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H-Intuitionistic fuzzy sets

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Abstract: In this paper, we introduce Heyting valued intuitionistic fuzzy set (H-IFS) using intuitionistic fuzzy Heyting algebra. Some operations and modal operators are defined over H-IFSs.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy Heyting algebra, Heyting valued intuitionistic fuzzy sets.
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1 Introduction

The original concept of fuzzy sets in Zadeh [7] was introduced as an extension of crisp sets by enlarging the truth value set to the real unit interval $[0, 1]$. In fuzzy set theory, if the membership degree of an element x is $\mu(x)$ then the non-membership degree is $1 - \mu(x)$ and thus it is fixed.

Intuitionistic fuzzy sets introduced by Atanassov in 1983 [1] and form an extension of fuzzy sets by enlarging the truth value set to the lattice $[0, 1] \times [0, 1]$ is defined as following.

Definition 1. *Let* $L = [0, 1]$ *then*

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".

For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows:

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2)),$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2)).$$

For each $J \subseteq L^*$,

$$\sup J = (\sup\{x : (x, y \in [0,1]), ((x,y) \in J)\}, \inf\{y : (x, y \in [0,1])((x,y) \in J)\})$$

and

$$\inf J = (\inf\{x : (x, y \in [0,1])((x,y) \in J)\}, \sup\{y : (x, y \in [0,1])((x,y) \in J)\}).$$

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x)$, $(\nu_A : X \to [0,1])$ is called the "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 3. *[1] An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1] Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is called the complement of A:*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

Heyting algebras were introduced by A. Heyting, in 1930 to formalize intuitionistic logic [5]. This algebra has many studying areas as topos theory, linguistics, quantum theory, etc.

Definition 5. *[5] A Heyting algebra is an algebra $(H, \vee, \wedge, \to, 0_H, 1_H)$ such that $(H, \vee, \wedge, 0, 1)$ is an lattice and for all $a, b, c \in H$,*

$$a \leq b \to c \Leftrightarrow a \wedge b \leq c.$$

$(H, \vee, \wedge, 0_H, 1_H)$ lattice is a Heyting algebra with $\to$ binary operation. $\forall a, b \in H$,

$$a \to b = \bigvee \{c : a \wedge c \leq b, c \in H\}.$$

Proposition 1. *An algebra $(H, \vee, \wedge, \to, 0_H, 1_H)$ is a Heyting algebra if and only if $(H, \vee, \wedge, 0_H, 1_H)$ is an lattice and the following identities hold for all $a, b, c \in H$,*

1. $a \to a = 1$
2. $a \wedge (a \to b) = a \wedge b$
3. $b \wedge (a \to b) = b$
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

E. Eslami introduced the Intuitionistic Fuzzy Residuated Lattice (IFRL) in 2012, [3], and examined some properties of these lattices.

Definition 6. *[6] A residuated lattice is an algebra* $L = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *such that;*

1. $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
2. $(L, *, 1)$ *is a commutative monoid*
3. *The operation* $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.,*

$$x * y \leq z \text{ if and only if } x \leq y \rightarrow z$$

for all $x, y, z \in L$.

A residuated lattice L is called a Heyting Algebra, in which $x * y = x \wedge y$, for all $x, y \in L$.

In [3], determined that if $L = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice and $\sim: L \rightarrow L$ by $\sim x = x \rightarrow 0$, then $\sim$is a negator on L which has not to be involutive. We can obtain a symmetric residuated lattice by adding an involutive negator, as follows.

Definition 7. *[3] A residuated lattice* L *is called a symmetric residuated lattice if it is equipped with unary operation* $\sim$ *satisfying;*

$$\begin{aligned} \sim & \sim x = x \\ \sim & (x \vee y) = \sim x \wedge \sim y \\ \sim & (x \wedge y) = \sim x \vee \sim y \end{aligned}$$

for all $x, y \in L$.

Definition 8. *[3] Let* $L = (L, \vee, \wedge, *, \rightarrow, \sim, 0, 1)$ *be a symmetric residuated lattice. A intuitionisitc fuzzy residuated lattice ia an algebra* $\widetilde{L} = \left(\widetilde{L}, \vee, \wedge, T, I, \widetilde{0}, \widetilde{1}\right)$ *where*

1. $\widetilde{L} = \{(x, y) \in L^2 : x \leq \sim y\}$
2. $(x, y) \wedge (u, v) = (\min(x, u), \max(y, v))$
3. $(x_1, y_1) \vee (x_2, y_2) = (\max(x, u), \min(y, v))$
4. $T((x, y), (u, v)) = (x * u, S(y, v))$ *, where* $S(y, v) = \sim (\sim y * \sim v)$
5. $I((x, y), (u, v)) = ((x \rightarrow u) \wedge (\sim y \rightarrow \sim v), \sim (\sim y \rightarrow \sim v))$
6. $\widetilde{0} = (0, 1), \widetilde{1} = (1, 0)$

Theorem 1. *[4] Let* L *and* $\widetilde{L}$ *defined as above definition. Then* L *is a Heyting algebra if and only if* $\widetilde{L}$ *is a Heyting algebra.*

2 H-Intuitionistic fuzzy sets

In this section, we studied a special case of intuitionisitc fuzzy residuated lattice.

Definition 9. *Let $L = ([0, 1], \vee, \wedge, \rightarrow, 0, 1)$ be a Heyting algebra and $\sim x = 1 - x$ is defined for all $x \in L$. An intuitionistic fuzzy Heyting algebra (IFHA) is an algebra $L^* = (L^*, \vee, \wedge, I, 0^*, 1^*)$ where*

1. $L^* = \{(x, y) \in [0, 1]^2 : x \leq \sim y\}$
2. $(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
3. $(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$
4. $I((x_1, y_1), (x_2, y_2)) = ((x_1 \rightarrow x_2) \wedge (\sim y_1 \rightarrow \sim y_2), \sim(\sim y_1 \rightarrow \sim y_2))$
5. $0^* = (0, 1), 1^* = (1, 0)$

Definition 10. *Let $L^* = (L^*, \vee, \wedge, I, 0^*, 1^*)$ be an intuitionistic fuzzy Heyting algebra. A Heyting valued Intuitionistic fuzzy set on a set X called H-Intuitionistic Fuzzy Set (shortly $H - IFS(X)$) is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where μ_A and ν_A are called "degree of membership of x in A"and "degree of non-membership of x in A", resp. which defined from X to $L = ([0, 1], \vee, \wedge, \rightarrow, 0, 1)$ Heyting algebra and satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The following fundamental operations and relations on $H-IFS(X)$ are similar with $IFS(X)$.

Definition 11. *Let $A, B \in H - IFS(X)$ then*

1. $A \sqsubseteq B$ *iff* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$, *for all* $x \in X$.
2. $A = B$ *iff* $A \sqsubseteq B$ *and* $B \sqsubseteq A$
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$
4. $A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$
5. $A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$

Two intuitionistic fuzzy modal operators called "necessity" and "possibility" were defined by Atanassov, as follows:

Definition 12. *[1] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.*

1. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x))\rangle : x \in X\}$
2. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$

Now, we will redefine these intuitionistic fuzzy modal operators on $H - IFS(X)$ and we will prove some properties of them. We will use $\Box_H$ and $\Diamond_H$ expressions to show that study on Heyting valued intuitionistic fuzzy sets.

Definition 13. *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in H - IFS(X)$.*

1. $\Box_H A = \{\langle x, 1 \to \mu_A(x), \sim (1 \to \mu_A(x))\rangle : x \in X\}$
2. $\Diamond_H A = \{\langle x, \sim (1 \to \nu_A(x)), 1 \to \nu_A(x)\rangle : x \in X\}$

$\Box_H A$ and $\Diamond_H A$ are called Heyting valued intuitionistic fuzzy modal operators.

Theorem 2. *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in H - IFS(X)$.*

1. $\Box_H A = I(1^*, \Box A)$
2. $\Diamond_H A = I(1^*, \Diamond A)$.

Proof. (1) For $x \in X$,

$$\begin{aligned} I(1^*, \Box A) &= I((1,0), (\mu_A(x), 1 - \mu_A(x))) \\ &= ((1 \to \mu_A(x)) \wedge (\sim 0 \to \sim (1 - \mu_A(x))), \sim (\sim 0 \to \sim (1 - \mu_A(x)))) \\ &= (1 \to \mu_A(x), \sim (1 \to \mu_A(x))) \\ &= \Box_H A \end{aligned}$$

(2) If we use $1 \to \sim a = \sim (1 \to a)$ for all $a \in [0,1]$ then it is clear that $\Diamond_H A = I(1^*, \Diamond A)$. □

Theorem 3. *Let X be a set and $A \in H - IFS(X)$.*

1. $(\Box_H A)^c = \Diamond_H A$
2. $(\Diamond_H A)^c = \Box_H A^c$
3. $\Box_H A \sqsubseteq \Diamond_H A$
4. $\Box_H \Box_H A = \Box_H A$
5. $\Diamond_H \Diamond_H A = \Diamond_H A$
6. $\Diamond_H \Box_H A = \Box_H A$
7. $\Box_H \Diamond_H A = \Diamond_H A$

Proof. (1) We know that $\Diamond_H A^c = I(1^*, \Diamond A^c)$. For $x \in X$,

$$\begin{aligned} I(1^*, \Diamond A^c) &= ((1 \to \sim \mu_A(x)) \wedge (\sim 0 \to \sim \mu_A(x)), \sim (\sim 0 \to \sim \mu_A(x))) \\ &= ((1 \to \sim \mu_A(x)) \wedge (1 \to \sim \mu_A(x)), \sim (1 \to \sim \mu_A(x))) \\ &= ((1 \to \sim \mu_A(x)), \sim (1 \to \sim \mu_A(x))) \\ &= (\sim (1 \to \mu_A(x)), (1 \to \mu_A(x))) \end{aligned}$$

So, $\Diamond_H A^c = I\left(1^*, \Diamond A^c\right) = \left(\Box_H A\right)^c$.

(2) If we use $\Box A \sqsubseteq \Diamond A$ then $\mu_A(x) \leq \sim \nu_A(x)$ and $\sim \mu_A(x) \geq \nu_A(x)$.

$$\begin{aligned} 1 &\rightarrow \mu_A(x) \leq 1 \rightarrow \sim \nu_A(x) \text{ and } 1 \rightarrow \sim \mu_A(x) \geq 1 \rightarrow \nu_A(x) \\ &\Rightarrow 1 \rightarrow \mu_A(x) \leq \sim (1 \rightarrow \nu_A(x)) \text{ and } \sim (1 \rightarrow \mu_A(x)) \geq 1 \rightarrow \nu_A(x) \end{aligned}$$

We obtained that $\Box_H A \sqsubseteq \Diamond_H A$.

Other properties can be seen similarly. □

Proposition 2. *Let $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ be a Heyting algebra.*

1. *If $a \leq b$ for $a, b \in H$ then $c \rightarrow a \leq c \rightarrow b$ for all $c \in H$.*
2. *$c \rightarrow (a \vee b) = (c \vee a) \rightarrow (c \vee b)$ for all $a, b, c \in H$.*

Theorem 4. *Let X be a set and $A, B \in H - IFS(X)$.*

1. *$\Box_H(A \sqcup B) = \Box_H A \sqcup \Box_H B$*
2. *$\Box_H(A \sqcap B) = \Box_H A \sqcap \Box_H B$*

Proof. (1)

$$\begin{aligned} \Box_H(A \sqcup B) &= \left\{\left\langle x, 1 \rightarrow \max\left\{\mu_A(x), \mu_B(x)\right\}, \sim \left(1 \rightarrow \max\left\{\mu_A(x), \mu_B(x)\right\}\right)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \max\left\{1 \rightarrow \mu_A(x), 1 \rightarrow \mu_B(x)\right\}, \sim \max\left\{1 \rightarrow \mu_A(x), 1 \rightarrow \mu_B(x)\right\}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \max\left\{1 \rightarrow \mu_A(x), 1 \rightarrow \mu_B(x)\right\}, \min\left\{\sim \left(1 \rightarrow \mu_A(x)\right), \sim \left(1 \rightarrow \mu_B(x)\right)\right\}\right\rangle : x \in X\right\} \\ &= \Box_H A \sqcup \Box_H B \end{aligned}$$

(2) It can be proved easily. □

Theorem 5. *Let X be a set and $A, B \in H - IFS(X)$.*

1. *$\Diamond_H(A \sqcap B) = \Diamond_H A \sqcap \Diamond_H B$*
2. *$\Diamond_H(A \sqcup B) = \Diamond_H A \sqcup \Diamond_H B$*

Proof. Let us prove the second characteristic.

$$\begin{aligned} \Diamond_H(A \sqcup B) &= \left\{\left\langle x, \sim \left(1 \rightarrow \min\left\{\nu_A(x), \nu_B(x)\right\}\right), 1 \rightarrow \min\left\{\nu_A(x), \nu_B(x)\right\}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \sim \min\left\{1 \rightarrow \nu_A(x), 1 \rightarrow \nu_B(x)\right\}, \min\left\{1 \rightarrow \nu_A(x), 1 \rightarrow \nu_B(x)\right\}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \min\left\{\sim \left(1 \rightarrow \nu_A(x)\right), \sim \left(1 \rightarrow \nu_B(x)\right)\right\}, \min\left\{1 \rightarrow \nu_A(x), 1 \rightarrow \nu_B(x)\right\}\right\rangle : x \in X\right\} \\ &= \Diamond_H A \sqcup \Diamond_H B \end{aligned}$$

□

Theorem 6. *Let X be a set and $A, B \in H - IFS(X)$.*

1. *$A \sqsubseteq B$ iff $\Box_H A \sqsubseteq \Box_H B$*
2. *$A \sqsubseteq B$ iff $\Diamond_H A \sqsubseteq \Diamond_H B$*

Proof. (1)

$$\begin{aligned} A &\sqsubseteq B \Rightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \sim \mu_A(x) \geq \sim \mu_B(x) \\ &\Rightarrow 1 \to \mu_A(x) \leq 1 \to \mu_B(x) \text{ and } 1 \to \sim \mu_A(x) \geq 1 \to \sim \mu_B(x) \\ &\Rightarrow 1 \to \mu_A(x) \leq 1 \to \mu_B(x) \text{ and } \sim (1 \to \mu_A(x)) \geq \sim (1 \to \mu_B(x)) \\ &\Rightarrow \Box_H A \sqsubseteq \Box_H B \end{aligned}$$

(2) It is clear. □

3 Conclusion

Heyting valued new intuitionistic fuzzy modal operators can be defined on H-Intuitionistic fuzzy set. Basic algebraic properties of H-Intuitionistic fuzzy sets can be studied.

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