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Decomposition theorems of an intuitionistic fuzzy set

Shiny Jose[1] **and Sunny Kuriakose**[2]

[1] St. George's College Aruvithura, Kerala, India
email: shinyjosedavis@gmail.com

[2] Principal, BPC College, Piravom, Kerala, India
email: asunnyk@gmail.com

Abstract: The (α, β) cut of an Intuitionistic Fuzzy Set (IFS) was studied by P. K. Sharma [4]. In this paper we introduce a new type of cut - strong (α, β) cut - of IFS and study their properties. Also three decomposition theorems are discussed.
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1 Introduction

The concept of the α cut of a fuzzy set was used by Zadeh, and several decomposition theorems were discussed [3, 5]. Following the introduction of intuitionistic fuzzy set (IFS) by Atanassov [1], P. K. Sharma defined (α, β) cut of an IFS and studied its properties [4].

In this paper, we propose the notion of strong (α, β) cut of an IFS. Using this, we develop three decomposition theorems of an IFS analogous to those in the fuzzy sets.

In Section 2, we recall the necessary definitions and theorems which are already in the literature. We define strong (α, β) cut of an IFS and study its properties in Section 3. We further develop three decomposition theorems for IFS. Section 4 contains an illustration.

2 Preliminaries

Definition 2.1 [2, 3]. Let X be any non-empty set, a fuzzy subset A of X is of the form $A = \{(x, \mu_A(x)) : x \in X\}$, $\mu_A : X \to [0, 1]$ is called the membership function and $\mu_A(x)$ is the degree of membership of the element $x \in A$.

Definition 2.2 [3]. Let A be any fuzzy set in X. Then for any $\alpha \in [0, 1]$, α cut of A, denoted by ${}^{\alpha}A$, is defined as ${}^{\alpha}A = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha\}$ and the strong α cut of A denoted by ${}^{\alpha+}A$, is defined as ${}^{\alpha+}A = \{x : x \in X \text{ such that } \mu_A(x) > \alpha\}$

Definition 2.3 [3]. Let A be any fuzzy set in X, then denote ${}_{\alpha}A$ by ${}_{\alpha}A(x) = \alpha.\ {}^{\alpha}A(x)$. Also ${}_{\alpha+}A(x) = \alpha.\ {}^{\alpha+}A$(x); where ${}_{\alpha}A$ and ${}_{\alpha+}A$ are fuzzy sets.

Definition 2.4 [3]. Let A be any fuzzy set in X. Then the level set of A denoted by $\wedge(A)$, is defined as $\wedge(A) = \{\alpha \mid A(x) = \alpha : x \in X\}$

Theorem 2.1 [3]. First Decomposition Theorem of fuzzy sets. For any fuzzy set A,

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A.$$

Theorem 2.2 [3]. Second Decomposition Theorem of fuzzy sets. For any fuzzy set A,

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha+}A$$

Theorem 2.3 [3]. Third Decomposition Theorem of fuzzy sets. For any fuzzy set A,

$$A = \bigcup_{\alpha \in \bigwedge(A)} {}_{\alpha}A;$$

where $\wedge(A)$ is the level set of A.

Definition 2.5 [1]. Let X be a given set. An intuitionistic fuzzy set A in X is given by

$$A = \{(x,\ \mu_A(x),\ \nu_A(x))\},$$

where $\mu_A, \nu_A : X \to [0, 1]$, $\mu_A(x)$ is the degree of membership of the element x in A and $\nu_A(x)$ is the degree of non-membership of x in A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.6 [1]. Let A and B be two IFSs then

$$A \cup B = \{(x,\ max(\mu_A(x),\ \mu_B(x)),\ min(\nu_A(x),\ \nu_B(x))) : x \in X\},$$

$$A \cap B = \{(x,\ min(\mu_A(x),\ \mu_B(x)),\ max(\nu_A(x),\ \nu_B(x))) : x \in X\}.$$

Let us denote maximum by $\vee$ and minimum by $\wedge$ for convenience.

Definition 2.7 [4]. Let A be any IFS then its (α, β) cut is defined by

$$C_{\alpha,\beta} = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\}$$

denoted by ${}^{(\alpha,\beta)}A$ for short.

Properties of (α, β) cut of an IFS [4].

If A and B are two IFSs, then the following properties hold

(i) ${}^{(\alpha,\beta)}A \subseteq {}^{(\delta,\theta)}A$ if $\alpha \geq \delta$ and $\beta \leq \theta$;

(ii) ${}^{(1-\beta,\beta)}A \subseteq {}^{(\alpha,\beta)} A \subseteq {}^{(\alpha,1-\alpha)} A$;

(iii) $A \subseteq B \Rightarrow {}^{(\alpha,\beta)} A \subseteq {}^{(\alpha,\beta)} B$;

(iv) ${}^{(\alpha,\beta)}(A \cap B) = {}^{(\alpha,\beta)} A \cap {}^{(\alpha,\beta)} B$;

(v) ${}^{(\alpha,\beta)}(A \cup B) \supseteq {}^{(\alpha,\beta)} A \cup {}^{(\alpha,\beta)} B$, equality holds if $\alpha + \beta = 1$;

(vi) ${}^{(\alpha,\beta)}(\cap A_i) = \cap {}^{(\alpha,\beta)} A_i$;

(vii) ${}^{(0,1)}A = X$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

3 Strong (α, β) cut of an IFS

Now we introduce a new type of cut–strong (α, β) cut–and study three decomposition theorems of IFS. First we define strong (α, β) cut of an IFS as follows.

Definition 3.1. Let A be an IFS in X and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then strong (α, β), cut of A denoted by ${}^{(\alpha,\beta)+}A$, is defined as follows

$$ {}^{(\alpha,\beta)+}A = \{x : x \in X \text{ such that } \mu_A(x) > \alpha,\ \nu_A(x) < \beta\}. $$

Now we discuss various properties of the strong (α, β) cut.

3.1 Properties of the strong (α, β) cut of IFS

(i) ${}^{(\alpha,\beta)+}A \subseteq {}^{(\delta,\theta)+}A$ if $\alpha \geq \delta$ and $\beta \leq \theta$

(ii) ${}^{(1-\beta,\beta)+}A \subseteq {}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,1-\alpha)+}A$

(iii) $A \subseteq B \Rightarrow {}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,\beta)+}B$

(iv) ${}^{(\alpha,\beta)+}(A \cap B) = {}^{(\alpha,\beta)+}A \cap {}^{(\alpha,\beta)+}B$

(v) ${}^{(\alpha,\beta)+}(A \cup B) \supseteq {}^{(\alpha,\beta)+}A \cup {}^{(\alpha,\beta)+}B$ equality holds if $\alpha + \beta$ = 1

(vi) ${}^{(\alpha,\beta)+}(\cap A_i) = \cap ({}^{(\alpha,\beta)+}A_i)$

(vii) ${}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,\beta)}A$

Proof (i) Let $x \in {}^{(\alpha,\beta)+}A \Rightarrow \mu_A(x) > \alpha$ and $\nu_A(x) < \beta$
Since $\alpha \geq \delta$ and $\beta \leq \theta$ we have $\mu_A(x) > \alpha \geq \delta$ and $\nu_A(x) < \beta \leq \theta$
$\Rightarrow \mu_A(x) > \delta$ and $\nu_A(x) < \theta$ and so x $\in {}^{(\delta,\theta)+}A$
And hence ${}^{(\alpha,\beta)+}A \subseteq {}^{(\delta,\theta)+}A$ if $\alpha \geq \delta$ and $\beta \leq \theta$

Proof (ii) Since $\alpha + \beta \leq 1, 1 - \beta \geq \alpha$ and $\beta \leq \beta$. So by (i)

$$ {}^{(1-\beta,\beta)+}A \subseteq {}^{(\alpha,\beta)+}A. \tag{3.1} $$

Again $\alpha + \beta \leq 1$ so $\alpha \geq \alpha$ and$\beta \leq 1 - \alpha$. Therefore by (i)

$$ {}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,1-\alpha)+}A. \tag{3.2} $$

From (3.1) and (3.2) we get ${}^{(1-\beta,\beta)+}A \subseteq {}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,1-\alpha)+}A$.

Proof (iii) Let $x \in {}^{(\alpha,\beta)+}A \Rightarrow \mu_A(x) > \alpha$ and $\nu_A(x) < \beta$
But $B \supseteq A \Rightarrow \mu_B(x) \geq \mu_A(x) > \alpha$ and $\nu_B(x) \leq \nu_A(x) < \beta$
$\Rightarrow \mu_B(x) > \alpha$ and $\nu_B(x) < \beta \Rightarrow$ x $\in {}^{(\alpha,\beta)+}B$
Therefore, ${}^{(\alpha,\beta)+}A \subseteq {}^{(\alpha,\beta)+}B$

Proof (iv) Since $A \cap B \subseteq A$ and

$$ A \cap B \subseteq B \Rightarrow {}^{(\alpha,\beta)+}(A \cap B) \subseteq {}^{(\alpha,\beta)+}A $$

and

$$ {}^{(\alpha,\beta)+}(A \cap B) \subseteq {}^{(\alpha,\beta)+}B \Rightarrow {}^{(\alpha,\beta)+}(A \cap B) \subseteq {}^{(\alpha,\beta)+}A \cap {}^{(\alpha,\beta)+}B \tag{3.3} $$

Also let $x \in {}^{(\alpha,\beta)+}A \cap {}^{(\alpha,\beta)+}B \Rightarrow x \in {}^{(\alpha,\beta)+}A$ and $x \in {}^{(\alpha,\beta)+}B$
$\Rightarrow \mu_A(x) > \alpha$, $\nu_A(x) < \beta$ and $\mu_B(x) > \alpha, \nu_B(x) < \beta$

$\Rightarrow \mu_A(x) > \alpha, \mu_B(x) > \alpha$ and $\nu_A(x) < \beta, \nu_B(x) < \beta$
$\Rightarrow \ \mu_A(x) \wedge \mu_B(x) > \alpha$ and $\nu_A(x) \vee \nu_B(x) < \beta \Rightarrow x \in^{(\alpha,\beta)+} (A \cap B)$.
Therefore,

$$^{(\alpha,\beta)+}A \cap^{(\alpha,\beta)+} B \subseteq^{(\alpha,\beta)+} (A \cap B). \tag{3.4}$$

From (3.3) and (3.4) we have $^{(\alpha,\beta)+}(A \cap B) = \ ^{(\alpha,\beta)+}A \cap \ ^{(\alpha,\beta)+}B$
Proof (v) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B \Rightarrow^{(\alpha,\beta)+} A \ \subseteq \ ^{(\alpha,\beta)+}(A \cup B)$ and $^{(\alpha,\beta)+}B \subseteq \ ^{(\alpha,\beta)+}(A \cup B)$

$$\Rightarrow^{(\alpha,\beta)+} (A \cup B) \ \supseteq \ ^{(\alpha,\beta)+}A \cup \ ^{(\alpha,\beta)+}B. \tag{3.5}$$

If $\alpha + \beta = 1$, we show that $^{(\alpha,\beta)+}(A \cup B) \subseteq \ ^{(\alpha,\beta)+}A \cup \ ^{(\alpha,\beta)+}B$.
Let $x \in \ ^{(\alpha,\beta)+}(A \cup B) \Rightarrow \mu_A(x) \vee \mu_B(x) > \alpha$ and $\nu_A(x) \wedge \nu_B(x) < \beta$.
If $\mu_A(x) > \alpha$ then $\nu_A(x) < 1 - \alpha = \beta \Rightarrow x \in \ ^{(\alpha,\beta)+}A \ \subseteq \ ^{(\alpha,\beta)+}A \cup \ ^{(\alpha,\beta)+}B$. Also, if $\mu_B(x) > \alpha$ then $\nu_B(x) < 1 - \alpha = \beta \Rightarrow \ x \in \ ^{(\alpha,\beta)+}B \subseteq \ ^{(\alpha,\beta)+}A \cup \ ^{(\alpha,\beta)+}B$. And so

$$^{(\alpha,\beta)+}(A \cup B) \subseteq \ ^{(\alpha,\beta)+}A \ \cup \ ^{(\alpha,\beta)+}B. \tag{3.6}$$

From (3.5) and (3.6)$^{(\alpha,\beta)+}(A \cup B) \ = \ ^{(\alpha,\beta)+}A \cup \ ^{(\alpha,\beta)+}B$.
Proof (vi) Let $x \in \ ^{(\alpha,\beta)+}(\cap A_i) \ \Rightarrow \ \wedge \ \mu_{Ai}(x) > \alpha$ and $\vee \ \nu_{Ai}(x) < \beta$
$\Rightarrow x \in \ ^{(\alpha,\beta)+}A_i$ for all i
$\Rightarrow x \in \cap(^{(\alpha,\beta)+}A_i) \Rightarrow^{(\alpha,\beta)+} (\cap A_i) \subseteq \ \cap (^{(\alpha,\beta)+}A_i)$.
The converse part is clear.
Therefore, $^{(\alpha,\beta)+}(\cap A_i) = \ \cap (^{(\alpha,\beta)+}A_i)$
Proof (vii) Let $x \in \ ^{(\alpha,\beta)+}A \Rightarrow \mu_A(x) > \alpha$ and $\nu_A(x) < \beta$
$\Rightarrow \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta \Rightarrow$ x $\in \ ^{(\alpha,\beta)}A$ Therefore, $^{(\alpha,\beta)+}A \ \subseteq \ ^{(\alpha,\beta)}A$
Definition 3.2 Using Definitions 2.7 and 3.1, and for $\alpha, \ \beta \in$ [0, 1] with $\alpha + \beta \ \leq \ 1$, we define new IFSs denoted by $_{(\alpha,\beta)}A$ and $_{(\alpha,\beta)+}A$ as $_{(\alpha,\beta)}A(x) \ = \ (\alpha, \beta)$, if $x \in \ ^{(\alpha,\beta)}A$ and $(0, 1)$ if $x \notin \ ^{(\alpha,\beta)}A$. Also $_{(\alpha,\beta)+}A(x) \ = \ (\alpha, \beta)$, if x $\in \ ^{(\alpha,\beta)+}A$ and (0,1) if $x \ \notin \ ^{(\alpha,\beta)+}A$.
Definition 3.3 We can define Level set of an IFS A as $\wedge(A) \ = \ \{(\alpha, \beta)$ such that $A(x) \ = \ (\alpha, \beta)\}$.

3.2 Decomposition theorems of intuitionistic fuzzy sets

Theorem 3.1 First Decomposition Theorem of IFS. Let X be any non-empty set. For an Intuitionistic fuzzy subset A in X,

$$A = \bigcup_{\alpha,\beta \in [0,1]} {}_{(\alpha,\beta)}A,$$

where $\bigcup$ denotes union given in Definition 2.6 and $_{(\alpha,\beta)}A$ given in Definition 3.2
Proof: For each particular $x \in X$ let us denote $(\mu_A(x), \ \nu_A(x)) = (a, \ b) = A(x)$, where a denotes the degree of belongingness of x in A and b denotes the degree of non-belongingness of x in A. Then
$(\bigcup_{\alpha,\beta \in [0,1]} {}_{(\alpha,\beta)}A)(x)$= $(\sup_{\alpha \in [0,1]}, \inf_{\beta \in [0,1]} {}_{(\alpha,\beta)}A(x))$

= max [($\sup_{\alpha\in[0,a]}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)}A(x)$), ($\sup_{\alpha\in(a,1]}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)}A(x)$)]
= ($\sup_{\alpha\in[0,a]}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)}A(x)$) (by Definition 3.2)
= max[($\sup_{\alpha\in[0,a]}$, $\inf_{\beta\in[0,b)}$ $_{(\alpha,\beta)}A(x)$), ($\sup_{\alpha\in[0,a]}$, $\inf_{\beta\in[b,1]}$ $_{(\alpha,\beta)}A(x)$)]
= ($\sup_{\alpha\in[0,a]}$, $\inf_{\beta\in[b,1]}$ $_{(\alpha,\beta)}A(x)$) = (a, b) = A(x)
Since the same argument is valid for each $x \in X$, the theorem is proved.

Theorem 3.2 Second Decomposition Theorem of IFS. Let X be any non-empty set. For an intuitionistic fuzzy subset A in X,

$$A = \bigcup_{\alpha,\beta\in[0,1]} {}_{(\alpha,\beta)+}A,$$

where $\bigcup$ denotes union given in Definition 2.6 and $_{(\alpha,\beta)+}A$ given in Definition 3.2.

Proof: For each particular $x \in X$ let us denote $(\mu_A(x), \nu_A(x)) = (a, b) = A(x)$, where a denotes the degree of belongingness of x in A and b denotes the degree of non-belongingness of x in A. Then
($\bigcup_{\alpha,\beta\in[0,1]}$ $_{(\alpha,\beta)+}A)(x)$ = ($\sup_{\alpha\in[0,1]}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)+}A(x)$)
= max [($\sup_{\alpha\in[0,a)}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)+}A(x)$), ($\sup_{\alpha\in[a,1]}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)+}A(x)$)]
= ($\sup_{\alpha\in[0,a)}$, $\inf_{\beta\in[0,1]}$ $_{(\alpha,\beta)+}A(x)$)
= max[($\sup_{\alpha\in[0,a)}$, $\inf_{\beta\in[0,b]}$ $_{(\alpha,\beta)+}A(x)$), ($\sup_{\alpha\in[0,a)}$, $\inf_{\beta\in(b,1]}$ $_{(\alpha,\beta)+}A(x)$)]
= ($\sup_{\alpha\in[0,a)}$, $\inf_{\beta\in(b,1]}$ $_{(\alpha,\beta)+}A(x)) = (a, b) = A(x)$.
Since the same argument is valid for each $x \in X$, the theorem is proved.

Theorem 3.3 Third Decomposition Theorem of IFS. Let X be any non-empty set. For an intuitionistic fuzzy subset A in X,

$$A = \bigcup_{\alpha,\beta\in\wedge A} {}_{(\alpha,\beta)}A,$$

where $\wedge A$ denotes the level set of A given in Definition 3.3. The proof is similar to the one in Theorem 3.1.

4 Illustrative Example

Let A be any intuitionistic fuzzy set of a set X, given by

$$A = \{(x_1, .5, .2),\ (x_2, .6, .3),\ (x_3, .4, .5),\ (x_4, .9, .1),\ (x_5, 0, 1),\ (x_6, 1, 0)\}.$$

Let us denote A for convenience as

$$A = \frac{(.5, .2)}{x_1} + \frac{(.6, .3)}{x_2} + \frac{(.4, .5)}{x_3} + \frac{(.9, .1)}{x_4} + \frac{(0, 1)}{x_5} + \frac{(1, 0)}{x_6}.$$

Then ${}^{(.5,.2)}A = \frac{(1,0)}{x_1} + \frac{(0,1)}{x_2} + \frac{(0,1)}{x_3} + \frac{(1,0)}{x_4} + \frac{(0,1)}{x_5} + \frac{(1,0)}{x_6}$

By Definition 3.2,

$$_{(.5,.2)}A = \frac{(.5,.2)}{x_1} + \frac{(0,1)}{x_2} + \frac{(0,1)}{x_3} + \frac{(.5,.2)}{x_4} + \frac{(0,1)}{x_5} + \frac{(.5,.2)}{x_6} \tag{4.1}$$

Similarly,

$$_{(.6,.3)}A = \frac{(0,1)}{x_1} + \frac{(.6,.3)}{x_2} + \frac{(0,1)}{x_3} + \frac{(.6,.3)}{x_4} + \frac{(0,1)}{x_5} + \frac{(.6,.3)}{x_6} \tag{4.2}$$

$$_{(.4,.5)}A = \frac{(.4,.5)}{x_1} + \frac{(.4,.5)}{x_2} + \frac{(.4,.5)}{x_3} + \frac{(.4,.5)}{x_4} + \frac{(0,1)}{x_5} + \frac{(.4,.5)}{x_6} \tag{4.3}$$

$$_{(.9,.1)}A = \frac{(0,1)}{x_1} + \frac{(0,1)}{x_2} + \frac{(0,1)}{x_3} + frac(.9,.1)x_4 + \frac{(0,1)}{x_5} + \frac{(.9,.1)}{x_6} \tag{4.4}$$

$$_{(0,1)}A = \frac{(0,1)}{x_1} + \frac{(0,1)}{x_2} + \frac{(0,1)}{x_3} + \frac{(0,1)}{x_4} + \frac{(0,1)}{x_5} + \frac{(0,1)}{x_6} \tag{4.5}$$

$$_{(1,0)}A = \frac{(0,1)}{x_1} + \frac{(0,1)}{x_2} + \frac{(0,1)}{x_3} + \frac{(0,1)}{x_4} + \frac{(0,1)}{x_5} + \frac{(1,0)}{x_6} \tag{4.6}$$

Using equations (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6)

$$A = \bigcup_{\alpha,\beta\in[0,1]} {}_{(\alpha,\beta)}A.$$

5 Conclusion

In this paper, we introduced the concept of strong (α, β) cut of an IFS and studied its properties. Further, using these (α, β) cuts, we defined two special IFS and proved three decomposition theorems for IFS.

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