Notes on Intuitionistic Fuzzy Sets
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Individual ergodic theorem for intuitionistic fuzzy observables using IF-probability

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Abstract: The aim of this paper is to formulate the individual ergodic theorem for intuitionistic fuzzy observables using $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ is an intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability can be decomposed to two intuitionistic fuzzy states, we can use the results holding for intuitionistic fuzzy states.
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1 Introduction

In [1, 2], K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. Later P. Grzegorzewski and E. Mrówka defined the probability on the family of intuitionistic fuzzy events

$$\mathcal{N} = \{(\mu_A, \nu_A)\ ;\ \mu_A, \nu_A \text{ are } \mathcal{S}\text{-measurable and } \mu_A + \nu_A \leq 1_\Omega\}$$

as a mapping $\mathcal{P}$ from the family $\mathcal{N}$ to the set of all compact intervals in R by the formula

$$\mathcal{P}((\mu_A, \nu_A)) = \left[\int_\Omega \mu_A \, dP, 1 - \int_\Omega \nu_A \, dP\right],$$

where $(\Omega, \mathcal{S}, P)$ is the probability space, see [7]. This intuitionistic fuzzy probability was axiomatically characterized by B. Riečan (see [10]).

In this paper, we formulate the Individual ergodic theorem for intuitionistic fuzzy observables, using $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ is an intuitionistic fuzzy probability. Recall that the formulation of the individual ergodic theorem for intuitionistic fuzzy events with product first appeared in the paper [3]. There we used a separating intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposed to two intuitionistic fuzzy states, we can use the results holding for intuitionistic fuzzy states, which were proved in [6].

Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 IF-events, IF-states, IF-observables and IF-mean value

In this section we explain the basic notions from IF-probability theory, see [1, 2, 13, 14, 15].

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \le 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0, 1]$ will be called the membership function, $\nu_A : \Omega \longrightarrow [0, 1]$ will be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Łukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1) \vee 0_\Omega)), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)) \end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \le \mathbf{B} \Longleftrightarrow \mu_A \le \mu_B, \nu_A \ge \nu_B.$$

In the paper, we use max–min connectives defined by

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$

$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the de Morgan rules

$$(a \vee b)^* = a^* \wedge b^*,$$

$$(a \wedge b)^* = a^* \vee b^*,$$

where $a^* = 1 - a$.

Example 2.3. *A fuzzy set $f : \Omega \longrightarrow [0, 1]$ can be regarded as an IF-set if we put*

$$\mathbf{A} = (f, 1_\Omega - f).$$

If $f = \chi_A$, then the corresponding IF-set has the form

$$\mathbf{A} = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the intersection of sets and $\leq$ to the set inclusion.

Consider a probability space $(\Omega, \mathcal{S}, P)$. Then in [7] the IF-probability $\mathcal{P}(\mathbf{A})$ of an IF-event $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$ has been defined as a compact interval by the equality

$$\mathcal{P}(\mathbf{A}) = \left[\int_\Omega \mu_A \, dP, 1 - \int_\Omega \nu_A \, dP \right].$$

Let $\mathcal{J}$ be the family of all compact intervals. Then the mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ can be defined axiomatically similarly as in [10].

Definition 2.4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ is called an IF-probability if the following conditions hold:*

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1, 1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0, 0]$;

(ii) If $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$;

(iii) If $\mathbf{A}_n \nearrow \mathbf{A}$, then $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.
(Recall that $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ means that $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$, but $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ means $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$.)

IF-probability $\mathcal{P}$ is called **separating**, if

$$\mathcal{P}\big((\mu_A, \nu_A)\big) = [\mathcal{P}^\flat(\mu_A), 1 - \mathcal{P}^\sharp(\nu_A)],$$

where the functions $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{T} \to [0, 1]$ are probabilities.

Of course, each $\mathcal{P}(\mathbf{A})$ is an interval, denote it by $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$. By this way we obtain two functions

$$\mathcal{P}^\flat : \mathcal{F} \to [0, 1], \mathcal{P}^\sharp : \mathcal{F} \to [0, 1]$$

and some properties of $\mathcal{P}$ can be characterized by some properties of $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see [11].

Theorem 2.5. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ and $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. Then $\mathcal{P}$ is an IF-probability if and only if $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ are IF-states.*

Proof. See [11, Theorem 2.3] □

Recall that by an **intuitionistic fuzzy state (IF-state) m** we understand each mapping $\mathbf{m} : \mathcal{F} \to [0, 1]$ which satisfies the following conditions (see [12]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Now we introduce the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted by $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets (see [16]).

Definition 2.6. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) If $A \cap B = \emptyset$ *and* $A, B \in \mathcal{B}(R)$*, then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*

(iii) If $A_n \nearrow A$ *and* $A_n, A \in \mathcal{B}(R)$*,* $n \in N$*, then* $x(A_n) \nearrow x(A)$*.*

Similarly, we can define the notion of n-dimensional IF-observable.

Definition 2.7. *By an n-dimensional IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) If $A \cap B = \emptyset$ *and* $A, B \in \mathcal{B}(R^n)$*, then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*

(iii) If $A_n \nearrow A$ *and* $A_n, A \in \mathcal{B}(R^n)$*,* $n \in N$*, then* $x(A_n) \nearrow x(A)$*.*

Similarly, as in the classical case the following theorem can be proved ([9, 15]).

Theorem 2.8. *Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \longrightarrow [0, 1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0, 1]$ by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0, 1]$ is a probability measure.

Since $\mathbf{m}_x : \mathcal{B}(R) \to [0, 1]$ plays now an analogous role as $P_\xi : \mathcal{B}(R) \to [0, 1]$, we can define **IF-expected value** $\mathbf{E}(x)$ by the same formula (see [9]).

Definition 2.9. *We say that an IF-observable x is an integrable IF-observable if the integral $\int_R t \, d\mathbf{m}_x(t)$ exists. In this case, we define the IF-expected value*

$$\mathbf{E}(x) = \int_R t \, d\mathbf{m}_x(t).$$

If the integral $\int_R t^2 \, d\mathbf{m}_x(t)$ exists, then we define IF-dispersion $\mathbf{D}^2(x)$ by the formula

$$\mathbf{D}^2(x) = \int_R t^2 \, d\mathbf{m}_x(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \, d\mathbf{m}_x(t).$$

3 Product operation, joint IF-observable and function of several IF-observables

In [8] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$, and showed an example of this operation.

Definition 3.1. *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfies the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$*;*

(ii) The operation $\cdot$ is commutative and associative;

(iii) If $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$*, then*

$$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$$

and

$$\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$$

for each $(c_1, c_2) \in \mathcal{F}$*;*

(iv) If $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$*,* $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$*, then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$*.*

The following theorem defines the product operation for IF-events.

Theorem 3.2. *The operation $\cdot$ defined by*

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ *is product operation on* $\mathcal{F}$*.*

Proof. See [8, Theorem 1]. □

In [13] B. Riečan defined the notion of a joint IF-observable and he proved its existence.

Definition 3.3. *Let* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be two IF-observables. The joint IF-observable of the IF-observables* x, y *is a mapping* $h : \mathcal{B}(R^2) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$*,* $h(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) If $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$*, then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$*;*

(iii) If $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$*, then* $h(A_n) \nearrow h(A)$*;*

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$*.*

Theorem 3.4. *For each two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF-observable.*

Proof. See [13, Theorem 3.3]. □

***Remark* 3.5.** *The joint IF-observable of the IF-observables x, y from Definition 3.3 is a two-dimensional IF-observable.*

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. About this says the following definition.

Definition 3.6. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n be their joint IF-observable and $g_n : R^n \to R$ be a Borel measurable function. Then, we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

4 Lower and upper limits, $\mathcal{P}$-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables.

Definition 4.1. *We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup\limits_{n\to\infty}$ if there exists an IF-observable $\overline{x} : \mathcal{B}(R) \to \mathcal{F}$ such that*

$$\overline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for every $t \in R$. We write $\overline{x} = \limsup\limits_{n\to\infty} x_n$.

Note that if another IF-observable y satisfies the above condition, then $\mathbf{m} \circ y = \mathbf{m} \circ \overline{x}$.

Definition 4.2. *A sequence $(x_n)_n$ of IF-observables has $\liminf\limits_{n\to\infty}$ if there exists an IF-observable $\underline{x}$ such that*

$$\underline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for all $t \in R$. Notation: $\underline{x} = \liminf\limits_{n\to\infty} x_n$.

In paper [5] we showed the connection between two kinds of $\mathcal{P}$-almost everywhere convergence.

Definition 4.3. *Let $(x_n)_n$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathcal{P})$. We say that $(x_n)_n$ converges $\mathcal{P}$-almost everywhere to 0, if*

$$\mathcal{P}\Big(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathcal{P}\Big(\bigwedge_{n=k}^{k+i} x_n\Big(\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big)\Big) = [1, 1] = 1.$$

***Remark* 4.4.** *The defining formula is equivalent to the following equality*

$$\mathcal{P}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(R\backslash\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = [0,0] = 0.$$

Theorem 4.5. *A sequence $(x_n)_n$ of IF-observables converges $\mathcal{P}$-almost everywhere to 0 if and only if it converges $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere to 0.*

Proof. See [5, Theorem 5]. □

Proposition 4.1. *A sequence $(x_n)_n$ of IF-observables converges $\mathcal{P}$-almost everywhere to 0 if and only if*

$$\begin{aligned}\mathcal{P}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) &= \mathcal{P}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \\ &= \mathcal{P}\big(0_{\mathcal{F}}((-\infty,t))\big),\end{aligned}$$

for every $t \in R$.

Proof. See [5, Proposition 2]. □

In accordance to Proposition 4.1, we can extend the notion of $\mathcal{P}$-almost everywhere convergence in the following way.

Definition 4.6. *A sequence $(x_n)_n$ of IF-observables converges $\mathcal{P}$-almost everywhere to an IF-observable x, if*

$$\begin{aligned}\mathcal{P}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) &= \mathcal{P}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \\ &= \mathcal{P}\big(x((-\infty,t))\big),\end{aligned}$$

for every $t \in R$.

Sometimes we need to work with a sequence of IF-observables induced by a Borel measurable function.

Recall, that the corresponding probability spaces are $(R^N, \sigma(\mathcal{C}), P^\flat)$ and $(R^N, \sigma(\mathcal{C}), P^\sharp)$, where $\mathcal{C}$ is the family of all sets of the form

$$\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\},$$

and $P^\flat, P^\sharp$ are the probability measures determined by the equalities

$$P^\flat\big(\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\}\big) = \mathcal{P}^\flat\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big),$$

$$P^\sharp\big(\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\}\big) = \mathcal{P}^\sharp\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big).$$

The corresponding projections $\xi_n : R^N \to R$ are defined by the equality

$$\xi_n\big((t_i)_{i=1}^{\infty}\big) = t_n.$$

Theorem 4.7. *Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ be the sequence of corresponding projections, $(g_n)_n$ be a sequence of Borel measurable functions $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1, \dots, \xi_n)\big)_n$ converges $P^\flat$-almost everywhere and $P^\sharp$-almost everywhere, then the sequence $\big(g_n(x_1, \dots, x_n)\big)_n$ converges $\mathcal{P}$-almost everywhere and*

$$\mathcal{P}\Big(\limsup_{n\to\infty} g_n(x_1, \dots, x_n)\big((-\infty, t)\big)\Big) = \mathcal{P}\Big(\liminf_{n\to\infty} g_n(x_1, \dots, x_n)\big((-\infty, t)\big)\Big)$$

for each $t \in R$. Moreover

$$\mathcal{P}\Big(\limsup_{n\to\infty} g_n(x_1, \dots, x_n)\big((-\infty, t)\big)\Big) \quad = \quad \Big[P^\flat\big(E\big), P^\sharp\big(E\big)\Big]$$

for each $t \in R$, where $E = \{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u), \dots, \xi_n(u)\big) < t\}$.

Proof. See [5, Theorem 6]. □

5 Individual Ergodic Theorem

In paper [5] we proved the modification of the classical Individual Ergodic Theorem using $\mathbf{m}$-almost everywhere convergence. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposed to two intuitionistic fuzzy states $\mathbf{m}$ (see [11, 14]), then we try to formulate the modification of the classical Individual Ergodic Theorem using $\mathcal{P}$-almost everywhere convergence.

Now, we recall the modification of the Individual Ergodic Theorem for the IF-state (see [6]).

Theorem 5.1. (Individual Ergodic Theorem) *Let $(\mathcal{F}, \cdot)$ be a family of IF-events with product, and $\mathbf{m}$ be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Then there exists an integrable IF-observable x^* such that*

(i) $\mathbf{E}(x) = \mathbf{E}(x^*)$,

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^* \quad$ $\mathbf{m}$*-almost everywhere.*

Proof. See [6, Theorem 6.3] . □

We defined the IF-mean value of an IF-observable and $\mathcal{P}$-almost everywhere convergence in the previous sections. Now we must define a transformation preserving an intuitionistic probability $\mathcal{P}$.

Definition 5.2. *Let $(\mathcal{F}, \cdot)$ be a family of IF-events with product, $\mathcal{P}$ be an IF-probability. Then, a mapping $\tau : \mathcal{F} \to \mathcal{F}$ is said to be a $\mathcal{P}$-preserving transformation if the following conditions are satisfied:*

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) *If $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ and* $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;

(iii) *If* $\mathbf{A}_n \nearrow \mathbf{A}$, $\mathbf{A}_n, \mathbf{A} \in \mathcal{F}$, $n \in N$, *then* $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$*;*

(iv) $\mathcal{P}\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) = \mathcal{P}(\mathbf{A} \cdot \mathbf{B})$ *for each* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$.

Now we show the connection to the $\mathbf{m}$-preserving transformation. Recall that by **$\mathbf{m}$-preserving transformation** we understand each mapping $\tau : \mathcal{F} \to \mathcal{F}$ if the following conditions are satisfied:

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) If $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ and $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;

(iii) If $\mathbf{A}_n \nearrow \mathbf{A}$, $\mathbf{A}_n, \mathbf{A} \in \mathcal{F}$, $n \in N$, then $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$;

(iv) $\mathbf{m}\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) = \mathbf{m}(\mathbf{A} \cdot \mathbf{B})$ for each $\mathbf{A}, \mathbf{B} \in \mathcal{F}$.

See [6].

Theorem 5.3. *Let* $(\mathcal{F}, \cdot)$ *be a family of IF-events with product,* $\mathcal{P}$ *be an IF-probability. The mapping* $\tau : \mathcal{F} \to \mathcal{F}$ *is the* $\mathcal{P}$*-preserving transformation if and only if the mapping* τ *is the* $\mathcal{P}^\flat$*-preserving transformation and the* $\mathcal{P}^\sharp$*-preserving transformation, where* $\mathcal{P}^\flat, \mathcal{P}^\sharp$ *are the IF-states.*

Proof. "$\Rightarrow$" Let $\mathcal{P}$ be an IF-probability. Then by *Theorem 2.5* it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$ such that $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. If the mapping $\tau : \mathcal{F} \to \mathcal{F}$ is the $\mathcal{P}$-preserving transformation, then by (iv) from *Definition 5.2* we have

$$\begin{aligned}\big[\mathcal{P}^\flat(\mathbf{A} \cdot \mathbf{B}), \mathcal{P}^\sharp(\mathbf{A} \cdot \mathbf{B})\big] &= \mathcal{P}(\mathbf{A} \cdot \mathbf{B}) = \mathcal{P}\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) \\ &= \big[\mathcal{P}^\flat\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big), \mathcal{P}^\sharp\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big)\big].\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{P}^\flat\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) &= \mathcal{P}^\flat(\mathbf{A} \cdot \mathbf{B}), \\ \mathcal{P}^\sharp\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) &= \mathcal{P}^\sharp(\mathbf{A} \cdot \mathbf{B}),\end{aligned}$$

for each $\mathbf{A}, \mathbf{B} \in \mathcal{F}$. Therefore, τ is a $\mathcal{P}^\flat$-preserving transformation and a $\mathcal{P}^\sharp$-preserving transformation.

"$\Leftarrow$" The opposite direction can be proved similarly. □

Theorem 5.4. (Individual Ergodic Theorem) *Let* $(\mathcal{F}, \cdot)$ *be a family of IF-events with product,* $\mathcal{P}$ *be an IF-probability. Let* x *be an integrable IF-observable and* τ *be an* $\mathcal{P}$*-preserving transformation. Then there exists an integrable IF-observable* x^* *such that*

(i) $\mathbf{E}^\flat(x) = \mathbf{E}^\flat(x^*)$, $\mathbf{E}^\sharp(x) = \mathbf{E}^\sharp(x^*)$

(ii) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} (\tau^i \circ x) = x^*$, $\mathcal{P}$*-almost everywhere.*

Proof. Let $\mathcal{P}$ be an IF-probability. By *Theorem 2.5* it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$, such that $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. Let τ be the $\mathcal{P}$-preserving transformation. Then from *Theorem 5.3* we obtain that τ is the $\mathcal{P}^\flat$-preserving transformation and the $\mathcal{P}^\sharp$-preserving transformation, where $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are the IF-states. Hence by *Theorem 5.1* there exists an integrable IF-observable x^* such that

(i) $\mathbf{E}^\flat(x) = \mathbf{E}^\flat(x^*)$, $\mathbf{E}^\sharp(x) = \mathbf{E}^\sharp(x^*)$

(ii) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} (\tau^i \circ x) = x^*$, $\mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere.

Finally by *Theorem 5.3* we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*, \quad \mathcal{P}\text{-almost everywhere.} \qquad \square$$

6 Conclusion

The paper is concerned in ergodic theory for family of intuitionistic fuzzy events. We proved the Individual ergodic theorem for intuitionistic fuzzy observables using $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ is an intuitinistic fuzzy probability. The results are a generalization of results given in [3].

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