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Intuitionistic fuzzy fractional equation

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Abstract: In this paper, we discuss the existence and uniqueness of mild solution for intuitionistic fuzzy fractional equation using the concept of semigroup in the intuitionistic fuzzy theory and the theorem of fixed point in the complete metric space.
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1 Introduction

Fractional calculus is a generalization of differentiation and integration to an arbitrary order. First works, devoted exclusively to the subject of fractional calculus, are the books [14, 15]. Many recently developed models in areas like rheology, viscoelasticity, electrochemistry, diffusion processes, etc. for details, see [7, 10].

Initial value problems for fractional differential equations have been considered by some authors recently [10, 11, 17].

For significant results from the theory of fuzzy differential equations and their applications we refer to the books [3, 9] and the papers [1, 2, 6, 11].

The initial value crisp problem

$$\begin{cases} D^q x(t) & = Ax(t) + f(t, x(t)) \ , \ t \in [0, T] \\ x(0) & = x_0 \end{cases} \tag{1}$$

has a unique mild solution under assumption some conditions, if A is the generator of a $C_0 - semigroup$, $(S(t))_{t\geq 0}$ on a Banach space X, the system (1) has a unique mild solution $x \in \mathcal{C}([0, T])$. In [4], C. G. Gal and S. G. Gal studied, with more details, fuzzy linear and semi-linear (additive and positive homogeneous) operators theory, introduced semigroups of operators of fuzzy-number-valued functions, and gave various applications to fuzzy differential equation. In this work, we study the existence and uniqueness of mild solution for fractional differential equation with intuitionistic fuzzy data of the following form:

$$\begin{cases} D^q x(t) & = Ax(t) + f(t, x(t)) \ , \ t \in [0, T] \\ x(0) & = x_0 \ \in IF_1 \end{cases} . \tag{2}$$

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$. we denote by

$$\mathbb{F}^1 = \mathrm{IF}(\mathbb{R}) = \left\{ \langle u, v \rangle \ : \ \mathbb{R} \to [0,1]^2 \ , \ |\forall x \in \mathbb{R} \ , \ 0 \leq u(x) + v(x) \leq 1 \right\} .$$

An element $\langle u, v \rangle$ of $\mathbb{F}^1$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v \rangle$ is normal i.e there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp\, \langle u, v \rangle = \mathrm{cl}\{x \in \mathbb{R} : v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by $\mathbb{F}_1$.
For $\alpha \in [0, 1]$ and $\langle u, v \rangle \in \mathbb{F}^1$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$\Big[\langle u, v \rangle \Big]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

and

$$\Big[\langle u, v \rangle \Big]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$$

Remark 1. *If $\langle u, v \rangle$ is an intuitionistic fuzzy number, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{(1,0)} \in \mathbb{F}_1$ as

$$0_{(1,0)}(t) = \begin{cases} (1, 0) & t = 0 \\ (0, 1) & t \neq 0 \end{cases}$$

Let $\langle u, v \rangle$,$\langle u', v' \rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by:

$$\Big(\langle u, v \rangle \oplus \langle u', v' \rangle \Big)(z) = \Big(\sup_{z=x+y} \min \left(u(x), u'(y)\right), \inf_{z=x+y} \max \left(v(x), v'(y)\right) \Big)$$

$$\lambda \langle u, v \rangle = \begin{cases} \langle \lambda u, \lambda v \rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u,v\rangle$, $\langle z,w\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$, the addition and scale-multiplication are defined as follows

$$\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha + \Big[\langle z,w\rangle\Big]^\alpha, \quad \Big[\lambda\,\langle z,w\rangle\Big]^\alpha = \lambda\Big[\langle z,w\rangle\Big]^\alpha$$
$$\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha + \Big[\langle z,w\rangle\Big]_\alpha, \quad \Big[\lambda\,\langle z,w\rangle\Big]_\alpha = \lambda\Big[\langle z,w\rangle\Big]_\alpha$$

Definition 1. *Let $\langle u,v\rangle$ an element of $\mathbb{F}_1$ and $\alpha \in [0,1]$, we define the following sets:*

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^+(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha) = \inf\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^-(\alpha) = \sup\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}$$

Remark 2.

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha), \Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle\Big]^\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha), \Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]$$

Proposition 1. *For all α, $\beta \in [0,1]$ and $\langle u,v\rangle \in \mathbb{F}_1$*

(i) $\Big[\langle u,v\rangle\Big]_\alpha \subset \Big[\langle u,v\rangle\Big]^\alpha$

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$

(iii) if $\alpha \leq \beta$ then $\Big[\langle u,v\rangle\Big]_\beta \subset \Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta \subset \Big[\langle u,v\rangle\Big]^\alpha$

(iv) If $\alpha_n \nearrow \alpha$ then $\Big[\langle u,v\rangle\Big]_\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha \in [0,1]$ we denote by

$$M_\alpha = \{x \in \mathbb{R} \ : \ u(x) \geq \alpha\} \quad \text{and} \quad M^\alpha = \{x \in \mathbb{R} \ : \ v(x) \leq 1-\alpha\}$$

Lemma 1. *Let $\Big\{M_\alpha,\ \alpha \in [0,1]\Big\}$ and $\Big\{M^\alpha,\ \alpha \in [0,1]\Big\}$ two families of subsets of $\mathbb{R}$ satisfies (i)–(iv) in proposition 1, if u and v define by*

$$u(x) = \begin{cases} 0 & \text{if } x \notin M_0 \\ \sup\{\alpha \in [0,1] \ : \ x \in M_\alpha\} & \text{if } x \in M_0 \end{cases}$$
$$v(x) = \begin{cases} 1 & \text{if } x \notin M^0 \\ 1 - \sup\{\alpha \in [0,1] \ : \ x \in M^\alpha\} & \text{if } x \in M^0 \end{cases}$$

Then $\langle u,v\rangle \in \mathbb{F}_1$.

Proof. See [12]. □

The space $\mathbb{F}_1$ is metrizable by the distance of the following form :

$$d_p\Big(\langle u,v\rangle,\langle z,w\rangle\Big)=\Bigg(\frac{1}{4}\int_0^1\Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha)-\Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big|^p d\alpha$$
$$+\frac{1}{4}\int_0^1\Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha)-\Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big|^p d\alpha$$
$$+\frac{1}{4}\int_0^1\Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha)-\Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big|^p d\alpha$$
$$+\frac{1}{4}\int_0^1\Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha)-\Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|^p d\alpha\Bigg)^{\frac{1}{p}}$$

and

$$d_\infty\Big(\langle u,v\rangle,\langle z,w\rangle\Big)=\frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha)-\Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha)-\Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha)-\Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha)-\Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|$$

Proposition 2. $\Big(I\!F_1, d_p\Big)$ *is a metric space.*

Theorem 1. *For* $p\in[1,\infty)$*,*
$(I\!F_1, d_p)$ *is a complete and separable metric space.*
$(I\!F_1, d_\infty)$ *is a complete and not separable metric space.*

Proof. See [12]. □

We can see that $d_1 \le d_\infty$
Let $[a,b]\subset\mathbb{R}$
$F:[a,b]\to IF_1$ is called integrable bounded if there exists an integrable function $h:[a,b]\to\mathbb{R}$ such that $\mid y \mid\le h(t)$ holds for any $y\in supp(F(t)), t\in[a,b]$.

Definition 2. *we say that a mapping* $F:[a,b]\to IF_1$ *is strongly measurable if for all* $\alpha\in[0,1]$ *the set-valued mappings* $F_\alpha:[a,b]\to P_k(\mathbb{R})$ *defined by* $F_\alpha(t)=[F(t)]_\alpha$ *and* $F^\alpha:[a,b]\to P_k(\mathbb{R})$ *defined by* $F^\alpha(t)=[F(t)]^\alpha$ *are (Lebesgue) measurable, when* $P_k(\mathbb{R})$ *is endowed with the topology generated the Hausdorff metric* d_H

Lemma 2. *Let* $F:[a,b]\to I\!F_1$ *be strongly measurable and denote* $F_\alpha(t)=[\lambda_\alpha(t),\lambda^\alpha(t)]$*,* $F^\alpha(t)=[\mu_\alpha(t),\mu^\alpha(t)]$ *for* $\alpha\in[0,1]$*. Then* $\lambda_\alpha,\lambda^\alpha,\mu_\alpha,\mu^\alpha$ *are measurable.*

Proof. See [12]. □

Definition 3. *Suppose $A=[a,b]$, $F:A\to I\!F_1$ is integrable bounded and strongly measurable for each $\alpha\in(0,1]$ write*

$$\left[\int_A F(t)dt\right]_\alpha=\int_A [F(t)]_\alpha\,dt=\left\{\int_A fdt|f:A\to\mathbb{R}\text{ is a measurable selection for }F_\alpha\right\}$$
$$\left[\int_A F(t)dt\right]^\alpha=\int_A [F(t)]^\alpha\,dt=\left\{\int_A fdt|f:A\to\mathbb{R}\text{ is a measurable selection for }F^\alpha\right\}$$

if there exists $\langle u,v\rangle\in I\!F_1$ such that $\Big[\langle u,v\rangle\Big]^\alpha=\Big[\int_A F(t)dt\Big]^\alpha$ and $\Big[\langle u,v\rangle\Big]_\alpha=\Big[\int_A F(t)dt\Big]_\alpha$ $\forall\alpha\in(0,1]$. Then F is called integrable on A, write $\langle u,v\rangle=\int_A F(t)dt$.

Remark 3.

1. *If $F(t)=\langle u_t,v_t\rangle$ is integrable, then $\int\langle u_t,v_t\rangle=\big\langle\int u_t,\int v_t\big\rangle$*

2. *If $F:[a,b]\to IF_1$is integrable then in view of Lemma (2) $\int F$ is obtained by integrating the α-level curves, that is*

$$\left[\int F\right]_\alpha=\left[\int\lambda_\alpha,\int\lambda^\alpha\right]\text{ and }\left[\int F\right]^\alpha=\left[\int\mu_\alpha,\int\mu^\alpha\right],\alpha\in[0,1]$$

with $F_\alpha(t)=[F(t)]_\alpha=\Big[\lambda_\alpha(t),\lambda^\alpha(t)\Big]$, $F^\alpha(t)=[F(t)]^\alpha=\Big[\mu_\alpha(t),\mu^\alpha(t)\Big]$ for $\alpha\in[0,1]$.

3 The embedding theorem

Definition 4. *An intuitionistic fuzzy set $\langle u,v\rangle$ is called convex intuitionistic fuzzy set if and only if u is convex fuzzy set and v is concave fuzzy set.*

The question that arises, is what $\mathbb{F}_1$ with addition and multiplication by a scalar is a vector space. The answer is negative. The embedding theorem of Radstrom will be extended to $\mathbb{F}_1$. To do this, a linear structure is defined in $\mathbb{F}_1$ by

1.
$$\begin{aligned}(\langle u,v\rangle\oplus\langle u',v'\rangle)(x)&=\Bigg(\sup\Big\{\alpha\in[0,1],x\in\Big[\langle u,v\rangle\Big]_\alpha+\Big[\langle u',v'\rangle\Big]_\alpha\Big\},\\&1-\sup\Big\{\alpha\in[0,1],x\in\Big[\langle u,v\rangle\Big]^\alpha+\Big[\langle u',v'\rangle\Big]^\alpha\Big\}\Bigg)\end{aligned}$$

2.
$$(\lambda\langle u,v\rangle(x))=\begin{cases}\langle u(x/_\lambda),v(x/_\lambda)\rangle & if\ \lambda\neq0\\ 0 & if\ \lambda=0\ x\neq0\\ \langle\sup_{y\in\mathbb{R}}u(y),1-\sup_{y\in\mathbb{R}}v(y)\rangle & if\ \lambda=0\ ,\ x=0\end{cases}$$

Theorem 2. *There exists a normed space $\mathcal{X}$ and a function $j:I\!F_1\to\mathcal{X}$ with properties*

1. *j is an isometry i.e. $||j(\langle u,v\rangle)-j(\langle u',v'\rangle)||=d_1\Big(\langle u,v\rangle;\langle u',v'\rangle\Big)\le d_\infty\Big(\langle u,v\rangle;\langle u',v'\rangle\Big)$*

2. *$j\Big(\langle u,v\rangle\oplus\langle u',v'\rangle\Big)=j(\langle u,v\rangle)+j(\langle u',v'\rangle)$*

3. $j\left(\lambda\left\langle u,v\right\rangle\right)=\lambda j\left(\left\langle u,v\right\rangle\right) \quad \lambda\geq 0$

Proof. Define an equivalence relation in $IF_1 \times IF_1$ by

$$\Big(\left\langle u,v\right\rangle;\left\langle u',v'\right\rangle\Big)\mathcal{R}\Big(\left\langle z,w\right\rangle;\left\langle z',w'\right\rangle\Big)\Leftrightarrow\left\langle u,v\right\rangle\oplus\left\langle z',w'\right\rangle=\left\langle u',v'\right\rangle\oplus\left\langle z,w\right\rangle$$

The space of equivalence classe $\overline{\Big(\left\langle u,v\right\rangle;\left\langle u',v'\right\rangle\Big)}$ of pairs $\Big(\left\langle u,v\right\rangle;\left\langle u',v'\right\rangle\Big)$ is denoted by $\mathcal{X}$. The norm in $\mathcal{X}$ is define by $||\overline{\Big(\left\langle u,v\right\rangle;\left\langle u',v'\right\rangle\Big)}||=d_1\Big(\left\langle u,v\right\rangle;\left\langle u',v'\right\rangle\Big)$. It is easy to check that $j: IF_1\to\mathcal{X}$ defined by

$$j\left(\left\langle u,v\right\rangle\right)=\overline{\left(\left\langle u,v\right\rangle;O_{(1,0)}\right)}$$

is an isometry, and properties (2), (3) follow the definition. □

Proposition 3. *$\mathcal{X}$ is a Banach real space.*

Proof. See [8]. □

4 Intuitionistic fuzzy strongly continuous semigroup

In this section, we give the approach of the concept of intuitionistic fuzzy Semigroup.

Definition 5. *We called an intuitionistic fuzzy $\mathcal{C}^0$-Semigroup (one parameter, strongly continuous, nonlinear) the whole family $\{T(t),t\geq 0\}$ of operators from $I\!F_1$ into itself satisfying the following conditions*

- $T(0)=i$*, the identity mapping on* $I\!F_1$*.*
- $T(t+s)=T(t)T(s),\ \forall t,s\geq 0.$
- *the function* $g:[0,+\infty[\to I\!F_1$*, defined by* $g(t)=T(t)\left\langle u,v\right\rangle$ *is continuous at* $t=0$ *for all* $\left\langle u,v\right\rangle\in I\!F_1$ *i.e*

$$\lim_{t\to 0^+}T(t)\left\langle u,v\right\rangle=\left\langle u,v\right\rangle$$

- *There exist two constants* $M>0$ *and* $\omega\in\mathbb{R}$ *such that*

$$d_\infty\Big(T(t)\left\langle u,v\right\rangle,T(t)\left\langle u',v'\right\rangle\Big)\leq Me^{\omega t}d_\infty\Big(\left\langle u,v\right\rangle,\left\langle u',v'\right\rangle\Big),\ for\ t\geq 0\,,\ \left(\left\langle u,v\right\rangle,\left\langle u',v'\right\rangle\right)\in I\!F_1^2$$

In particular if $M=1$ and $\omega=0$, we say that $\{T(t);t\geq 0\}$ is a contraction intuitionistic fuzzy semigroup.

Remark 4. *The continuity of g at 0, implies the continuity of g at $t_0\geq 0$.*

Definition 6. *Let $\{T(t),t\geq 0\}$ be an intuitionistic fuzzy $\mathcal{C}^0$-semigroup on $I\!F_1$ and $\left\langle u,v\right\rangle\in I\!F_1$, if for $h>0$ sufficiently small, the Hukuhara difference $T(h)\left\langle u,v\right\rangle-_H\left\langle u,v\right\rangle$ exists, we define*

$$\lim_{h\to 0}d_\infty\left(\frac{T(h)\left\langle u,v\right\rangle-_H\left\langle u,v\right\rangle}{h},A\left(\left\langle u,v\right\rangle\right)\right)=0$$

whenever this limit exists in the metric space $(I\!F_1, d_\infty)$. Then the operator $A : \langle u, v\rangle \to A\langle u, v\rangle$ defined on

$$D(A) = \left\{ \langle u, v\rangle \in I\!F_1, \lim_{h\to 0} \left\{ \frac{T(h)\langle u, v\rangle -_H \langle u, v\rangle}{h} \right\} \; exists \right\}$$

is called the infinitesimal generator of the intuitionistic fuzzy semigroup $\{T(t), t \geq 0\}$.

Proposition 4. *Let $A : IF_1 \to I\!F_1$ and $A_1 = jAj^{-1} : \mathcal{X} \to \mathcal{X}$ two operators. A is the infinitesimal generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on $I\!F_1$ if and only if A_1 is the infinitesimal generator of the semigroup $\{T_1(t); t \geq 0\}$ defined on $\mathcal{X}$ by $T_1(t) = jT(t)j^{-1}$ for $t \geq 0$*

Proof. We assume that A is the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on $\mathbb{F}_1$, then we have For all $\langle u, v\rangle \in j^{-1}(D(A))$

$$\begin{aligned}
\lim_{h\to 0^+} \frac{T_1(h)\langle u, v\rangle - \langle u, v\rangle}{h} &= \lim_{h\to 0^+} \frac{jT(h)j^{-1}\langle u, v\rangle - jj^{-1}\langle u, v\rangle}{h} \\
&= j \lim_{h\to 0^+} \frac{T(h)j^{-1}\langle u, v\rangle -_H j^{-1}\langle u, v\rangle}{h} \\
&= jAj^{-1}\langle u, v\rangle \\
&= A_1\langle u, v\rangle
\end{aligned}$$

Conversely, if A_1 is the generator of an semigroup $\{T_1(t); t \geq 0\}$ on $\mathcal{X}$, then for all $\langle u, v\rangle \in D(A)$

$$\begin{aligned}
\lim_{h\to 0^+} \frac{T(h)\langle u, v\rangle -_H \langle u, v\rangle}{h} &= \lim_{h\to 0^+} \frac{j^{-1}T_1(h)j\langle u, v\rangle -_H j^{-1}j\langle u, v\rangle}{h} \\
&= j^{-1} \lim_{h\to 0^+} \frac{T_1(h)j\langle u, v\rangle - j\langle u, v\rangle}{h} \\
&= j^{-1}A_1 j\langle u, v\rangle \\
&= A\langle u, v\rangle .
\end{aligned}$$

This completes the proof. □

Remark 5. *Since the infinitesimal generator A_1 of $\{T_1(t); t \geq 0\}$ is unique, we deduce that the infinitesimal generator A of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ is also unique.*

Lemma 3. *Let A be the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on IF_1, then for all $\langle u, v\rangle \in IF_1$ such that $T(t)\langle u, v\rangle \in D(A)$ for all $t \geq 0$, the mapping $t \to g(t) = T(t)\langle u, v\rangle$ is differentiable and*

$$g'(t) = AT(t)\langle u, v\rangle$$

Proof. Let $\langle u, v\rangle \in IF_1$, for $t, h \geq 0$ we have

$$T(t+h)\langle u, v\rangle = T(h)T(t)\langle u, v\rangle$$

Since $T(t)\langle u, v\rangle \in D(A)$ then

$$\begin{aligned}\lim_{h\to 0^+}\frac{g(t+h)-_H g(t)}{h} &= \lim_{h\to 0^+}\frac{T(t+h)\langle u,v\rangle -_H T(t)\langle u,v\rangle}{h}\\ &= \lim_{h\to 0^+}\frac{T(h)T(t)\langle u,v\rangle -_H T(t)\langle u,v\rangle}{h}\\ &= AT(t)\langle u,v\rangle\end{aligned}$$

Denote $A_h = \frac{T(h)-i}{h}$, for $h > 0$. Using the continuity of g and the definition of A, we have

$$\begin{aligned}\lim_{h\to 0^+}\frac{g(t-h)-_H g(t)}{-h} &= \lim_{h\to 0^+}\frac{T(t-h)\langle u,v\rangle -_H T(t)\langle u,v\rangle}{-h}\\ &= \lim_{h\to 0^+}\frac{T(h)T(t-h)\langle u,v\rangle -_H T(t-h)\langle u,v\rangle}{h}\\ &= \lim_{h\to 0^+} A_h T(t-h)\langle u,v\rangle\\ &= AT(t)\langle u,v\rangle\end{aligned}$$

Hence, g is differentiable and $g'(t) = AT(t)\langle u,v\rangle$, for all $t \geq 0$. □

5 Intuitionistic fuzzy fractional integral and fuzzy fractional derivative

Let $\langle u,v\rangle \in \mathbb{F}_1$ such that $\Big[\langle u,v\rangle(t)\Big]_\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), \Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\Big]$, $\Big[\langle u,v\rangle(t)\Big]^\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), \Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big]$ for all $t \in (0,T]$ and $q \in \mathbb{R}_+$ with $[q]$ is the largest integer less or equal to q.

Suppose that

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha), \Big[\langle u,v\rangle\Big]_r^+(\alpha), \Big[\langle u,v\rangle\Big]_l^-(\alpha), \Big[\langle u,v\rangle\Big]_r^-(\alpha) \in C((0,T],\mathbb{R}) \cap L^1((0,T),\mathbb{R})$$

for all $\alpha \in [0,1]$ and let

$$\begin{aligned}A_\alpha &:= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)ds\Big]\\ &:= \Big[\phi_q(t) * \Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), \phi_q(t) * \Big[\langle u,v\rangle\Big]_l^+(\alpha)(t)\Big] \qquad (3)\end{aligned}$$

$$\begin{aligned}A^\alpha &:= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)ds\Big]\\ &:= \Big[\phi_q(t) * \Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), \phi_q(t) * \Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big] \qquad (4)\end{aligned}$$

Lemma 4. *The family $\{A_\alpha, A^\alpha; \alpha \in [0,1]\}$, given by (3) and (4), defined an intuitionistic fuzzy number $\langle u,v\rangle \in IF_1$ such that $\Big[\langle u,v\rangle\Big]_\alpha = A_\alpha$ and $\Big[\langle u,v\rangle\Big]^\alpha = A^\alpha$.*

We define

$$\phi_q(t) = \begin{cases} \frac{t^{q-1}}{\Gamma(q)} & t > 0, \\ 0, & t \leq 0 \end{cases}$$

and

$$\begin{aligned} \phi_{-q}(t) &= \phi_{1+k-q}(t) * \delta^{1+k}(t) \qquad k = [q] \\ \phi_{-n}(t) &= \delta^{n}(t) \qquad n = 0, 1, 2, \ldots \end{aligned}$$

with the property $\phi_q(t) * \phi_p(t) = \phi_{q+p}(t)$ for $p > 0$, where $\delta^n(t)$ is the n^{th} derivative of the delta function and $\Gamma(.)$ is the gamma function (for the properties of $\phi_q(t)$ see [5] and [16]).

Let $\langle u, v\rangle \in \mathcal{C}\Big((0, T], \mathbb{F}_1\Big) \cap L^1\Big((0, T), \mathbb{F}_1\Big)$. Define the intuitionistic fuzzy fractional primitive of order $q > 0$ of $\langle u, v\rangle$

$$I^q \langle u, v\rangle (t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \langle u, v\rangle (s) ds, \quad t \in (0, T)$$

by

$$\begin{aligned} \big[I^q \langle u, v\rangle (t)\big]_\alpha &= \frac{1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} \Big[\langle u, v\rangle\Big]_l^+ (\alpha)(s) ds, \int_0^t (t-s)^{q-1} \Big[\langle u, v\rangle\Big]_r^+ (\alpha)(s) ds\Big] \quad (5) \\ &= \Big[\Big[\langle u, v\rangle\Big]_l^+ (\alpha)(t) * \phi_q(t), \Big[\langle u, v\rangle\Big]_r^+ (\alpha)(t) * \phi_q(t)\Big], \quad t \in (0, T) \\ \big[I^q \langle u, v\rangle (t)\big]^\alpha &= \frac{1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} \Big[\langle u, v\rangle\Big]_l^- (\alpha)(s) ds, \int_0^t (t-s)^{q-1} \Big[\langle u, v\rangle\Big]_r^- (\alpha)(s) ds\Big] \quad (6) \\ &= \Big[\Big[\langle u, v\rangle\Big]_l^- (\alpha)(t) * \phi_q(t), \Big[\langle u, v\rangle\Big]_r^- (\alpha)(t) * \phi_q(t)\Big] \end{aligned}$$

for $t \in (0, T)$

(i) $I^q(c\langle u, v\rangle)(t) = cI^q(\langle u, v\rangle)(t)$ for each constant $c \in \mathbb{F}_1$

(ii) $I^q(\langle u, v\rangle \oplus \langle u', v'\rangle)(t) = I^q(\langle u, v\rangle)(t) \oplus I^q(\langle u, v\rangle)(t)$

Proposition 5. *If* $\langle u, v\rangle \in \mathcal{C}\Big((0, T], \mathbb{F}_1\Big) \cap L^1\Big((0, T), \mathbb{F}_1\Big)$ *and* p, $q > 0$, *then we have*

$$I^p I^q \langle u, v\rangle = I^{p+q} \langle u, v\rangle .$$

Proof. Let $g \in L^1\Big([0, T], \mathbb{F}_1\Big)$ be such that $[g(t)]_\alpha = \big[g_l^+(\alpha)(t), g_r^+(\alpha)(t)\big]$, $[g(t)]^\alpha = \big[g_l^-(\alpha)(t), g_r^-(\alpha)(t)\big]$ for all $t \in [0, T]$ and $\alpha \in [0, 1]$, where

$$\begin{aligned} \big[g_l^+(\alpha)(t), g_r^+(\alpha)(t)\big] &= \frac{1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} [(u, v)]_l^+ (\alpha)(s) ds, \int_0^t (t-s)^{q-1} [(u, v)]_r^+ (\alpha)(s) ds\Big] \\ \big[g_l^-(\alpha)(t), g_r^-(\alpha)(t)\big] &= \frac{1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} [(u, v)]_l^- (\alpha)(s) ds, \int_0^t (t-s)^{q-1} [(u, v)]_r^- (\alpha)(s) ds\Big] \end{aligned}$$

Then $[I^q f(t)]_\alpha = \big[g_l^+(\alpha)(t), g_r^+(\alpha)(t)\big]$ and $[I^q f(t)]^\alpha = \big[g_l^-(\alpha)(t), g_r^-(\alpha)(t)\big]$, that is $I^q f(t) = g(t)$. It follows that

$$
\begin{aligned}
[I^pI^q(u,v)(t)]_\alpha &= [I^pg(t)]_\alpha\\
&= \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}g_l^+(\alpha)(s)ds, \int_0^t (t-s)^{p-1}g_r^+(\alpha)(s)ds\right]\\
&= \frac{1}{\Gamma(p)\Gamma(q)}\left[\int_0^t (t-s)^{p-1}\int_0^s (s-\tau)^{q-1}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(\tau)d\tau ds,\right.\\
&\left.\int_0^t (t-s)^{p-1}\int_0^s (s-\tau)^{q-1}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(\tau)d\tau ds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
[I^pI^q(u,v)(t)]^\alpha &= [I^pg(t)]^\alpha\\
&= \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}g_l^-(\alpha)(s)ds, \int_0^t (t-s)^{p-1}g_r^-(\alpha)(s)ds\right]\\
&= \frac{1}{\Gamma(p)\Gamma(q)}\left[\int_0^t (t-s)^{p-1}\int_0^s (s-\tau)^{q-1}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(\tau)d\tau ds,\right.\\
&\left.\int_0^t (t-s)^{p-1}\int_0^s (s-\tau)^{q-1}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(\tau)d\tau ds\right]
\end{aligned}
$$

The Dirichlet formula and the substitution $s=\tau+\theta(t-\tau)$ yield

$$
\begin{aligned}
&\int_0^t\int_\tau^s (t-s)^{p-1}(s-\tau)^{q-1}\Big[\langle u,v\rangle\Big]_i^j(\alpha)(\tau)dsd\tau\\
&\qquad= \int_0^t (s-\tau)^{p+q-1}\Big[\langle u,v\rangle\Big]_i^j(\alpha)(\tau)d\tau\int_0^1 \theta^{q-1}(1-\theta)^{p-1}d\theta,\\
&\qquad\qquad i=\{l,r\}\ ,\quad j=\{+,-\}
\end{aligned}
$$

Since $\displaystyle\int_0^1 \theta^{q-1}(1-\theta)^{p-1}d\theta=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ we obtain that

$$
\begin{aligned}
[I^pI^q\langle u,v\rangle(t)]_\alpha &= \frac{1}{\Gamma(p+q)}\left[\int_0^t (t-s)^{p+q-1}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)ds, \int_0^t (t-s)^{p+q-1}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)ds\right]\\
&= \left[I^{p+q}\langle u,v\rangle(t)\right]_\alpha\\
[I^pI^q\langle u,v\rangle(t)]^\alpha &= \frac{1}{\Gamma(p+q)}\left[\int_0^t (t-s)^{p+q-1}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)ds, \int_0^t (t-s)^{p+q-1}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)ds\right]\\
&= \left[I^{p+q}\langle u,v\rangle(t)\right]^\alpha
\end{aligned}
$$

As $[I^pI^q\langle u,v\rangle(t)]_\alpha \subset [I^pI^q\langle u,v\rangle(t)]^\alpha$, then $[I^{p+q}\langle u,v\rangle(t)]_\alpha \subset [I^{p+q}\langle u,v\rangle(t)]^\alpha$, which implies

$$
[I^pI^q\langle u,v\rangle(t)]_\alpha = \left[I^{p+q}\langle u,v\rangle(t)\right]_\alpha,\quad [I^pI^q\langle u,v\rangle(t)]^\alpha = \left[I^{p+q}\langle u,v\rangle(t)\right]^\alpha
$$

This completes the proof. □

Definition 7. *Let $\langle u,v\rangle \in \mathcal{C}^{1+k}\Big((0,T],\mathbb{F}_1\Big)\cap L^1\Big((0,T),\mathbb{F}_1\Big)$. The intuitionistic fuzzy fractional differential operator is defined*

$$\begin{aligned} D^q\langle u,v\rangle(t) &= D^{1+k}\langle u,v\rangle(t) * \phi_{1+k-q}(t)\\ &= \frac{1}{\Gamma(1+k-q)}\int_0^t D^{1+k}\langle u,v\rangle(s)(t-s)^{k-q}ds \end{aligned}$$

by

$$[D^q\langle u,v\rangle(t)]_\alpha = \Big[D^{1+k}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t)*\phi_{1+k-q}(t), D^{1+k}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)*\phi_{1+k-q}(t)\Big]$$
$$= \frac{1}{\Gamma(1+k-q)}\Big[\int_0^t D^{1+k}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)(t-s)^{k-q}ds, \int_0^t D^{1+k}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)(t-s)^{k-q}ds\Big]$$

$$[D^q\langle u,v\rangle(t)]^\alpha = \Big[D^{1+k}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t)*\phi_{1+k-q}(t), D^{1+k}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)*\phi_{1+k-q}(t)\Big]$$
$$= \frac{1}{\Gamma(1+k-q)}\Big[\int_0^t D^{1+k}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)(t-s)^{k-q}ds, \int_0^t D^{1+k}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)(t-s)^{k-q}ds\Big]$$

for all $t \in (0,T]$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} \Big[D^q\langle u,v\rangle(t)\Big]_\alpha &= \Big[D^q\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), D^q\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\Big]\\ \Big[D^q\langle u,v\rangle(t)\Big]^\alpha &= \Big[D^q\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), D^q\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big] \end{aligned}$$

6 Fractional differential equations with intuitionistic fuzzy data

In the sequel we choose $\omega > 0$. We consider the intuitionistic fuzzy initial value problem

$$\begin{cases} D^q x(t) = Ax(t) + f\Big(t,x(t)\Big) & t \in [0,T]\\ x(0) = x_0 \end{cases} \tag{7}$$

with $x_0 \in \mathbb{F}_1$

Definition 8. *We say that x is a mild solution of (7) if :*

$$x(t) = T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds$$

Where $T(t)$ is an intuitionistic fuzzy $\mathcal{C}^0$-semigroup generating by A.
We assume that :

- (H_1): $f:[0,T]\times\mathbb{F}_1 \to \mathbb{F}_1$ is continuous and $d_\infty\Big(f(t,x(t)),f(t,y(t))\Big) \leq K\|x-y\|_\infty$ for all $x,y \in \mathcal{C}\Big([0,T],\mathbb{F}_1\Big)$.

- (H_2): $\quad x_0 \in D(A)$
- (H_3): $\dfrac{MK}{\Gamma(q)\omega^q}\displaystyle\int_0^{\omega T} s^{q-1}e^s ds < 1$

Theorem 3. *Under assumption* $(H_1)-(H_3)$*, if* A *is the generator of a* C_0*-semigroup* $\Big(T(t)\Big)_{t\geq 0}$. *on* $\mathbb{F}_1$*, the system* (7) *has a unique mild solution* $x \in \mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$.

Proof. Denote $\mathcal{C}_T = \mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$ and define a mapping $P : \mathcal{C}_T \to \mathcal{C}_T$ by

$$Px(t) = T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds$$

Step 1.
For $x \in C_T$; $t \in [0,T]$ and h very small

$$\begin{aligned}
&d_\infty\left(Px(t+h), Px(t)\right)\\
&= d_\infty\Bigg(T(h)T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^{h+t}(t+h-s)^{q-1}T(t+h-s)f(s,x(s))ds,\\
&\qquad T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds\Bigg)\\
&= d_\infty\Bigg(T(h)T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t+h-s)f(s,x(s))ds \oplus\\
&\qquad \frac{1}{\Gamma(q)}\int_h^{t+h}(t+h-s)^{q-1}T(t-s)f(s,x(s))ds,\\
&\qquad T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds\Bigg)\\
&\leq d_\infty\Big(T(h)T(t)x_0, T(t)x_0\Big)\\
&\quad + d_\infty\Bigg(\frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t+h-s)f(s,x(s))ds, O_{(1,0)}\Bigg)\\
&\quad + d_\infty\Bigg(\frac{1}{\Gamma(q)}\int_h^{t+h}(t+h-s)^{q-1}T(t+h-s)f(s,x(s))ds,\\
&\qquad \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds\Bigg)\\
&\leq d_\infty\Big(T(h)T(t)x_0, T(t)x_0\Big) + d_\infty\Bigg(\frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t-s)f(s,x(s))ds, O_{(1,0)}\Bigg)\\
&\quad + d_\infty\Bigg(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s+h,x(s+h))ds,\\
&\qquad \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds\Bigg)
\end{aligned}$$

$$\leq d_\infty\Big(T(h)T(t)x_0, T(t)x_0\Big) + d_\infty\Big(\frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t-s)f(s,x(s))ds, O_{(1,0)}\Big)$$
$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} d_\infty\Big(T(t-s)f(s+h,x(s+h)), T(t-s)f(s,x(s))\Big)ds$$
$$\leq d_\infty\Big(T(h)T(t)x_0, T(t)x_0\Big) + d_\infty\Big(\frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t-s)f(s,x(s))ds, O_{(1,0)}\Big)$$
$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} M e^{\omega t} d_\infty\Big(f(s+h,x(s+h)), f(s,x(s))\Big)ds$$

It is clear that

$$d_\infty\Big(T(h)T(t)x_0, T(t)x_0\Big) \to 0$$
$$d_\infty\Big(\frac{1}{\Gamma(q)}\int_0^h (t+h-s)^{q-1}T(t-s)f(s,x(s))ds, O_{(1,0)}\Big) \to 0$$

And by the dominated convergence theorem:

$$\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} M e^{\omega t} d_\infty\Big(f(s+h,x(s+h)), f(s,x(s))\Big) \to 0$$

Step 2. Let $x, y \in C_T$, we have

$$d_\infty\Big(Px(t), Py(t)\Big) = d_\infty\Big(T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,x(s))ds,$$
$$T(t)x_0 \oplus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,y(s))ds\Big)$$
$$\leq \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{\omega(t-s)} d_\infty\Big(f(s,x(s)), f(s,y(s))\Big)ds$$
$$\leq \frac{MK}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{\omega(t-s)}ds||x-y||_\infty \leq \frac{MK}{\Gamma(q)\omega^q}\int_0^{\omega t} s^{q-1}e^s ds||x-y||_\infty$$

The space $\Big(\mathbb{F}_1, d_\infty\Big)$ is a complete metric space, by the point fixe theorem of Banach, P have a unique fixed point wich a mild solution of (7). □

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