New properties of the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$

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Abstract: Some new properties of the intuitionistic fuzzy modal operator $\otimes_{a,b,\gamma,\delta}$ are formulated and proved. Some open problems are formulated.

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1 Introduction

In the previous papers of the authors [4, 5], a new type of intuitionistic fuzzy modal operator, marked by $\otimes_{a,b,\gamma,\delta}$ is introduced and some of its properties are studied. Now, new properties of this operator are studied.
Let a set $E$ be fixed. The Intuitionistic Fuzzy Set (IFS) $A$ in $E$ is defined by (see, e.g., [1]):

$$A = \{ (x, \mu_A(x), \nu_A(x)) | x \in E \},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Different relations, operations and operators are introduced over the IFSs. One of them is the classical neegation, defined by

$$\neg A = \{ (x, \nu_A(x), \mu_A(x)) | x \in E \}.$$  

For the needs of the present research, we introduce the extended modal operators, defined over IFSs (see, e.g. [1, 2]). Let $\alpha, \beta \in [0, 1]$ and let:

$$F_{\alpha, \beta}(A) = \{ (x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + \beta \cdot \pi_A(x)) | x \in E \}, \text{ where } \alpha + \beta \leq 1,$$

$$G_{\alpha, \beta}(A) = \{ (x, \alpha \cdot \mu_A(x), \beta \cdot \nu_A(x)) | x \in E \},$$

$$H_{\alpha, \beta}(A) = \{ (x, \alpha \cdot \mu_A(x), \nu_A(x) + \beta \cdot \pi_A(x)) | x \in E \},$$

$$H^*_{\alpha, \beta}(A) = \{ (x, \alpha \cdot \mu_A(x), \nu_A(x) + \beta \cdot (1 - \alpha \cdot \mu_A(x) - \nu_A(x))) | x \in E \},$$

$$J_{\alpha, \beta}(A) = \{ (x, \mu_A(x) + \alpha \cdot \pi_A(x), \beta \cdot \nu_A(x)) | x \in E \},$$

$$J^*_{\alpha, \beta}(A) = \{ (x, \mu_A(x) + \alpha \cdot (1 - \mu_A(x) - \beta \cdot \nu_A(x)), \beta \cdot \nu_A(x)) | x \in E \}.$$  

These operators are partial cases of the following operator (see [2, 3])

$$X_{a,b,c,d,e,f}(A) = \{ (x, \alpha \cdot \mu_A(x) + b \cdot (1 - \mu_A(x) - c \cdot \nu_A(x)),$$

$$d \cdot \nu_A(x) + e \cdot (1 - f \cdot \mu_A(x) - \nu_A(x)) | x \in E \}$$

where $a, b, c, d, e, f \in [0, 1]$ and there, the following two conditions are given:

$$a + e - e \cdot f \leq 1, \quad (1)$$

$$b + d - b \cdot c \leq 1, \quad (2)$$

$$b + e \leq 1. \quad (3)$$

In [4], we introduced the following new operator from modal type:

$$\otimes_{\alpha, \beta, \gamma, \delta} A = \{ (x, \alpha \cdot \mu_A(x) + \gamma \cdot \nu_A(x), \beta \cdot \mu_A(x) + \delta \cdot \nu_A(x)) | x \in E \},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.

According to this definition, on one hand, the operator reduces by $\alpha$ the degree of membership $\mu_A(x)$ original IFS $A$’s and sums it up with a part of the degree of non-membership ($\gamma \cdot \nu_A(x)$), and in the same time it reduces the original $A$’s degree of non-membership ($\nu_A(x)$) by $\delta$ and sums it up with a part of the degree of membership ($\beta \cdot \mu_A(x)$).
As it is mentioned in [4], it is easy to see that

$$\otimes_{1,0,0,1} A = A,$$

$$\otimes_{0,1,1,0} A = \neg A,$$

and the set $$\otimes_{\alpha,\beta,\gamma,\delta} A$$ is an IFS. There, they are proved also that for every IFS $$A$$ and for every four real numbers $$\alpha, \beta, \gamma, \delta \in [0, 1]$$ such that $$\alpha + \beta \leq 1, \gamma + \delta \leq 1$$

$$\neg \otimes_{\alpha,\beta,\gamma,\delta} \neg A = \otimes_{\delta,\gamma,\beta,\alpha} A.$$

In [5] it is proved that for the IFS $$A$$ and for $$a, b, c, d, e, f, g, h \in [0, 1]$$, so that $$a + b, c + d, e + f, g + h \in [0, 1]$$.

Then

$$\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A) = \otimes_{a e + b g, a f + b h, c e + d g, c f + d h}(A).$$

2 Main results

Here, we formulate and prove some new assertions, related to the intuitionistic fuzzy modal operator $$\otimes_{\alpha,\beta,\gamma,\delta}$$.

First, we mention that in [1, 2] the following five operations are defined for the IFSs

$$A = \{\langle x, \mu_{A}(x), \nu_{A}(x)\rangle | x \in E\}$$

and

$$B = \{\langle x, \mu_{B}(x), \nu_{B}(x)\rangle | x \in E\} :$$

$$A \cap B = \{\langle x, \min(\mu_{A}(x), \mu_{B}(x)), \max(\nu_{A}(x), \nu_{B}(x))\rangle | x \in E\},$$

$$A \cup B = \{\langle x, \max(\mu_{A}(x), \mu_{B}(x)), \min(\nu_{A}(x), \nu_{B}(x))\rangle | x \in E\},$$

$$A + B = \{\langle x, \mu_{A}(x) + \mu_{B}(x) - \mu_{A}(x)\mu_{B}(x), \nu_{A}(x)\nu_{B}(x) \rangle | x \in E\},$$

$$A \cdot B = \{\langle x, \mu_{A}(x)\mu_{B}(x), \nu_{A}(x) + \nu_{B}(x) - \nu_{A}(x)\nu_{B}(x) \rangle | x \in E\},$$

$$A \otimes B = \{\langle x, \frac{\mu_{A}(x)}{2} + \frac{\mu_{B}(x)}{2}, \frac{\nu_{A}(x)}{2} + \frac{\nu_{B}(x)}{2} \rangle | x \in E\}.$$

Theorem 1. Let $$A$$ be an IFS and let $$\alpha, \beta, \gamma, \delta \in [0, 0.25], \epsilon, \zeta, \eta, \theta \in [0, 1]$$. Then

$$J_{1.0}(H_{4a,\epsilon}(A) \otimes H_{4\gamma,\zeta}(\neg A)) @ H_{0.1}(J_{n,4\beta}(\neg A) @ J_{\theta,4\delta} A) = \otimes_{\alpha,\beta,\gamma,\delta} A. \quad (4)$$

Proof. Let the IFS $$A$$ and the real numbers $$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$$ satisfy the conditions from Theorem 1.

Then

$$J_{1.0}(H_{4a,\epsilon}(A) \otimes H_{4\gamma,\zeta}(\neg A)) @ H_{0.1}(J_{n,4\beta}(\neg A) @ J_{\theta,4\delta})$$

$$= J_{1.0}(H_{4a,\epsilon}\{\langle x, \mu_{A}(x), \nu_{A}(x)\rangle | x \in E\} @ H_{0.1}\{\langle x, \mu_{A}(x), \nu_{A}(x)\rangle | x \in E\})$$

$$@ H_{0.1}(J_{n,4\beta}\{\langle x, \mu_{A}(x), \nu_{A}(x)\rangle | x \in E\} @ J_{\theta,4\delta}\{\langle x, \mu_{A}(x), \nu_{A}(x)\rangle | x \in E\})$$

$$= J_{1.0}\{\langle x, 4\alpha\mu_{A}(x), \nu_{A}(x) + \epsilon\pi_{A}(x)\rangle | x \in E\} @ \{\langle x, 4\gamma\nu_{A}(x), \mu_{A}(x) + \zeta\pi_{A}(x)\rangle | x \in E\})$$

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@H_{0,1}(\{\langle x, \nu_A(x) + \eta \pi_A(x), 4\beta \mu_A(x)\rangle | x \in E\} @ \{\langle x, \mu_A(x) + \theta \pi_A(x), 4\delta \nu_A(x)\rangle | x \in E\})

= J_{1,0}(\{\langle x, 2\alpha \mu_A(x) + 2\gamma \nu_A(x), \frac{\mu_A(x) + \nu_A(x) + (\varepsilon + \zeta) \pi_A(x)}{2} \rangle | x \in E\})

@H_{0,1}(\{\langle x, \frac{\mu_A(x) + \nu_A(x) + (\eta + \theta) \pi_A(x)}{2}, 2\beta \mu_A(x) + 2\delta \nu_A(x)\rangle | x \in E\})

= \{\langle x, 2\alpha \mu_A(x) + 2\gamma \nu_A(x), 0 \rangle | x \in E\} @ \{\langle x, 0, 2\beta \mu_A(x) + 2\delta \nu_A(x)\rangle | x \in E\}

= \{\langle x, \alpha \mu_A(x) + \gamma \nu_A(x), \beta \mu_A(x) + \delta \nu_A(x)\rangle | x \in E\}

= \otimes_{\alpha, \beta, \gamma, \delta} A.

Therefore, (4) is valid. So, the Theorem is proved. \(\square\)

**Theorem 2.** Let \(A\) be an IFS and let \(\alpha, \beta, \gamma, \delta \in [0, 0.25], \varepsilon, \zeta, \eta, \theta \in [0, 1]\). Then

\[J^*_{1,0}((H_{4, \varepsilon} A @ H_{4, \zeta} \neg A) @ H^*_{0,1}((J_{4, \eta \beta \neg A @ J_{4, \delta} A) = \otimes_{\alpha, \beta, \gamma, \delta} A.\]

The proof is similar to the above one. The same is valid for Theorem 3.

**Theorem 3.** Let \(A\) be an IFS and let \(\alpha, \beta, \gamma, \delta \in [0, 0.5]\). Then

\[\otimes_{\alpha, \beta, \gamma, \delta} A = G_{2\alpha, 2\gamma} A @ G_{2\beta, 2\delta} A.\]

These three theorems give answer to the Open Problem 1 from [5]: Can operator \(\otimes_{\alpha, \beta, \gamma, \delta}\) be represented by the extended modal operators?

On the other hand, we can check directly that for each IFS \(A\) and for \(\alpha, \beta \in [0, 1]\),

\[G_{\alpha, \beta} A = \otimes_{\alpha, 0, 0, \beta} A,\]

\[-G_{\alpha, \beta} A = \otimes_{0, \beta, 0, \alpha} A.\]

In [4] it is proved that for every two IFSs \(A\) and \(B\) and for every four real numbers \(\alpha, \beta, \gamma, \delta \in [0, 1]\) such that \(\alpha + \beta \leq 1, \gamma + \delta \leq 1\), it holds that

(a) \(\otimes_{\alpha, \beta, \gamma, \delta} (A \cup B) = \otimes_{\alpha, \beta, \gamma, \delta} A \cup \otimes_{\alpha, \beta, \gamma, \delta} B,\)

(b) \(\otimes_{\alpha, \beta, \gamma, \delta} (A \cap B) = \otimes_{\alpha, \beta, \gamma, \delta} A \cap \otimes_{\alpha, \beta, \gamma, \delta} B,\)

(c) \(\otimes_{\alpha, \beta, \gamma, \delta} (A + B) = \otimes_{\alpha, \beta, \gamma, \delta} A + \otimes_{\alpha, \beta, \gamma, \delta} B,\)

(d) \(\otimes_{\alpha, \beta, \gamma, \delta} (A \cdot B) = \otimes_{\alpha, \beta, \gamma, \delta} A \cdot \otimes_{\alpha, \beta, \gamma, \delta} B.\)

Here, we shall continue this sequence of equalities, proving the following theorem.

**Theorem 4.** Let \(A\) and \(B\) be two IFS and let \(\alpha, \beta, \gamma, \delta \in [0, 1]\), so that \(\alpha + \beta \leq 1, \gamma + \delta \leq 1\). Then,

\[\otimes_{\alpha, \beta, \gamma, \delta} (A @ B) = \otimes_{\alpha, \beta, \gamma, \delta} A @ \otimes_{\alpha, \beta, \gamma, \delta} B.\]

**Proof.** Let the IFSs \(A\) and \(B\) and the real numbers \(\alpha, \beta, \gamma, \delta\) satisfy the conditions from Theorem 3. Then

\[\otimes_{\alpha, \beta, \gamma, \delta} A @ \otimes_{\alpha, \beta, \gamma, \delta} B\]
Theorem 5.
Proof. Let the IFS $a, b, c, d, e, f$ be an IFS, let $\alpha, \beta, \gamma, \delta \in (0, 1]$, so that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and let $a, b, c, d, e, f \in (0, 1]$, so that: $b, e > 0$ and (1) – (3) are valid. Then

$$X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta} A) = X_{b+a-\alpha-bc, b+a-\alpha-\beta-c\delta, b+e+\delta, c, e, \delta, e, \delta, e, \delta, e, \delta, e}.$$ (6)

Proof. Let the IFS $A$ and real numbers $\alpha, \beta, \gamma, \delta, a, b, c, d, e, f$ satisfying the conditions in the Theorem, are given. Then

$$X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta} A) = X_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta} \{\{x, \mu_A(x), \nu_A(x)\} | x \in E\})$$

$$= \{\{x, \alpha \mu_A(x) + \gamma \nu_A(x) + b(1 - \alpha \mu_A(x) - \gamma \nu_A(x) - c(\beta \mu_A(x) + \gamma \nu_A(x)))\} | x \in E\}$$

$$d(\beta \mu_A(x) + \gamma \nu_A(x)) + e(1 - f(\alpha \mu_A(x) + \gamma \nu_A(x)) - (\beta \mu_A(x) + \gamma \nu_A(x))) | x \in E\}$$

$$= \{\{x, \alpha \mu_A(x) + \gamma \nu_A(x) + b - \alpha \mu_A(x) - b \gamma \nu_A(x) - bc \beta \mu_A(x) - bc \gamma \nu_A(x))\} | x \in E\}$$

$$d\beta \mu_A(x) + d\gamma \nu_A(x) + e - ef \alpha \mu_A(x) - ef \gamma \nu_A(x) - e\beta \mu_A(x) - e\gamma \nu_A(x)) | x \in E\}$$

Having in mind that

$$X_{p,q,r,s,t,u}(A) = \{\{x, p \mu_A(x) + q(1 - \mu_A(x) - r \nu_A(x))\},$$
we see that now
\[ p - q = a\alpha - b\alpha - bc\beta, \]
\[ q = b, \]
\[ -qr = a\gamma - b\gamma - bc\delta, \]
\[ s - t = d\delta - ef\gamma - e\delta, \]
\[ t = e, \]
\[ -tu = d\beta - ef\alpha - e\beta. \]

Therefore, from \( b, e > 0 \) it follows that
\[ p = b + a\alpha - b\alpha - bc\beta, \]
\[ r = \frac{b\gamma - a\gamma + bc\delta}{b}, \]
\[ s = e + d\delta - ef\gamma - e\delta, \]
\[ u = \frac{e\beta - d\beta + ef\alpha}{e}. \]

Hence, (6) is valid. Now, we must check the validity of conditions (1) – (3) for \( p, q, r, s, t, u \).

For condition (1) we see that
\[ p + t - t.u = b + a\alpha - b\alpha - bc\beta + e - e. \frac{e\beta - d\beta + ef\alpha}{e} \]
\[ = b + a\alpha - b\alpha - bc\beta + e - e\beta + d\beta - ef\alpha \]
\[ = b + e + (a - b)\alpha - (e - d)\beta - ef\alpha - bc\beta \]
\[ = b + e + (a - b - ef)\alpha + (-e + d - bc)\beta \]

(Because \( a, b, c, d, e, f \) satisfy (1) – (3), from (1) we have that \( a - ef \leq 1 - e \) and from (2) – that \( d - bc \leq 1 - b \). Therefore,
\[ a - b - ef \leq 1 - e - b, \]
\[ -e + d - bc \leq 1 - b - e. \]
\[ \leq b + e + (1 - b - e)\alpha + (1 - b - e)\beta \]
\[ = b + e + (1 - b - e)(\alpha + \beta) \]
\[ \leq b + e + 1 - b - e = 1. \]

Hence, (1) is checked for \( p, q, r \). By analogy, we check (3) for \( s, t, u \), while the validity of (2) for \( q \) and \( r \) follows directly from (3).

So, the Theorem is proved.

\[ \square \]

**Theorem 6.** Let \( A \) be an IFS, let \( \alpha, \beta, \gamma, \delta \in [0, 1] \), so that \( \alpha + \beta \leq 1, \gamma + \delta \leq 1 \) and let \( a, b, c, d, e, f \in [0, 1] \), so that: \( ba + e\gamma, b\beta + e\delta > 0 \) and (1) – (3) are valid. Then,
\[ \otimes_{\alpha,\beta,\gamma,\delta}(X_{a,b,c,d,e,f}(A)) = X_{aa+e\gamma-ef\gamma,ba+e\gamma,\frac{bc}{ba+e\gamma},\frac{b\beta+e\delta}{ba+e\gamma},\frac{b\beta+e\delta}{ba+e\gamma},\frac{b\beta+e\delta}{ba+e\gamma}}. \]

The proof is similar to the above one.
3 Conclusion

In the present paper, new properties of the intuitionistic fuzzy modal operator $\otimes_{\alpha,\beta,\gamma,\delta}$ are given. It is different from the rest modal operators, defined over IFSs. Following [4] we mention that it gives rise to some open problems. The solution of the first of them was given here. The rest two are:

Open Problem 2. Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be represented by the modal operator $\Box_{a,b,c,d,e,f}$?

Open Problem 3. Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be used for representation of some type of modal operators?

References


