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Menger's theorem for intuitionistic fuzzy graphs

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Abstract: In this paper, the concept of strength reducing set of vertices and edges have been introduced. Using it, Menger's theorem for intuitionistic fuzzy graphs has been proved.
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1 Introduction

Fuzzy set theory invented by L. A. Zadeh [6] in 1965, generalises 0 and 1 membership values of a crisp set to membership function of a fuzzy set [7, 8]. Rosenfeld [5] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs. The connectivity of fuzzy graph is entirely different from classical graphs. The concept of connectivity in crisp graph is used to disconnect a graph into two or more components. But, the fuzzy ideas deals with the reduction in the strength of connectedness between pair of vertices. The theory of intuitionistic fuzzy graphs (IFGs) was introduced by K. T. Atanassov in [4]. In [1], Karunambigai and Parvathi introduced intuitionistic fuzzy graph as a special case of Atanassov's IFG. In [2], the edges in intuitionistic fuzzy graphs were classified into α-strong, β-strong and δ-weak depending on the strength of connectedness between two vertices. Sunil Mathew and Sunitha introduced Menger's theorem for fuzzy graphs in [3]. In this way, the authors got motivated to introduce the Menger's theorem for intuitionistic fuzzy graphs.

The paper is organised as follows. In Section 2, preliminaries required for this research work are given. In Section 3, strength reducing set of vertices and edges, minimum strength reducing set of vertices and edges are defined. Strength reducing set of vertices are characterized in Theorem 3.1 and strength reducing set of edges are characterized in Theorem 3.2. In Section 4, Menger's theorem for intuitionistic fuzzy graphs is proved for vertex version and its edge version is stated without proof.

2 Preliminaries

In this section, some basic definitions and theorems which are useful in constructing the properties relating to this study, are given.

Definition 2.1. [1] *Minmax intuitionistic fuzzy graph* (IFG) is of the form $G = (V, E)$, where
(i) $V = \{v_1, v_2, \ldots v_n\}$ such that $\mu_i : V \to [0,1]$ and $\nu_i : V \to [0,1]$ denote the degrees of membership and non - membership of the element $v_i \in V$ respectively and $0 \leq \mu_i + \nu_i \leq 1$, for every $v_i \in V$ $(i = 1, 2, \ldots, n)$.
(ii) $E \subset V \times V$ where $\mu_{ij} : V \times V \to [0,1]$ and $\nu_{ij} : V \times V \to [0,1]$ are such that

$$\mu_{ij} \leq \min[\mu_i, \mu_j]$$

$$\nu_{ij} \leq \max[\nu_i, \nu_j]$$

and $0 \leq \mu_{ij} + \nu_{ij} \leq 1$ for every $e_{ij} \in E$.
Here the triple (v_i, μ_i, ν_i) denotes the degree of membership and degree of non - membership of the vertex v_i. The triple $(e_{ij}, \mu_{ij}, \nu_{ij})$ denotes the degree of membership and degree of non - membership of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.
For each Intuitionistic Fuzzy Graph G, the degree of hesitance (hesitation degree) of the vertex $v_i \in V$ in G is $\Pi_i = 1 - \mu_i - \nu_i$ and the degree of hesitance (hesitation degree) of an edge $e_{ij} = (v_i, v_j) \in E$ in G is $\Pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$.

Notation: Here after an IFG, $G = (V, E)$ means a *minmax* IFG $G = (V, E)$.

Definition 2.2. [1] An IFG, $G = \langle V, E \rangle$ is said to be *strong* IFG if $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\nu_{ij} = \max(\nu_i, \nu_j)$ for all $e_{ij} \in E$.

Definition 2.3. [1] An IFG, $G = \langle V, E \rangle$ is said to be *complete* IFG if $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\nu_{ij} = \max(\nu_i, \nu_j)$ for every $v_i, v_j \in V$.

Definition 2.4. [1] A path P in an IFG is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ for all $i, j = 1, 2, \ldots, n$ such that either one of the following conditions is satisfied. A path P between two vertices v_i and v_j is denoted by $[v_i, v_j]$-path.

i) $\mu_{ij} > 0$ and $\nu_{ij} = 0$ for some i and j.
ii) $\mu_{ij} > 0$ and $\nu_{ij} > 0$ for some i and j.

Definition 2.5. [1] The $\mu - strength$ of a path $P = v_1, v_2, \ldots, v_n$ is defined as $\min\{\mu_{ij}\}$ for all $i, j = 1, 2, \ldots, n$ and it is denoted by S_μ.

Definition 2.6. [1] The $\nu - strength$ of a path $P = v_1, v_2, \ldots, v_n$ is defined as $\max\{\nu_{ij}\}$ for all $i, j = 1, 2, \ldots, n$ and it is denoted by S_ν.

Note 1. If same edge possess both the values S_μ and S_ν, then it is the *strength of the path P* and is denoted by S_P.

Definition 2.7. [2] If $v_i, v_j \in V \subseteq G$, the μ - strength of connectedness between two vertices v_i and v_j is $\text{CONN}_{\mu(G)}(v_i, v_j) = \max\{S_\mu\}$ and ν - strength of connectedness between two vertices v_i and v_j is $\text{CONN}_{\nu(G)}(v_i, v_j) = \min\{S_\nu\}$ of all possible paths between v_i and v_j.

Note 2. *[2]* $\text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j), \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$ *is the strength of connectedness between* v_i *and* v_j *in the IFG obtained from G by deleting the edge* e_{ij}.

Definition 2.8. *[2] An edge* e_{kl} *is said to be a cut edge or bridge in G, if either* $\text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ *and* $\text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j) \geq \text{CONN}_{\nu(G)}(v_i, v_j)$ *or* $\text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j) \leq \text{CONN}_{\mu(G)}(v_i, v_j)$ *and* $\text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j) > \text{CONN}_{\nu(G)}(v_i, v_j)$ *for some* $v_i, v_j \in V$.

In other words, deleting an edge e_{kl} *reduces the strength of connectedness between some pair of vertices* v_i *and* v_j *or* e_{kl} *is a bridge if there exist vertices* $v_i, v_j \in V$ *such that* e_{kl} *is an edge of every strongest path from* v_i *to* v_j.

Definition 2.9. *[2] A vertex* v_l *is said to be a cut-vertex in G, if either* $\text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ *and* $\text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j) \geq \text{CONN}_{\nu(G)}(v_i, v_j)$ *or* $\text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j) \leq \text{CONN}_{\mu(G)}(v_i, v_j)$ *and* $\text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j) > \text{CONN}_{\nu(G)}(v_i, v_j)$ *for some* $v_i, v_j \in V$.

In other words, a vertex v_l *is said to be a cut-vertex in G if deleting a vertex* v_l *reduces the strength of connectedness between some pair of vertices or* v_l *is a cut vertex if and only if there exists* $v_i, v_j \in V$ *such that* v_l *is a vertex of every strongest path from* v_i *to* v_j.

Definition 2.10. *[2] An edge* e_{ij} *is said to be strong edge if* $\mu_{ij} \geq \text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j)$ *and* $\nu_{ij} \leq \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$ *for every* $v_i, v_j \in V$.

Definition 2.11. *[2] An edge* e_{ij} *is said to be weak edge if* $\mu_{ij} < \text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j)$ *and* $\nu_{ij} > \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$ *for every* $v_i, v_j \in V$.

Definition 2.12. [2] In an IFG $G = (V, E)$, a path P between any two vertices is called the *strongest path* if its strength equals the strength of connectedness $\text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G)}(v_i, v_j)$ and both the values lie in the same edge of P.

Definition 2.13. [2] A $[v_i, v_j]$-path P in an IFG $G = (V, E)$ is called a *strong path* if P contains only strong edges.

Definition 2.14. [2] An edge e_{ij} in G is called α-strong if $\mu_{ij} > \text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j)$ and $\nu 2ij < \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$.

Definition 2.15. [2] An edge e_{ij} in G is called β-strong if $\mu_{ij} = \text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j)$ and $\nu_{2ij} = \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$.

Definition 2.16. [5] An edge e_{ij} in G is called δ-weak if $\mu_{ij} < \text{CONN}_{\mu(G)-(v_i,v_j)}(v_i, v_j)$ and $\nu_{2ij} > \text{CONN}_{\nu(G)-(v_i,v_j)}(v_i, v_j)$.

Definition 2.17. [2] A path in an IFG $G = (V, E)$ is called an α-strong path if all its edges are α-strong and also a path is β-strong path if all its edges are β-strong.

3 Strength reducing sets

In graph theory, a $[v_i, v_j]$-separating set S of vertices is a collection of vertices in G whose removal disconnects the graph G and, v_i and v_j belonging to different components of $G - S$. Similarly, a $[v_i, v_j]$-separating set of edges is defined. Now, we define strength reducing sets of vertices and edges in IFGs as follows.

Definition 3.1. Let v_i and v_j be any two vertices in an $IFGG = (V, E)$ such that the edge e_{ij} is not strong. A set $S \subseteq V$ of vertices is said to be a $[v_i, v_j]$-*strength reducing set of vertices* if $\text{CONN}_{\mu(G-S)}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-S)}(v_i, v_j) \geq \text{CONN}_{\nu(G)}(v_i, v_j)$ or $\text{CONN}_{\mu(G-S)}(v_i, v_j) \leq \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-S)}(v_i, v_j) > \text{CONN}_{\nu(G)}(v_i, v_j)$ for some $v_i, v_j \in V$ where $G - S$ is the IF subgraph of G obtained by removing the vertices of S from G.

In other words, a set $S \subseteq V$ of vertices is said to be a $[v_i, v_j]$-*strength reducing set of vertices* if deleting the set $S \subseteq V$ of vertices reduces the $\text{CONN}_{\mu(G)}(v_i, v_j)$ and increases $\text{CONN}_{\nu(G)}(v_i, v_j)$ between the vertices v_i and v_j.

Definition 3.2. A set of edges $D \subseteq E$ is said to be a $[v_i, v_j]$-*strength reducing set of edges* if $\text{CONN}_{\mu(G-D)}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-D)}(v_i, v_j) \geq \text{CONN}_{\nu(G)}(v_i, v_j)$ or $\text{CONN}_{\mu(G-D)}(v_i, v_j) \leq \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-D)}(v_i, v_j) > \text{CONN}_{\nu(G)}(v_i, v_j)$ for some $v_i, v_j \in V$ where $G - D$ is the IF subgraph of G obtained by removing the edges of D from G.

In other words, a set $D \subseteq E$ of vertices is said to be a $[v_i, v_j]$-*strength reducing set of edges* if deleting the set $D \subseteq E$ of edges reduces the $\text{CONN}_{\mu(G)}(v_i, v_j)$ and increases $\text{CONN}_{\nu(G)}(v_i, v_j)$ between the vertices v_i and v_j.

Definition 3.3. A $[v_i, v_j]$-strength reducing set of vertices (edges) with 'n' elements is said to be a *minimum* $[v_i, v_j]$-*strength reducing set of vertices (edges)* if there exist no $[v_i, v_j]$-strength reducing set of vertices (edges) with less than 'n' elements. A *minimum* $[v_i, v_j]$-*strength reducing set of vertices* is denoted by $S_G(v_i, v_j)$ and *minimum* $[v_i, v_j]$-*strength reducing set of edges* is denoted by $D_G(v_i, v_j)$

Example 3.1. In Figure 3.1, e_{25} is a weak edge, $S_G(v_2, v_5) = \{v_1, v_4, v_6\}$ and $D_G(v_2, v_5) = \{e_{15}, e_{34}, e_{26}\}$

Theorem 3.1. Let $G = (V, E)$ be a connected IFG and v_i, v_j any two vertices in G such that e_{ij} is not strong. Then a set S of vertices in G is a $[v_i, v_j]$-strength reducing set if and only if every strongest path from v_i to v_j contains at least one vertex of S.

Proof. Suppose that S is a $[v_i, v_j]$-strength reducing set of vertices in G and let P be a strongest $[v_i, v_j]$-path in G. If P contains no vertex of S, then removal of S keep P intact and hence $G - S$ contains P. Thus, $\text{CONN}_{\mu(G-S)}(v_i, v_j) = \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-S)}(v_i, v_j) = \text{CONN}_{\nu(G)}(v_i, v_j)$ which contradicts the fact that S is $[v_i, v_j]$-strength reducing set of vertices. Thus P must contains at least one member of S. It is obvious that this result is not true when the edge e_{ij} is strong. Any strong edge e_{ij} is a strongest $[v_i, v_j]$-path containing no vertex from S.

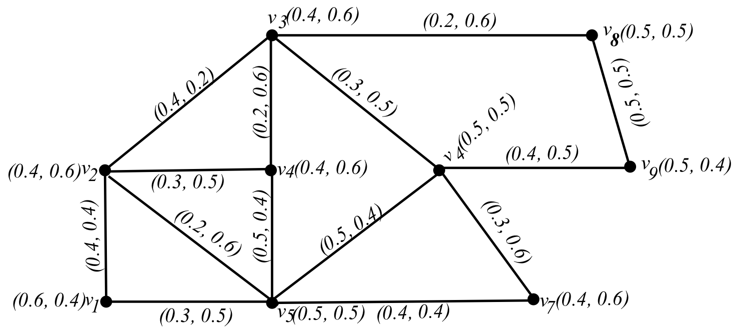

Figure 3.1

Conversely, suppose that every strongest path from v_i to v_j contain at least one vertex of S, where $S \subseteq V$ and v_i, v_j not in S. Then the removal of S destroys all strongest $[v_i, v_j]$-paths in G and hence $\text{CONN}_{\mu(G-S)}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-S)}(v_i, v_j) \geq \text{CONN}_{\nu(G)}(v_i, v_j)$ or $\text{CONN}_{\mu(G-S)}(v_i, v_j) \leq \text{CONN}_{\mu(G)}(v_i, v_j)$ and $\text{CONN}_{\nu(G-S)}(v_i, v_j) >$ $\text{CONN}_{\nu(G)}(v_i, v_j)$. Thus, it follows that S is a $[v_i, v_j]$-path strength reducing set of vertices in G. $\square$

Example 3.2. In Figure 3.2, $S_G(v_1, v_6) = \{v_3, v_7, v_9\}$. The strongest paths from v_1 to $_6$ are $v_1 - v_2 - v_3 - v_4 - v_5 - v_6$, $v_1 - v_7 - v_6$ and $v_1 - v_9 - v_8 - v_6$ and every strongest path contains at least one vertex of $S_G(v_1, v_6)$.

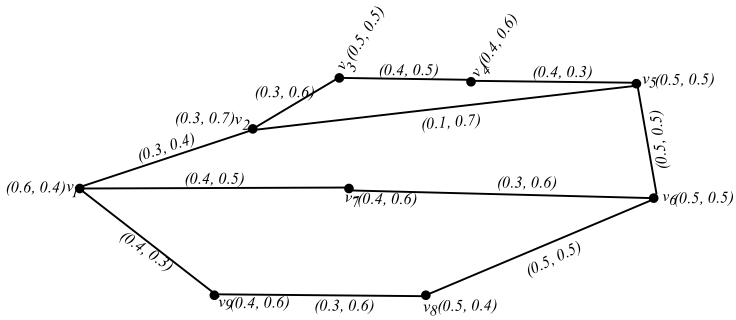

Figure 3.2

Theorem 3.2. Let $G = (V, E)$ be a connected IFG and v_i, v_j any two vertices in G. Then a set D of edges in G is a $[v_i, v_j]$-strength reducing set if and only if every strongest path from v_i to v_j contains at least one edges of D.

Proof is similar to that of Theorem 3.1.

4 Menger's theorem for intuitionistic fuzzy graphs

Next we present a generalization of one of the celebrated results in Graph theory due to Karl Menger (1927).

Theorem 4.1. *(Generalization of the vertex version of Menger's Theorem)* Let $G = (V, E)$ be an IFG. For any two vertices $v_i, v_j \in V$ such that e_{ij} is not strong, the maximum number of internally disjoint strongest $[v_i, v_j]$-paths in G is equal to the number of vertices in a minimal $[v_i, v_j]$-strength reducing set.

Proof. We shall prove the result by induction on the strong size $ss(G)$ of an IFG G, (the set contains number of strong edges of G). When $ss(G) = 0$, there is no edge between the vertices v_i and v_j. Hence, the result is trivially true for any pair of vertices $v_i, v_j \in V$.

Assume that the theorem is true for all intuitionistic fuzzy graphs $G = (V, E)$ with strong size less than m where $m \geq 1$. Let G be an IFG of strong size m. Let $v_i, v_j \in V$ such that e_{ij} is not strong. If v_i and v_j are in different components of $G = (V, E)$, the theorem is obviously true. So assume that v_i and v_j belongs to the same component of $G = (V, E)$. Then either e_{ij} is not in E or e_{ij} is a δ-edge. In both cases $[v_i, v_j]$-strength reducing set of vertices exists in G. (If e_{ij} is strong, then reduction of any number of vertices will not reduce the strength of connectivity between v_i and v_j and hence no strength reducing set of vertices exist).

Now suppose that $S_G(v_i, v_j)$ is a minimum strength reducing set of vertices in G with $|S_G(v_i, v_j)| = k \geq 1$. By Theorem 3.1, each strongest $[v_i, v_j]$-path must contain atleast one member from $S_G(v_i, v_j)$. Hence any $[v_i, v_j]$-strength reducing set must contain atleast as many vertices as the number of internally disjoint strongest $[v_i, v_j]$-paths. In other words, there exists atmost k internally disjoint $[v_i, v_j]$-paths. We show that G contains exactly k internally disjoint strongest $[v_i, v_j]$-paths. If $k = 1$, then $|S_G(v_i, v_j)| = 1$. Let $S_G(v_i, v_j) = \{w\}$. Then $\text{CONN}_{\mu(G)-\{w\}}(v_i, v_j) < \text{CONN}_{\mu(G)}(v_i, v_j)$ and
$\text{CONN}_{\nu(G)-\{w\}}(v_i, v_j) > \text{CONN}_{\nu(G)}(v_i, v_j)$.

That is w is an intuitionistic fuzzy cut vertex of G. So every strongest $[v_i, v_j]$-path must pass through w. Hence the number of internally disjoint $[v_i, v_j]$-paths is one and the result is true. So assume that $k \geq 2$.

Case 1. G has a minimum $[v_i, v_j]$-strength reducing set of vertices containing a vertex v_k such that both e_{ik} and e_{kj} are α-strong edge.

Let $S_G(v_i, v_j)$ be the minimum $[v_i, v_j]$-strength reducing set of vertices with the above mentioned property. Then $S_G(v_i, v_j) - \{v_k\}$ is a minimum $[v_i, v_j]$-strength reducing set in $G - \{v_k\}$ having $k-1$ vertices. Since both e_{ik} and e_{kj} are α-strong edges, they are clearly strong and hence $ss(G - \{v_k\}) < ss(G)$. By induction, it follows that $G - \{v_k\}$ contains $k - 1$ internally disjoint strongest $[v_i, v_j]$-paths. Since e_{ik} and e_{kj} are α-strong, by Proposition [3.1] of [2], P is a strongest $[v_i, v_j]$-path. Thus we have k internally disjoint strongest $[v_i, v_j]$-paths in G.

Case 2. For every minimum $[v_i, v_j]$-strength reducing set $S_G(v_i, v_j)$ in G, either every vertex in $S_G(v_i, v_j)$ is an α-strong neighbor of v_i (that is if v_l is a vertex in $S_G(v_i, v_j)$, then e_{il} is an edge which is the unique strongest $[v_i, v_l]$-path), but not of v_j or every vertex in $S_G(v_i, v_j)$ is an α-strong neighbor of v_j but not of v_i.

Suppose that every vertex in $S_G(v_i, v_j)$ is an α-strong neighbor of v_i, but not of v_j. Consider a strongest $[v_i, v_j]$-path P in G. Let v_k be the first vertex of P which is in $S_G(v_i, v_j)$. Then e_{ik} is α-strong and since e_{kj} is not α-strong, there exist atleast one vertex say v_l other than v_i and v_j such that e_{kl} is β-strong. Denote the edge by e_{kl}.

Claim. Every $[v_i, v_j]$-strength reducing set in $G - \{e_{kl}\}$ has exactly k vertices.

On the contrary assume that there exist a minimum $[v_i, v_j]$-strength reducing set in $G - \{e_{kl}\}$ with $k-1$ vertices say $Z = z_1, z_2, \ldots, z_{k-1}$. Then $Z \cup v_k$ is a minimum $[v_i, v_j]$-strength reducing set in G. Note that every $z_i, i = 1, 2, \ldots, k-1$ and v_k are α-strong neighbor of v_i. Since $Z \cup v_l$ also is a minimum $[v_i, v_j]$-strength reducing set in G, it follows that v_l is an α-strong neighbor of v_i contradicting the fact that edge e_{kl} is β-strong edge. (The edges (e_{ik}, e_{il} and e_{kl} forms a triangle with edge e_{kl} as the weakest edge. The unique weakest edge of a cycle is a δ-edge). Thus k is the minimum number of vertices in a $[v_i, v_j]$-strength reducing set in $G - \{e_{kl}\}$. Since $ss(G - \{e_{kl}\}) < ss(G)$, it follows by induction that there are k internally disjoint $[v_i, v_j]$-paths in $G - \{e_{kl}\}$ and hence in G.

Example 4.1. *In Figure 4.1,* $[v_1, v_7]$*-strength reducing set of vertices* $S_G(v_1, v_7) = \{v_3, v_4, v_6, v_8, v_{10}\}$. *Every vertex in* $S_G(v_1, v_7)$ *is an* α*-strong neighbor of* v_7 *but not of* v_1.

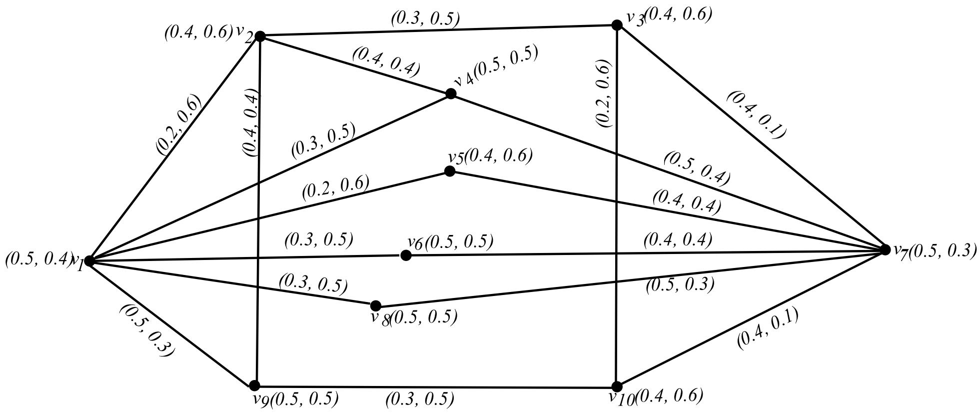

Figure 4.1

Case 3. There exists a $[v_i, v_j]$-strength reducing set W in G such that no member of W is an α-strong neighbor of both v_i and v_j and W contains atleast one vertex which is not an α-strong neighbor of v_i and atleast one vertex which is not an α-strong neighbor of v_j. (In Figure 4.2, $S_G(v_1, v_5) = \{v_3, v_{12}, v_8\}$).

Let W be a minimum $[v_i, v_j]$-strength reducing set with k elements having the above properties. Let $W = w_1, w_2, \ldots, w_k$. Consider all strongest paths from v_i to v_j. Then since W is minimum, $w_i, i = 1, 2, \ldots, k$ must belong to atleast one such path. Let G_{v_i} be the IF subgraph of G consisting of all $[v_i, w_i]$-sub paths of all strongest $[v_i, v_j]$-paths in which $w_i \in W$ is the only vertex of the path belonging to W. Note that if $\text{CONN}_{\mu(G)}(v_i, v_j) = s$ and $\text{CONN}_{\nu(G)}(v_i, v_j) = t$, then $\mu_{kl} \geq s$ and $\nu_{kl} \leq t$ for all edge e_{kl} in these paths.

Let G'_{v_i} be the IFG constructed from G_{v_i} by adding a new vertex v'_j and joining v'_j to each vertex w_i for $i = 1, 2, \ldots, k$ (see Figure 4.3). Let $\mu(v'_j) = 1$, $\nu(v'_j) = 0$, and $\mu(w_i, v'_j) = \mu(w_i)$, $\nu(w_i, v'_j) = \nu(w_i)$for every $i = 1, 2, \ldots, k$. The IFGs G_{v_j} and G'_{v_j} are defined similarly (see Figure 4.4).

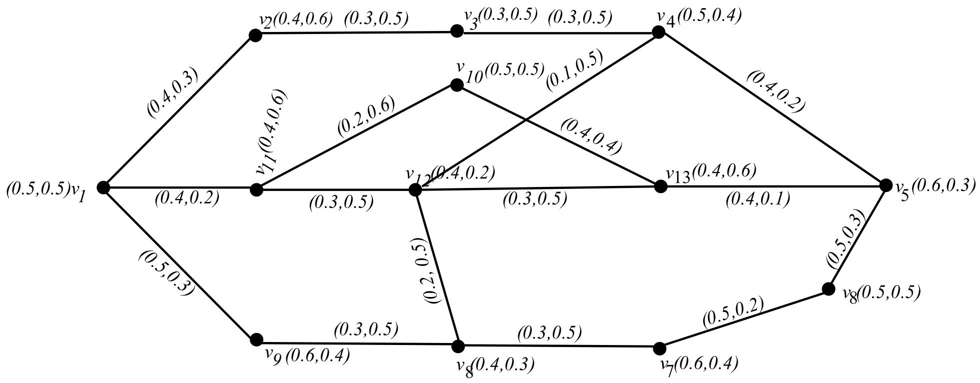

Figure 4.2

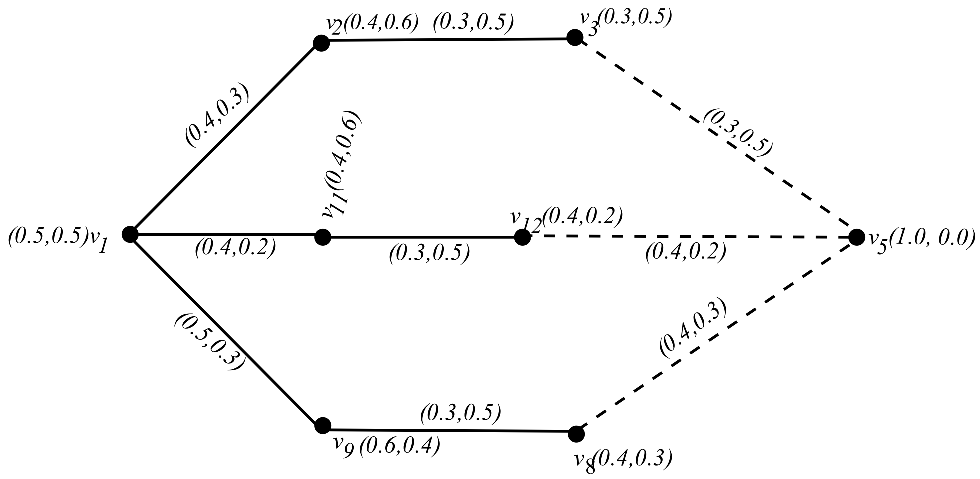

Figure 4.3

Since W contains a vertex that is not an α-strong neighbor of v_i and a vertex that is not an α-strong neighbor of v_j (Note that all newly introduced edges are strong), we have $ss(G'_{v_i}) < ss(G)$ and $ss(G'_{v_j}) < ss(G)$.

Clearly $S_{G'_{v_i}}(v_i, v'_j) = k$ and $S_{G'_{v_j}}(v'_i, v_j) = k$. So by induction G'_{v_i} contains k internally disjoint $[v_i, v'_j]$-paths say $A_i, i = 1, 2, \ldots, k$ where A_i contains w_i. Also, G'_{v_j} contains k internally disjoint $[v'_i, v_j]$-paths say $B_i, i = 1, 2, \ldots, k$ where B_i contains w_i. Let A'_i be the $[v_i, w_i]$-sub paths of A_i and B'_i be the $[w_i, v_j]$-subpaths of B_i for $1 \leq i \leq k$. Now k internally disjoint strongest $[v_i, v_j]$-paths can be formed by joining A_i and B_i for $i = 1, 2, \ldots, k$ and the theorem is proved by induction. □

Now we state the edge version of Theorem 4.1. The proof is very similar to that of Theorem 4.1.

Theorem 4.2. *(Generalization of the edge version of Menger's theorem)* Let $G = (V, E)$ be a connected IFG and let $v_i, v_j \in V$. Then the maximum number of edge disjoint strongest $[v_i, v_j]$-paths in G is equal to the number of edges in a minimum (with respect to the number of edges) $[v_i, v_j]$-strength reducing set.

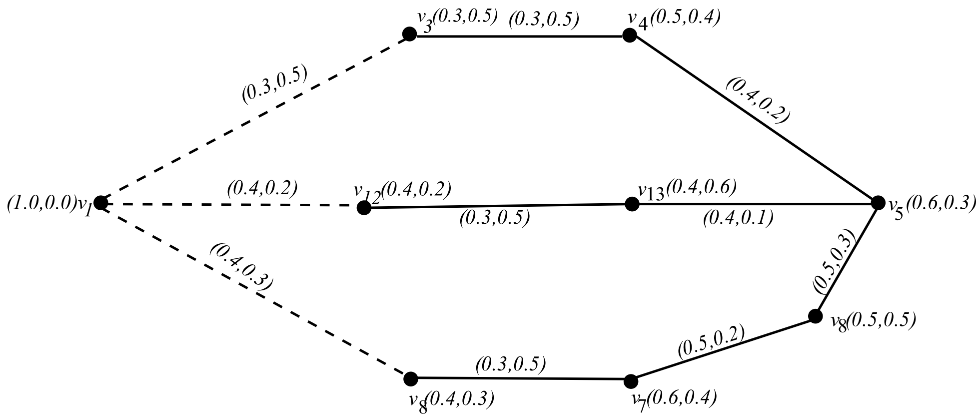

Figure 4.4

5 Conclusion

In this article, we presented the characteristics of strength reducing sets of vertices and edges. We generalized the Menger's theorem of vertex and edge version for intuitionistic fuzzy graphs.

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