Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 1, 54–61

On almost fuzzy strong precompactness in intuitionistic fuzzy topological spaces

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Abstract: The concept of almost fuzzy strong precompactnesss, near fuzzy strong precompactness, countable fuzzy strong precompactness and light fuzzy strong precompactness in Çoker's space have been introduced and studied. Also we investigate the behavior of fuzzy strong precompactness under several types of fuzzy continuous functions.
Keywords: Intuitionistic fuzzy topology, Intuitionistic fuzzy strong precompactnesss, Intuitionistic almost fuzzy strong precompactnesss, Intuitionistic near fuzzy strong precompactness, Intuitionistic countable fuzzy strong precompactness, Intuitionistic light fuzzy strong precompactness.
AMS Classification: 54A40, 03E72.

1 Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [13]. The concept of fuzzy topological spaces was introduced and developed by C. L. Chang [4]. In [7, 8] some weaker forms of fuzzy compactness are considered for the first time.

Since Atannasov [1, 2, 3] introduced the notion of intuitionistic fuzzy sets, Çoker [5] defined the intuitionistic fuzzy topological spaces. Çoker and Eş [6] introduced the investigate fuzzy almost compactness, fuzzy near compactness and fuzzy light compactness in intuitionistic fuzzy topological spaces. Krsteska and Abbas [9,10] introduced the concept of intuitionistic fuzzy strong precompactness which is strictly stronger than the concept of intuitionistic fuzzy compactness.

In this paper, we introduce the investigate intuitionistic almost fuzzy strong precompactness and light fuzzy strong precompactness in Çoker's space.

2 Preliminaries

Definition 2.1. [1] Let X be a non-empty fixed set and I the closed interval [0,1]. An intuitionistic fuzzy set (IFS) A is an object of following form

$$A = \{< x, \mu_A(x), \vartheta_A(x) > | x \in X\}$$

where the mappings $\mu_A: X \to I$ and $\vartheta_A: X \to I$ denote the degree of membership (namely) $\mu_A(x)$ and the degree of non-membership (namely) $\vartheta_A(x)$ for each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \vartheta_A(x) \leq 1$ for each $x \in X$.

Definition 2.2. [1] Let A and B be IFS'S of the form $A = \{< x, \mu_A(x), \vartheta_A(x) > | x \in X\}$ and $B = \{< x, \mu_B(x), \vartheta_B(x) > | x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\vartheta_A(x) \geq \vartheta_B(x)$;

(ii) $\bar{A} = \{< x, \vartheta_A(x)(x), \mu_A(x) > | x \in X\}$;

(iii) $A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x),\ \vartheta_A(x)(x) \vee \vartheta_A(x)(x) > | x \in X\}$;

(iv) $A \cup B = \{< x, \mu_A(x) \vee \mu_B(x),\ \vartheta_A(x)(x) \wedge \vartheta_A(x)(x) > | x \in X\}$.

Definition 2.3. [1] $0_\sim = \{< x, 0, 1 > x \in X\}$ and $1_\sim = \{< x, 1, 0 > x \in X\}$.

Definition 2.4. [5] An intuitionistic fuzzy topology (IFT for short) in Çoker's sense on a non-empty set X is a family τ of intuitionistic fuzzy sets in X satisfying the following axioms:

(T_1) $0_\sim, 1_\sim \in \tau$,

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(T_3) $\bigcup_{i \in I} G_i \in \tau$ for any arbitrary family $\{G_i : i \in I\} \subseteq \tau$.

In this case the pair (X, τ) is called an intuitionistic fuzzy topological space (IFT for short) and each IFS in τ is known as a intuitionistic fuzzy open set (IFOS for short) in X.

Definition 2.5. [5] The complement $\bar{A}$ of an IFOS A in an IFTS (X,τ) is called an intuitionistic fuzzy closed set (IFCS for short) in X.

Definition 2.6.[5] Let (X, τ) be an IFTS and $A = \{< x, \mu_A(x), \vartheta_A(x) > | x \in X\}$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are defined by

$cl(A) = \cap \{K : K$ is and IFCS in X and $A \subseteq K\}$

and

$int(A) = \cup \{G : G$ is an IFOS in X and $G \subseteq A\}$.

Definition 2.7. [5] Let X and Y be two nonempty sets and $f : X \to Y$ be a function. If $B = \{< y, \mu_B(y), \vartheta_B(y) > | y \in Y\}$ is an IFS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$ is the IFS in X defined by

$$f^{-1}(B) = \{< x,\ f^{-1}(\mu_B)(x),\ f^{-1}(\vartheta_B)(x) > |x \in X\}.$$

Similarly, if $A = \{< x, \mu_A(x), \vartheta_A(x) > |x \in X\}$ is an IFS in *X*, then the image of *A* under *f*, denoted by *f*(*A*), is the IFS in *Y* defined by

$$f(A) = \{< y, f(\mu_A)(y), f(\vartheta_A)(y) > |y \in Y\}$$

Proposition 2.8. [5] Let (*X*, *τ*) be an IFTS and *A*, *B* be IFS's in *X*. Then the following properties hold:

(i) $cl(\bar{A}) = \overline{(\imath nt(A))},\ int(\bar{A}) = \overline{(cl(A))}$;

(ii) $int(A) \subseteq A \subseteq cl(A)$.

Definition 2.9. [11] An IFS A in an IFTS X is called an intuitionistic fuzzy preopen set (IFPOS for short) if $A \subseteq int(cl(A))$. The complement $\bar{A}$ of an IFPOS in X is called an intuitionistic fuzzy preclosed set (IFPCS for short) in X.

Definition 2.10. [10] Let $\mathrm{A} = \{< x, \mu_A(x), \vartheta_A(x) > : x \in X\}$ be an IFS in IFTS *X*. Then $preint(A) = \cup \{G: G$ is an IFPOS in X and $\mathrm{G} \subseteq A\}$ is called an intuitionistic fuzzy preinterior of A;
$precl(A) = \cap \{K: K$ is and IFPCS in X and $\mathrm{A} \subseteq K\}$ is an intuitionistic fuzzy preclosure of A.

Definition 2.11. [10] An IFS A in an IFTS X is called an intuitionistic fuzzy strongly preopen set (IFSPOS for short) if $A \subseteq int(precl(A))$. The complement $\bar{A}$ of an IFSPOS in X is called an intuitionistic fuzzy preclosed set (IFSPCS for short) in X.

Definition 2.12. [10] Let f be a mapping from an IFTS X into an IFTS Y. The mapping f is called

(i) an intuitionistic fuzzy strong precontinuity if $f^{-1}(B)$ is an IFSPOS in *X*, for each IFOS *B* in *Y*.
(ii) an intuitionistic fuzzy strong irresolute precontinuity if $f^{-1}(B)$ is an IFSPOS in *X*, for each IFSPOS *B* in *Y*.

Definition 2.13. [5] Let *X* be an IFTS. A family $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ of IFOS's in *X* satisfies the condition $\cup \left\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\right\} = 1_{\sim}$ is called a fuzzy open cover of *X*.

A finite subfamily of a fuzzy open cover $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ which is also a fuzzy open cover of *X* is called a finite subcover of $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$.

An IFTS *X* is called fuzzy compact iff every fuzzy open cover has a finite subcover.

Definition 2.14[10] Let X be an IFTS. A family $\left\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\right\}$ of IFSPOS 's in *X* satisfies the condition $\cup \left\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\right\} = 1_{\sim}$ is called a fuzzy strongly preopen cover of *X*.

A finite subfamily of a fuzzy strongly preopen cover $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ which is also a fuzzy strongly preopen cover of X is called a finite subcover of $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$.

Definition 2.15. [10]] Let *X* be an IFTS. A family $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ of IFSPOS's in *X* has the finite intersection property (FIP for short) if every finite subfamily $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i = 1,2, \dots, n\}$ satisfies the condition $\bigcap_{i=1}^{n} < x,\ \mu_{G_i}, \vartheta_{G_i} > \neq\ 0_{\sim}$.

Definition 2.16. [10] An IFTS *X* is called fuzzy strongly precompact iff every fuzzy strongly preopen cover has a finite subcover.

Definition 2.17. [10] An IFTS *X* is called fuzzy strongly precompact iff every family $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ of IFSPOS's with the finite intersection property has a non-empty intersection.

Remark 2.18. [10] Since every IFOS is an IFSPOS, from the definition above we may conclude that every fuzzy strongly precompact IFTS is fuzzy compact.

Theorem 2.19. [10] Let *f*: *X* →*Y* be an intuitionistic fuzzy strongly precontinuous mapping from an IFTS *X* onto IFTS *Y*. If *X* is fuzzy stronly precompact, then *Y* is fuzzy compact.

Definition 2.20. [10] An IFTS *X* is called countably fuzzy strongly precompact iff every countable fuzzy strongly preopen cover of *X* has a finite subcover.

3 Intuitionistic almost fuzzy strong precompactness

Definition 3.1. (i) An IFTS (*X*,*τ*) is called almost fuzzy strongly precompact iff every fuzzy strongly preopen cover of *X* has a finite subcollection whose preclosures cover *X*.

(ii) An IFTS (*X*,*τ*) is called nearly fuzzy strongly precompact iff every fuzzy strongly preopen cover of *X* has a finite subcollection such that the preinteriors of preclosures of IFS's in this subcollection covers *X*.

It is clear that in IFTS's we have the following implications:

Fuzzy strong precompactness → nearly fuzzy strongly precompactness → almost fuzzy strongly precompactness.

But the reverse implications do not hold:

Example 3.2. Let $X = \{1,\ 2\}$ and consider the IFS's $\{G_n : n = 1,2,3, \dots\}$ as follows: $G_n =< x, \left(\frac{1}{1-\frac{1}{n}}, \frac{2}{1-\frac{1}{n+1}}\right), \left(\frac{1}{\frac{1}{n+1}}, \frac{2}{\frac{1}{n+2}}\right) >$.

The family $\tau = \{G_n : n = 1,2, \dots\} \cup \{\ 0_{\sim}\ ,\ 1_{\sim}\}$ is an IFT on *X* [6]. Since $precl(G_n) =< x, 1, 0 >$ and $preint(precl(G_n)) =\ 1_{\sim}$, (*X*, *τ*) is nearly fuzzy strongly pre compact. On the other hand, $\{G_n : n = 1,2, \dots\}$ is a fuzzy strongly preopen cover of *X*, but the family $\{G_n: n = 1,2, \dots\}$ cannot have a finite subcover. Thus (X, τ) is not fuzzy strongly precompact.

Theorem 3.3. An IFTS (X, τ) is almost fuzzy strongly precompact iff every family $\{G_i: i \in I\}$ where $G_i =< x, \mu_{G_i}, \vartheta_{G_i} >, i \in I$, of IFPOS's in X having the FIP we have $\bigcap_{i\in I} precl(G_i) \neq 0_\sim$.

Proof. Suppose X is almost fuzzy strongly precompact and let $\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\}$ be any family of IFPOS in X such that $\bigcap precl\{< x, \mu_{G_i}, \vartheta_{G_i} >: i \in I\} = 0_\sim$. Then we have $\bigcup preint\,\{< x,\ \vartheta_{G_i}, \mu_{G_i} >: i \in I\} = 1_\sim$.

Since X almost fuzzy strongly precompact, there exists a finite subfamily

$\{< x,\ \vartheta_{G_i}, \mu_{G_i} >: i = 1,2, . . , n\}$ such that $\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{precl}(\mathrm{preint}(\overline{\mathrm{G}_\mathrm{i}})) = 1_\sim$.

Hence $\bigcup_{i=1}^{n} precl(\overline{precl(G_i)}) = \bigcup_{i=1}^{n} \overline{preint(precl(G_i))} = 1_\sim$.

Now $\bigcap_{i=1}^{n} preint(precl(G_i) = 0_\sim$. But from $G_i \subseteq preint(precl(G_i))$, we see that $\bigcap_{i=1}^{n} G_i = 0_\sim$, which is a contradiction with the FIP of the family.

Conversely, $\{G_i: i \in I\}$ be a fuzzy preopen cover of X, and suppose that there exists no finite subfamily of $\{G_i: i \in I\}$, whose preclosures is not a cover of X. Now the family $\{\overline{precl(G_i)} : i \in I\}$, since $\overline{precl(G_i)} = preint(\overline{\mathrm{G}_\mathrm{i}})$, consists of IFPOS's in X and the FIP. According to the assumption, we have $\bigcap_{i\in I} precl(\,\left(\overline{precl(G_i)}\right) \neq 0_\sim$.

Hence $\bigcup_{i\in I} preint(precl(G_i)) \neq 1_\sim$, which is a contradiction with $\bigcup_{i\in I} G_i = 1_\sim$, since $G_i \subseteq preint(precl(G_i))$ for each $i \in I$. ☐

Definition 3.4. An intuitionistic fuzzy set A is called an intuitionistic fuzzy pre regular open set (IFPROS for short) iff $A = precl\big(preint(A)\big)$; an intuitionistic fuzzy set B is called an intuitionistic fuzzy pre regular closed set (IFPRCS for short) iff $B = precl\big(preint(B)\big)$.

Theorem 3.5. In an IFTS (X,τ) the following conditions are equivalent:

(i) (X,τ) is almost fuzzy strongly precompact.

(ii) For every family $\{K_i: i \in I\}$, where $K_i = \{< x, \mu_{K_i}, \vartheta_{K_i} >: i \in I\}$, of IFPRCS's such that $\bigcap_{i\in I} K_i = 0_\sim$, there exists a finite subfamily $\{K_i: i = 1,2, \dots , n\}$ such that $\bigcap_{i=1}^{n} preint(K_i) = 0_\sim$.

(iii) $\bigcap_{i\in I} precl(K_i) \neq 0_\sim$ holds for each family $\{K_i: i \in I\}$ of IFPROS's having the FIP, where $K_i = \{< x, \mu_{K_i}, \vartheta_{K_i} >: i \in I\}$.

(iv) Every intuitionistic fuzzy pre regular open cover of X contains a finite subfamily whose closures cover X.

Proof. The proof of this theorem follows a similar pattern to Theorem 3.3. ☐

Definition 3.6. Let (X, τ_1), (Y, τ_2) be two IFTS's and let f :X→Y be a function.Then f is said to be fuzzy strongly precontinuous iff for each IFS A in X, $f(precl(A)) \subseteq f(A)$.

Theorem 3.7. Let (X, τ_1), (Y, τ_2) be two IFTS's and let f :X→Y be a fuzzy strongly precontinuous surjection. If (X, τ_1) is almost fuzzy strongly precompact, then (Y, τ_2) is fuzzy strongly precompact.

Proof. Let $\{G_i : i \in I\}$ be a fuzzy preopen cover of Y, where $G_i =< y, \mu_{G_i}, \vartheta_{G_i} >: i \in I$. Since f is fuzzy strongly precontinuous and hence precontinuous, $\{f^{-1}(G_i) : i \in I\}$ is a fuzzy preopen cover of X. Since X is almost fuzzy strongly precompact, there exists a finite subfamily $\{G_i : i = 1,2, \dots, n\}$ such that $\bigcup_{i=1}^{n} precl(f^{-1}(G_i)) = 1_{\sim}$. From the surjectivity and fuzzy strong precontinuity of f, we obtain

$$f(\bigcup_{i=1}^{n} precl(f^{-1}(G_i))) = \bigcup_{i=1}^{n} f\left(precl\left(f^{-1}(G_i)\right)\right) \subseteq \bigcup_{i=1}^{n} f\left(f^{-1}(G_i)\right) = \bigcup_{i=1}^{n} G_i = 1_{\sim} .$$

Hence Y is fuzzy strongly precompact. □

Definition 3.8. Let (X, τ_1) and (Y, τ_2) be two IFTS's and let f :X→Y be a function. Then f is said to be almost fuzzy strongly precontinuous iff the preimage of each intuitionistic fuzzy preregular open set of Y is an intuitionistic fuzzy preopen set in X.

Theorem 3.9. Let (X, τ_1) and (Y, τ_2) be two IFTS's and let f :X→Y be an almost fuzzy strongly precontinuous surjection. If (X, τ_1) is almost fuzzy strongly precompact, then so is (Y, τ_2) .

Proof. It is similar to the proof of the Theorem 3.7. □

Definition 3.10. Let (X, τ_1), (Y, τ_2) be two IFTS's and let f :X→Y be a function. Then f is said to be weakly fuzzy strongly precontinuous iff for each IFSPOS B of Y, $f^{-1}(B) \subseteq preint\left(f^{-1}\left(precl(B)\right)\right)$.

Theorem 3.11. Let (X, τ_1), (Y, τ_2) be two IFTS's and let f :X→Y be a weakly fuzzy strongly precontinuous surjection. If (X, τ_1) is fuzzy strongly precompact, then (Y, τ_2) is almost fuzzy strongly precompact.

Proof. It is similar to the proof of the Theorem 3.7. □

Definition 3.12. An IFTS (X,τ) is called lightly fuzzy strongly precompact iff every countable strongly preopen cover of X has a finite subfamily whose preclosures cover X.

It is clear that in IFTS's we have

Countably fuzzy strongly precompact → lightly fuzzy strongly precompact.

The reverse implication does not hold:

Example 3.13. Let X ={1,2} and define the IFS's $\{A_n : n = 1,2,3, \dots\}$ as follows ;

$$A_n =< x, \left(\frac{1}{\frac{n}{n+1}}, \frac{2}{\frac{n+1}{n+2}}\right), \left(\frac{1}{\frac{1}{n+2}}, \frac{2}{\frac{1}{n+3}}\right) > .$$

In this case the family

$$\tau = \{A_n : n = 1,2, \dots\} \cup \{ 0_\sim , 1_\sim\}$$

is an IFT on X [6]. The IFTS (X,τ) is lightly fuzzy strongly precompact, but not fuzzy countably fuzzy strongly precompact.

Theorem 3.14. An IFTS (X,τ) is called lightly fuzzy strongly precompact iff every countable family $G_n : n \in N\}$, where $G_n =< x, \mu_{G_n}, \vartheta_{G_n} >, n \in N$, of IFPOS's in X having the FIP we have $\bigcap_{n \in N} precl(G_n) \neq 0_\sim$.

Proof. The proof of this theorem follows a similar pattern to Theorem 3.3. ☐

Theorem 3.15. Let (X, τ_1), (Y, τ_2) be two IFTS's and let f :X→Y be a weakly fuzzy strongly precontinuous surjection. If (X, τ_1) is countably fuzzy strongly precompact, then (Y, τ_2) is lightly fuzzy strongly precompact.

Proof. The proof is similar to Theorem 3.7. ☐

Theorem 3.16. Let (X, τ_1) and (Y, τ_2) be two IFTS's and let f : X→Y be a fuzzy strongly precontinuous surjection. If (X, τ_1) is lightly fuzzy strongly precompact, then (Y, τ_2) is countably fuzzy strongly precompact.

Proof. The proof is similar to Theorem 3.7. ☐

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