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On intuitionistic fuzzy quantifiers

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Abstract: Several intuitionistic fuzzy quantifiers are introduced and some of their properties are discussed.
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1 Short remarks on intuitionistic fuzzy propositional and predicate logics

In classical logic (e.g., [10]), to each proposition (sentence) we juxtapose its truth value: truth – denoted by 1, or falsity – denoted by 0. In the case of fuzzy logic [11], this truth value is a real number in the interval $[0, 1]$ and it is called "truth degree". In the intuitionistic fuzzy case (see [1, 2, 4, 6] we add one more value – "falsity degree" – which is again in interval $[0, 1]$. Thus, to the proposition p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

For propositions, a lot of operations were introduced in [1, 4]. We give the list of only these that will be used below. Firstly, let us define

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ \\ 0 & \text{if } x \leq 0 \end{cases},$$

and

$$\overline{\mathrm{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ \\ 1 & \text{if } x \leq 0 \end{cases}.$$

In [7], 185 intuitionistic fuzzy implications and in [8], 53 different intuitionistic fuzzy negations are defined. Below, we use the following from them:

$\neg_1$	$\langle b, a \rangle$
$\neg_2$	$\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a) \rangle$
$\neg_3$	$\langle b, a.b + a^2 \rangle$
$\neg_4$	$\langle b, 1 - b \rangle$
$\neg_8$	$\langle 1 - a, a \rangle$
$\neg_9$	$\langle \overline{\mathrm{sg}}(a), a \rangle$
$\neg_{12}$	$\langle b.(b + a), a.(b^2 + a + b.a) \rangle$
$\neg_{14}$	$\langle \mathrm{sg}(b), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{15}$	$\langle \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{18}$	$\langle b.\mathrm{sg}(a), a.\mathrm{sg}(b) \rangle$
$\neg_{19}$	$\langle b.\mathrm{sg}(a), 0 \rangle$
$\neg_{20}$	$\langle b, 0 \rangle$
$\neg_{21}$	$\langle \min(1 - a, \mathrm{sg}(a)), \min(a, \mathrm{sg}(1 - a)) \rangle$
$\neg_{22}$	$\langle \min(1 - a, \mathrm{sg}(a)), 0 \rangle$
$\neg_{23}$	$\langle 1 - a, 0 \rangle$
$\neg_{25}$	$\langle \min(b, \mathrm{sg}(1 - b)), 0 \rangle$
$\neg_{26}$	$\langle b, a.b + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{27}$	$\langle 1 - a, a.(1 - a) + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{28}$	$\langle b, (1 - b).b + \overline{\mathrm{sg}}(b) \rangle$
$\neg_{29}$	$\langle \max(0, b.a + \overline{\mathrm{sg}}(1 - b)), \min(1, a.(b.a + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(1 - a)) \rangle$
$\neg_{30}$	$\langle a.b,\ a.(a.b + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{31}$	$\langle \max(0, (1 - a).a + \overline{\mathrm{sg}}(a)), \min(1, a.((1 - a).a + \overline{\mathrm{sg}}(a)) + \overline{\mathrm{sg}}(1 - a)) \rangle$
$\neg_{32}$	$\langle (1 - a).a,\ a.((1 - a).a + \overline{\mathrm{sg}}(a)) + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, b + \varepsilon), \max(0, a - \eta) \rangle$, where $\varepsilon, \eta \in [0, 1]$ and $\varepsilon \leq \eta < 1$
$\neg_{52}$	$\langle 1 - a, \min(1, 1 - a) \rangle$
$\neg_{53}$	$\langle \overline{\mathrm{sg}}(a) + \mathrm{sg}(a)b, a \rangle$

Here, we define only the operations "disjunction", "conjunction" and "implication", originally introduced in [1], that have classical logic analogues, as follows:

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle,$$

$$V(p \to q) = \langle \max(\nu(p), \mu(q)), \min(\nu(p), \mu(q)) \rangle.$$

Thie idea for evaluation of the propositions was transformed for the predicates as follows (ses, e.g., [2, 4, 9, 5]).

Let x be a variable, obtaining values in set E and let $P(x)$ be a predicate with a variable x. Let

$$V(P(x)) = \langle \mu(P(x)), \nu(P(x)) \rangle.$$

2 Standard intuitionistic fuzzy quantifiers

The IF-interpretations of the quantifiers *for all* ($\forall$) and *there exists* ($\exists$) are introduced in [2, 9, 5] by

$$V(\exists x P(x)) = \langle \sup_{y \in E} \mu(P(y)), \inf_{y \in E} \nu(P(y)) \rangle,$$

$$V(\forall x P(x)) = \langle \inf_{y \in E} \mu(P(y)), \sup_{y \in E} \nu(P(y)) \rangle.$$

If E is a finite set, them we can use the denotations

$$V(\exists x P(x)) = \langle \max_{y \in E} \mu(P(y)), \min_{y \in E} \nu(P(y)) \rangle,$$

$$V(\forall x P(x)) = \langle \min_{y \in E} \mu(P(y)), \max_{y \in E} \nu(P(y)) \rangle.$$

In general, below, we use the first forms of both quantifiers.

Their geometrical interpretations are illustrated in Figs. 1 and 2, respectively, where $x_1, ..., x_5$ are the possible values of variable x and $V(x_1), ..., V(x_5)$, their IF-estimations.

The most important property of the two quantifiers is that each of them juxtaposes to predicate P a point (exactly one for each quantifier) in the IF-interpretational triangle.

In [9, 5], for implication $\to_4$ they are proved the following three theorems, where we use $\to$ instead of $\to_4$.

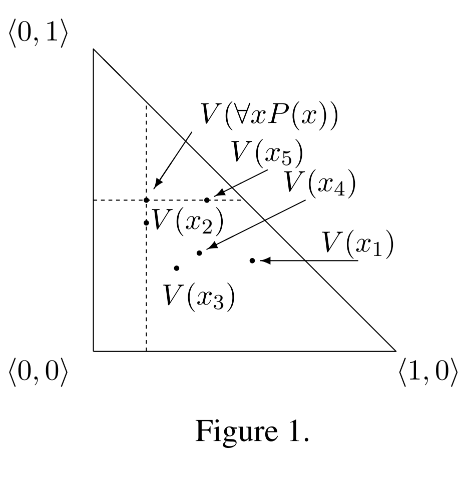

Figure 1.

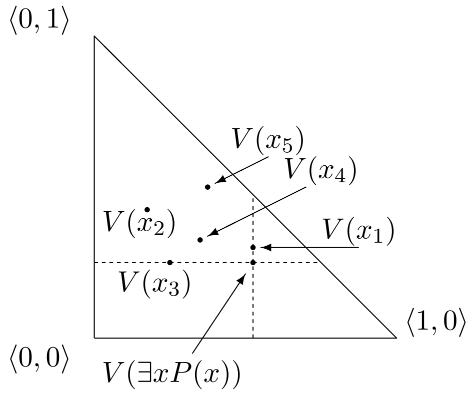

Figure 2.

Theorem 1. The logical axioms of the theory $\mathcal{K}$ (see [10]):

$$\begin{array}{ll} \text{(a)} & \forall x A(x) \to A(t), \text{ for the fixed variable } t, \\ \text{(b)} & \forall x(A \to B) \to (A \to \forall x B), \end{array}$$

are IFTs.

Proof. (a) Let the variable t be fixed.

Then

$$\begin{aligned} & V(\forall x A(x) \to A(t)) \\ &= \langle \inf_x \mu(A(x)), \sup_x \nu(A(x))\rangle \to \langle \mu(A(t)), \nu(A(t))\rangle \\ &= \langle \max(\sup_x \nu(A(x)), \mu(A(t))), \min(\inf_x \mu(A(x)), \nu(A(t)))\rangle \end{aligned}$$

and

$$\begin{aligned} & \max(\sup_x \nu(A(x)), \mu(A(t))) - \min(\inf_x \mu(A(x)), \nu(A(t))) \\ & \geq \mu(A(t)) - \inf_x \mu(A(x)) \geq 0, \end{aligned}$$

i.e., (d) is an IFT.

For (b) we obtain sequentially:

$$\begin{aligned} V(\forall x(A \to B) \to (A \to \forall x B)) &= V(\forall x(A \to B)) \to V(A \to \forall x B) \\ &= \langle \inf_x \max(\mu(B), \nu(A)), \sup_x \min(\mu(A), \nu(B))\rangle \\ &\to \langle \max(\nu(A), \inf_x \mu(B)), \min(\mu(A), \sup_x \nu(B))\rangle \\ &= \langle \max(\nu(A), \inf_x \mu(B), \sup_x \min(\mu(A), \nu(B))), \\ &\quad \min(\mu(A), \sup_x \nu(B), \inf_x \max(\mu(B), \nu(A)))\rangle \end{aligned}$$

and

$$\max(\nu(A), \inf_x \mu(B), \sup_x \min(\mu(A), \nu(B))) \geq \max(\nu(A), \inf_x \mu(B))$$

$$= \inf_x \max(\mu(B), \nu(A)) \geq \min(\mu(A), \sup_x \nu(B), \inf_x \max(\mu(B), \nu(A))),$$

i.e., (e) also is an IFT. $\square$

Below, we list some assertions, which are theorems of the classical first order logic (see, e.g. [10]).

Theorem 2. The following formulae are IFTs:

(a) $(\forall x A(x) \to B) \equiv \exists x(A(x) \to B)$,
(b) $\exists x A(x) \to B \equiv \forall x(A(x) \to B)$,
(c) $B \to \forall x A(x) \equiv \forall x(B \to A(x))$,
(d) $B \to \exists x A(x) \equiv \exists x(B \to A(x))$,
(e) $(\forall x A \wedge \forall x B) \equiv \forall x(A \wedge B)$,
(f) $(\forall x A \vee \forall x B) \to \forall x(A \vee B)$,
(g) $\neg \forall x A \equiv \exists x \neg A$,
(h) $\neg \exists x A \equiv \forall x \neg A$,
(i) $\forall x \forall y A \equiv \forall y \forall x A$,
(j) $\exists x \exists y A \equiv \exists y \exists x A$,
(k) $\exists x \forall y A \to \forall y \exists x A$,
(l) $\forall x(A \to B) \to (\forall x A \to \forall x B)$.

Theorem 3. For each predicate P and for negation $\neg_i, \forall x P(x) \vee \exists x \neg_i P(x)$ is an IFT for $\neg_1$, $\neg_3$, $\neg_4$, $\neg_8$, $\neg_9$, $\neg_{11}$, $\neg_{12}$, $\neg_{14}$, $\neg_{15}$, $\neg_{11}$, $\neg_{18}$, ..., $\neg_{23}$, $\neg_{25}$, ..., $\neg_{32}$, $\neg_{45}$, $\neg_{52}$, $\neg_{53}$.

The link between the interpretations of quantifiers and the topological operators C (closure) and I (interior) defined over IFSs see [2] is obvious.

3 Extended intuitionistic fuzzy quantifiers

In [4], we introduced the following six quantifiers and studied some of their properties.

$$V(\forall_\mu x P(x)) = \{\langle x, \inf_{y \in E} \mu(P(y)), \nu(P(x))\rangle | x \in E\},$$

$$V(\forall_\nu x P(x)) = \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x))), \sup_{y \in E} \nu(P(y))\rangle | x \in E\},$$

$$V(\exists_\mu x P(x)) = \{\langle x, \sup_{y \in E} \mu(P(y)), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(x)))\rangle | x \in E\},$$

$$V(\exists_\nu x P(x)) = \{\langle x, \mu(P(x)), \inf_{y \in E} \nu(P(y))\rangle | x \in E\},$$

$$V(\forall_\nu^* x P(x))$$

$$= \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x))), \min(\sup_{y \in E} \nu(P(y)), 1 - \mu(P(x)))\rangle | x \in E\},$$

$$V(\exists_\mu^* x P(x))$$
$$= \{\langle x, \min(\sup_{y\in E} \mu(P(y)), 1-\nu(P(x))), \min(1-\sup_{y\in E} \mu(P(y)), \nu(P(x)))\rangle | x \in E\}.$$

Let the possible values of variable x be a, b, c and let their IF-estimations $V(a), V(b), V(c)$ be shown on Fig. 3. The geometrical interpretations of the new quantifiers are shown in Figs. 4–9.

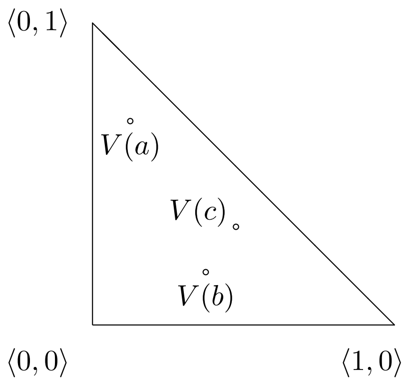

Figure 3.

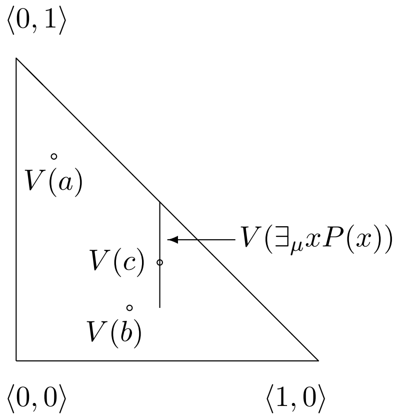

Figure 4.

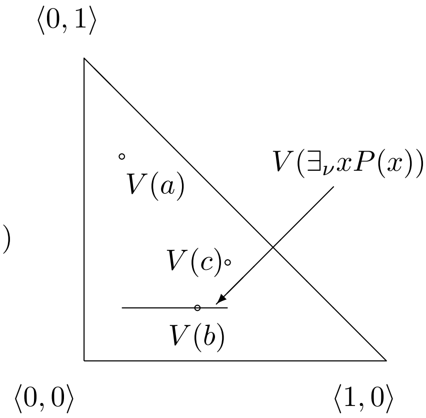

Figure 5.

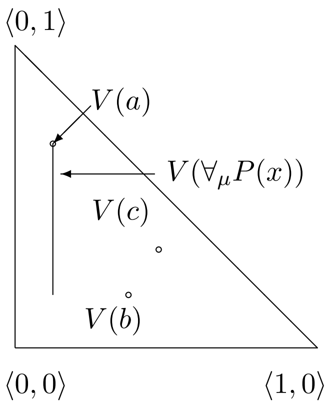

Figure 6.

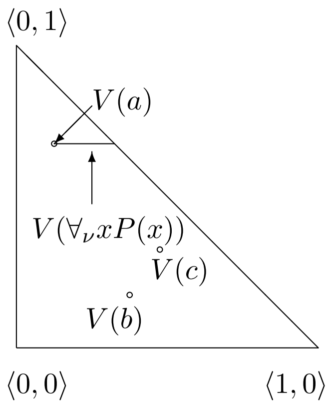

Figure 7.

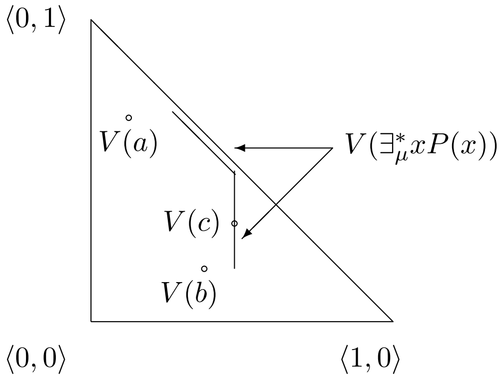

Figure 8.

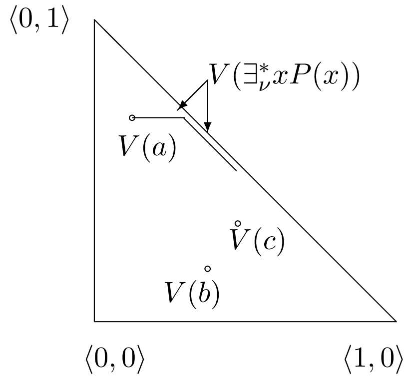

Figure 9.

Now, we see that we can change the forms of the first two quantifiers to the forms

$$V(\forall xP(x)) = \{\langle x, \inf_{y\in E}\mu(P(y)), \sup_{y\in E}\nu(P(y))\rangle | x \in E\},$$

$$V(\exists xP(x)) = \{\langle x, \sup_{y\in E}\mu(P(y)), \inf_{y\in E}\nu(P(y))\rangle | x \in E\}.$$

Obviously, for every predicate P,

$$V(\forall xP(x)) \subseteq V(\forall_\mu xP(x)) \subseteq V(\forall_\nu xP(x)) \subseteq V(\exists_\nu xP(x))$$

$$\subseteq V(\exists_\mu xP(x)) \subseteq V(\exists xP(x))$$

and

$$V(\forall xP(x)) \subseteq V(\forall_\nu xP(x)) \subseteq V(\forall^*_\nu xP(x))$$
$$\subseteq V(\exists^*_\mu xP(x)) \subseteq V(\exists_\mu xP(x)) \subseteq V(\exists xP(x)).$$

Now, we can modofy the new six operators, so to change their set form to the form of the first two operaotrs.

Let a be one of possible values for variable x. Then

$$V((\forall_\mu xP(x)), a) = \langle \inf_{y\in E}\mu(P(y)), \nu(P(a))\rangle,$$

$$V((\forall_\nu xP(x)), a) = \langle \min(1 - \sup_{y\in E}\nu(P(y)), \mu(P(a))), \sup_{y\in E}\nu(P(y))\rangle,$$

$$V((\exists_\mu xP(x)), a) = \langle \sup_{y\in E}\mu(P(y)), \min(1 - \sup_{y\in E}\mu(P(y)), \nu(P(a)))\rangle,$$

$$V((\exists_\nu xP(x)), a) = \langle \mu(P(a)), \inf_{y\in E}\nu(P(y))\rangle,$$

$$V((\forall^*_\nu xP(x)), a)$$
$$= \langle \min(1 - \sup_{y\in E}\nu(P(y)), \mu(P(a))), \min(\sup_{y\in E}\nu(P(y)), 1 - \mu(P(a)))\rangle,$$

$$V((\exists^*_\mu xP(x)), a)$$
$$= \langle \min(\sup_{y\in E}\mu(P(y)), 1 - \nu(P(a))), \min(1 - \sup_{y\in E}\mu(P(y)), \nu(P(a)))\rangle.$$

We finish this section with an example.

Let the universe comprise the members of the European Union and let for each country the degree of government approval and disapproval be known. Let the predicate $P(x)$ be "The government of country x is widely approved by the people of country x". The first quantifier $\forall$ will give the minimal degree of approval which exists in the countries of the EU, and the maximal degree of disapproval in the countries (not necessarily the same). Conversely, the second operator $\exists$ will give us the maximal degree of approval in one of these countries and the minimal degree of disapproval.

Let us assume that for some reason we do not have complete information about either the approval or disapproval for a fixed country a from the EU (but we have such information about the rest). If we are missing information about the degree of approval for a, then, the third operator $\forall_\mu$ will give us a lower bound for this degree of approval for a. The fifth operator $\exists_\mu$ will give us an upper bound for the degree of approval for a.

Conversely, if we are missing information about the degree of disapproval, the fourth operator will give us $\forall_\nu$ will give us the upper bound and the sixth $\exists_\nu$ will give us the lower bound for the degree of disapproval for a.

The seventh and eighth operators act exactly like the fourth and the fifth operators, respectively, but provide a more precise evaluation for the respective degree.

4 Ideas for new types of disjunctions and conjunctions

As it was mentioned in Section 1, now there are 185 different intuitionistic fuzzy implications and 53 different intuitionistic fuzzy negations. Therefore, using the standard formulas for defining of disjunctions and conjunctions

$$A \vee B = \neg A \to B, \tag{1}$$

$$A \wedge B = \neg(A \to \neg B), \tag{2}$$

we can construct 185 disjunctions and 185 conjunctions.

For example, if $V(A) = \langle a, b\rangle, V(B) = \langle c, d\rangle, a, b, c, d \in [0, 1], a + b \leq 1, c + d \leq 1$ and if we use second intuitionistic fuzzy negation $\neg_2$ and its related intuitionistic fuzzy implication $\to_2$, that are defined by

$$\neg_2\langle a, b\rangle = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle,$$

$$\langle a, b\rangle \to_2 \langle c, d\rangle = \langle \overline{\text{sg}}(a - c), d\text{sg}(a - c)\rangle,$$

we obtain sequentially:

$$\begin{aligned} V(A \vee_2 B) &= \langle a, b\rangle \vee_2 \langle c, d\rangle \\ &= \neg_2\langle a, b\rangle \to_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \to_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(\overline{\text{sg}}(a) - c), d\text{sg}(\overline{\text{sg}}(a) - c) \end{aligned}$$

and

$$V(A \wedge_2 B) = \langle a, b\rangle \wedge_2 \langle c, d\rangle$$

$$= \neg_2(\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)$$

$$= \neg_2(\langle a,b\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)$$

$$= \neg_2\langle \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(a - \overline{\mathrm{sg}}(c))\rangle$$

$$= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c)))\rangle$$

$$= \langle \mathrm{sg}(a - \overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a - \overline{\mathrm{sg}}(c))\rangle.$$

On the other hand, as we saw , e.g., in [3], the disjunctions and conjunctions can have the following other forms:

$$A \vee B = \neg A \rightarrow \neg\neg B, \tag{3}$$

$$A \wedge B = \neg(\neg\neg A \rightarrow \neg B). \tag{4}$$

For example, for the above formulas A and B, negation $\neg_2$ and implication $\rightarrow_2$, we obtain sequentially:

$$V(A \vee_2 B) = \neg_2\langle a,b\rangle \vee_2 \neg_2\neg_2\langle c,d\rangle$$

$$= \neg_2\langle a,b\rangle \rightarrow_2 \neg_2\langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle$$

$$= \neg_2\langle a,b\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(c)), \mathrm{sg}(\overline{\mathrm{sg}}(c))\rangle$$

(For every $x \in [0,1]$:

$$\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(x)) = \begin{cases} 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases} = \mathrm{sg}(x)$$

and

$$\mathrm{sg}(\overline{\mathrm{sg}}(x)) = \begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ if } x > 0 \end{cases} = \overline{\mathrm{sg}}(x). \quad)$$

$$= \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_2 \langle \mathrm{sg}(c), \overline{\mathrm{sg}}(c)\rangle$$

$$= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a) - \mathrm{sg}(c)), \overline{\mathrm{sg}}(c)\mathrm{sg}(\overline{\mathrm{sg}}(a) - \mathrm{sg}(c))\rangle$$

and

$$V(A \wedge_2 B) = \langle a,b\rangle \wedge_2 \langle c,d\rangle$$

$$= \neg_2(\neg_2\neg_2\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)$$

$$= \neg_2(\neg_2\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)$$

$$= \neg_2(\langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)), \mathrm{sg}(\overline{\mathrm{sg}}(a))\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)$$

$$= \neg_2(\langle \mathrm{sg}(a), \overline{\mathrm{sg}}(a)\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)$$

$$= \neg_2\langle \overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))\rangle$$

$$= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)))\rangle$$

$$= \langle \mathrm{sg}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(\mathrm{sg}(a) - \overline{\mathrm{sg}}(c))\rangle.$$

Therefore, formulas (1)–(4) must be rewritten to

$$A \vee_{i,1} B = \neg_{\varphi(i)} A \rightarrow_i B, \tag{5}$$

$$A \wedge_{i,1} B = \neg_{\varphi(i)}(A \rightarrow_i \neg_{\varphi(i)} B), \tag{6}$$

$$A \vee_{i,2} B = \neg_{\varphi(i)} A \rightarrow_i \neg_{\varphi(i)}\neg_{\varphi(i)} B, \tag{7}$$

$$A \wedge_{i,2} B = \neg_{\varphi(i)}(\neg_{\varphi(i)}\neg_{\varphi(i)} A \rightarrow_i \neg_{\varphi(i)} B), \tag{8}$$

where $\varphi(i)$ is the number of the negation that corresponds to the i-th implication.

Now, we see a possibility for constructing a third group of disjunctions and conjunctions. They have the forms

$$A \vee_{i,3} B = \neg_1 A \rightarrow_i B, \tag{9}$$

$$A \wedge_{i,3} B = \neg_1(A \rightarrow_i \neg_1 B). \tag{10}$$

For example, for the above formulas A and B, negation $\neg_2$ and implication $\rightarrow_2$, we obtain sequentially:

$$\begin{aligned} V(A \vee_{2,3} B) &= \neg_1\langle a, b\rangle \vee_2 \neg_1\neg_1\langle c, d\rangle \\ &= \langle b, a\rangle \vee_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(b-c), d.\text{sg}(b-c)\rangle \end{aligned}$$

and

$$\begin{aligned} V(A \wedge_{2,3} B) &= \neg_1(\neg_1\neg_1\langle a, b\rangle \rightarrow_2 \neg_1\langle c, d\rangle) \\ &= \neg_1(\langle a, b\rangle \rightarrow_2 \langle d, c\rangle) \\ &= \neg_1\langle \overline{\text{sg}}(a-d), c.\text{sg}(a-d)\rangle \\ &= \langle c.\text{sg}(a-d), \overline{\text{sg}}(a-d)\rangle. \end{aligned}$$

5 Ideas for new types of quantifiers

It is well known from classical logic that for each predicate P with argument x having interpretations $a_1, a_2, ..., a_n$ (finite number):

$$V(\forall x P(x)) = V(P(a_1) \wedge P(a_2) \wedge ... \wedge P(a_n)),$$

$$V(\exists x P(x)) = V(P(a_1) \vee P(a_2) \vee ... \vee P(a_n)).$$

Now, we can construct a lot of new quantifiers – for each new pair of conjunction and disjunction, we obtain a pair of quantifiers that have the forms

$$\begin{aligned} V(\forall_{i,j} P(x)) &= V(P(a_1) \wedge_{i,j} P(a_2) \wedge_{i,j} ... \wedge_{i,j} P(a_n), \\ V(\exists_{i,j} P(x)) &= V(P(a_1) \vee_{i,j} P(a_2) \vee_{i,j} ... \vee_{i,j} P(a_n), \end{aligned}$$

where $i(1 \leq i \leq 185)$ and $j(1 \leq j \leq 3)$ are the indices of the respective pair of conjunction and disjunction that generates the new pair of quantifiers.

Obviously, $\forall_{4,1}$ coincides with the standard quantifier $\forall$ and $\exists_{4,1}$ coincides with the standard quantifier $\exists$.

One special case is the following: using implication $\rightarrow_{139}$ and negation $\neg_1$ we obtain for $a, b, c, d \in [0, 1]$ and $a + b, c + d \leq 1$:

$$V(\langle a, b\rangle \vee_{139,3} \langle c, d\rangle) = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle = \langle a, b\rangle \wedge_{139,3} \langle c, d\rangle.$$

Therefore, if for each i: $V(P(x_i)) = \langle a_i, b_i\rangle$, then

$$V(\forall_{139,3} x P(x)) = \left\langle \frac{\sum_i a_i}{n}, \frac{\sum_i b_i}{n} \right\rangle = \exists_{139,3} P(x).$$

In this case, we check directly, that

$$\neg_1 \forall_{139,3} x \neg_1 P(x) = \forall_{139,3} x P(x).$$

Hence, there exists a quantifier's interpretation for which both quantifiers "$\forall$" and "$\exists$" coincide.

It is very interesting that the weight-operator W (see, e.g. [3]) is exact analogue of quantifier $\forall_{139,3}$.

6 Conclusion

The so defined quantifiers give us the possibility to clasify all of them in two groups.

- Global quantifiers: $\forall, \exists$,
- Local quantifiers: $\forall_\mu, \forall_\mu, \forall_\nu^*, \exists_\mu, \exists_\nu, \exists_\mu^*$.

Open Problem. Study in details the behaviour of these quantifiers.

In future, we plan to work over this problem.

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