes_{\alpha,\beta,\gamma,\delta} (A.B) = \otimes_{\alpha,\beta,\gamma,\delta} A. \otimes_{\alpha,\beta,\gamma,\delta} B$.

Here, we shall continue this sequence of equalities, proving the following theorem.

Theorem 4. Let A and B be two IFS and let $\alpha, \beta, \gamma, \delta \in [0, 1]$, so that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$. Then,

$$\otimes_{\alpha,\beta,\gamma,\delta}(A @ B) = \otimes_{\alpha,\beta,\gamma,\delta} A @ \otimes_{\alpha,\beta,\gamma,\delta} B. \quad (5)$$

Proof. Let the IFSs A and B and the real numbers $\alpha, \beta, \gamma, \delta$ satisfy the conditions from Theorem 3. Then

$$\otimes_{\alpha,\beta,\gamma,\delta} A @ \otimes_{\alpha,\beta,\gamma,\delta} B$$

$$= \otimes_{\alpha,\beta,\gamma,\delta}\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} @ \otimes_{\alpha,\beta,\gamma,\delta} \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}$$

$$= \{\langle x, \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x)\rangle | x \in E\}$$

$$@\{\langle x, \alpha.\mu_B(x) + \gamma.\nu_B(x), \beta.\mu_B(x) + \delta.\nu_A(B)\rangle | x \in E\}$$

$$= \{\langle x, \frac{\alpha.\mu_A(x) + \alpha.\mu_B(x)}{2} + \frac{\gamma.\nu_A(x) + \gamma.\nu_B(x)}{2},$$

$$\frac{\beta.\mu_A(x) + \beta.\mu_B(x)}{2} + \frac{\delta.\nu_A(x) + \delta.\nu_A(B)}{2}\rangle | x \in E\}$$

$$= \{\langle x, \alpha.\frac{\mu_A(x) + \mu_B(x)}{2} + \gamma.\frac{\nu_A(x) + \nu_B(x)}{2},$$

$$\beta.\frac{\mu_A(x) + \mu_B(x)}{2} + \delta.\frac{\nu_A(x) + \nu_A(B)}{2}\rangle | x \in E\}$$

$$= \otimes_{\alpha,\beta,\gamma,\delta}\{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2} + \frac{\nu_A(x) + \nu_B(x)}{2},$$

$$\frac{\mu_A(x) + \mu_B(x)}{2} + \frac{\nu_A(x) + \nu_A(B)}{2}\rangle | x \in E\}$$

$$= \otimes_{\alpha,\beta,\gamma,\delta}(A@B).$$

Therefore, (5) is valid. So, the Theorem is proved. □

By analogy with Theorem 4, we can prove Theorem 5 below.

Theorem 5. Let A be an IFS, let $\alpha, \beta, \gamma, \delta \in [0, 1]$, so that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and let $a, b, c, d, e, f \in [0, 1]$, so that: $b, e > 0$ and (1) – (3) are vaild. Then

$$X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta}A) = X_{b+a\alpha-b\alpha-bc\beta, b, \frac{b\gamma-a\gamma+bc\delta}{b}, e+d\delta-ef\gamma-e\delta, e, \frac{e\beta-d\beta+ef\alpha}{e}}. \qquad (6)$$

Proof. Let the IFS A and real numbers $\alpha, \beta, \gamma, \delta, a, b, c, d, e, f$ satisfying the conditions in the Theorem, are given. Then

$$X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta}A)$$

$$= X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta}\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\})$$

$$= X_{a,b,c,d,e,f}(\{\langle x, \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x)\rangle | x \in E\})$$

$$= \{\langle x, a(\alpha.\mu_A(x) + \gamma.\nu_A(x)) + b(1 - \alpha.\mu_A(x) - \gamma.\nu_A(x) - c(\beta.\mu_A(x) + \delta.\nu_A(x))),$$

$$d(\beta.\mu_A(x) + \delta.\nu_A(x)) + e(1 - f(\alpha.\mu_A(x) + \gamma.\nu_A(x)) - (\beta.\mu_A(x) + \delta.\nu_A(x)))\rangle | x \in E\}$$

$$= \{\langle x, a\alpha.\mu_A(x) + a\gamma.\nu_A(x) + b - b\alpha.\mu_A(x) - b\gamma.\nu_A(x) - bc\beta.\mu_A(x) - bc\delta.\nu_A(x))),$$

$$d\beta.\mu_A(x) + d\delta.\nu_A(x)) + e - ef\alpha.\mu_A(x) - ef\gamma.\nu_A(x)) - e\beta.\mu_A(x) - e\delta.\nu_A(x))\rangle | x \in E\}$$

$$= \{\langle x, a\alpha.\mu_A(x) + a\gamma.\nu_A(x) + b - b\alpha.\mu_A(x) - b\gamma.\nu_A(x) - bc\beta.\mu_A(x) - bc\delta.\nu_A(x))),$$

$$d\beta.\mu_A(x) + d\delta.\nu_A(x)) + e - ef\alpha.\mu_A(x) - ef\gamma.\nu_A(x)) - e\beta.\mu_A(x) - e\delta.\nu_A(x))\rangle | x \in E\}.$$

Having in mind that

$$X_{p,q,r,s,t,u}(A) = \{\langle x, p.\mu_A(x) + q.(1 - \mu_A(x) - r.\nu_A(x)),$$

$$s.\nu_A(x) + t.(1 - u.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$

we see that now

$$p - q = a\alpha - b\alpha - bc\beta,$$

$$q = b,$$

$$-qr = a\gamma - b\gamma - bc\delta,$$

$$s - t = d\delta - ef\gamma - e\delta,$$

$$t = e,$$

$$-tu = d\beta - ef\alpha - e\beta.$$

Therefore, from $b, e > 0$ it follows that

$$p = b + a\alpha - b\alpha - bc\beta,$$

$$r = \frac{b\gamma - a\gamma + bc\delta}{b},$$

$$s = e + d\delta - ef\gamma - e\delta,$$

$$u = \frac{e\beta - d\beta + ef\alpha}{e}.$$

Hence, (6) is valid. Now, we must check the validity of conditions (1) – (3) for p, q, r, s, t, u. For condition (1) we see that

$$\begin{aligned} p + t - t.u &= b + a\alpha - b\alpha - bc\beta + e - e.\frac{e\beta - d\beta + ef\alpha}{e} \\ &= b + a\alpha - b\alpha - bc\beta + e - e\beta + d\beta - ef\alpha \\ &= b + e + (a - b)\alpha - (e - d)\beta - ef\alpha - bc\beta \\ &= b + e + (a - b - ef)\alpha + (-e + d - bc)\beta \end{aligned}$$

(Because a, b, c, d, e, f satisfy (1) – (3), from (1) we have that $a - ef \leq 1 - e$ and from (2) – that $d - bc \leq 1 - b$. Therefore,

$$a - b - ef \leq 1 - e - b,$$

$$-e + d - bc \leq 1 - b - e.)$$

$$\begin{aligned} &\leq b + e + (1 - b - e)\alpha + (1 - b - e)\beta \\ &= b + e + (1 - b - e)(\alpha + \beta) \\ &\leq b + e + 1 - b - e = 1. \end{aligned}$$

Hence, (1) is checked for p, q, r. By analogy, we check (3) for s, t, u, while the validity of (2) for q and r follows directly from (3).

So, the Theorem is proved. $\square$

Theorem 6. Let A be an IFS, let $\alpha, \beta, \gamma, \delta \in [0, 1]$, so that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and let $a, b, c, d, e, f \in [0, 1]$, so that: $b\alpha + e\gamma, b\beta + e\delta > 0$ and (1) – (3) are vaild. Then,

$$\otimes_{\alpha,\beta,\gamma,\delta}(X_{a,b,c,d,e,f}(A)) = X_{a\alpha+e\gamma-ef\gamma, b\alpha+e\gamma, \frac{bc\alpha-d\gamma+e\gamma}{b\alpha+e\gamma}, b\beta+d\delta-bc\beta, b\beta+e\delta, \frac{b\beta-a\beta+ef\delta}{b\alpha+e\gamma}}.$$

The proof is similar to the above one.

3 Conclusion

In the present paper, new properties of the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$ are given. It is different from the rest modal operators, defined over IFSs. Following [4] we mention that it gives rise to some open problems. The solution of the first of them was given here. The rest two are:

Open Problem 2. Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be represented by the modal operator $\boxdot_{a,b,c,d,e,f}$?

Open Problem 3. Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be used for representation of some type of modal operators?

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