**Notes on Intuitionistic Fuzzy Sets**
**Print ISSN 1310–4926, Online ISSN 2367–8283**
**Vol. 23, 2017, No. 5, 85–101**

# Solving I-fuzzy number linear programming problems via Tanaka and Asai approach

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**Received:** 26 September 2017 **Accepted:** 2 November 2017

**Abstract:** This paper proposes an extension of Tanaka and Asai approach to study Atanassov's I-fuzzy linear programming problems where problem parameters are prescribed by I-fuzzy numbers. In literature, there are various indices based ranking function approaches for solving such I-fuzzy linear programming problems, e.g., Li [26], Li et al. [27], Dubey and Mehra [18] and Dubey et al. [19]. One major issue with these approaches is that the solution so obtained depends on the specific choice of the ranking function. The primary advantage of the proposed method is that, it is independent of any transformation and also provides the precise degrees of belief and disbelief of the optimal solution in achieving the goals set by the decision maker. It is shown that solving such an optimization problem is equivalent to solving a non-linear programming problem. A small numerical example is included as an illustration.

**Keywords:** I-fuzzy set, Triangular I-fuzzy numbers, I-fuzzy mathematical programming, I-fuzzy parameters.
**AMS Classification:** 90C72.

# 1 Introduction

Atanassov [5, 7, 8] integrated the notion of hesitancy degree in a fuzzy set by adding a new component which describes the degree of non-membership of an element in a given fuzzy set and called such a set an intutionistic fuzzy set. While the definition of fuzzy set provides the degree of membership of an element in a given set and its non-membership degree is understood as one minus its membership degree, the definition of intutionistic fuzzy set provides more-or-less independent degrees of membership and non-membership of an element in a given set. The only requirement in latter is that the sum of two degrees is less than or equal to one. As a result, an intutionistic fuzzy set exhibits characteristics of affirmation, negation and hesitation. For instance, in any confronting situation in decision making, beside support or positive response, objection or negative response, there could be an abstention which indicates hesitation or indeterminacy in response to the situation. Intutionistic fuzzy set thus occurs very naturally in real life decision making problems. For detailed description and properties of intutionistic fuzzy sets, we may refer to Atanassov [6], Szmidt and Kacprzyk [29] and other references cited therein.

The domain of intutionistic fuzzy set is not devoid of its share of controversies (see, Dubois et al. [15] and Grzegorzewski and Mrówka [22]). The nomenclature of intutionistic fuzzy set itself was an issue of debate because same nomenclature had also been used in intuitionistic fuzzy logic, and the two differ in their mathematical structure and treatment. It obviously makes sense to avoid using same terminology for two different concepts. As suggested in [15] and [22], in this paper, Atanassov intuitionistic fuzzy set is called *Atanassov's I-fuzzy set* or simply *I-fuzzy set*.

In the early study on fuzzy linear programming, two approaches have contributed significantly. These are due to Zimmermann [34] and, Tanaka and Asai [30]. While Zimmermann's approach is applied to linear programming with fuzzy goals (also called flexible linear programming problems), the approach of Tanaka and Asai [30] is useful for solving linear programming with fuzzy parameters (also called fuzzy number linear programming problems).

Compared to Tanaka and Asai [30] approach, Zimmermann's approach is more visible in the literature because most of the early engineering applications employed the same. The work on linear programming with fuzzy parameters followed a somewhat different direction. This has been mainly because there is no unique method of comparing fuzzy numbers. Therefore, depending on the choice of ordering, we have the corresponding solution concept for the given fuzzy linear programming problem. Most of the early work in this direction is based on Yager's ranking function approach [33]. This lead to several variants of the original work of Yager [33] e.g., Bector et al. [11], Li [26] and Li et al. [27]. Different from these ranking functions approaches, Clemente et al. [12] have recently defined fuzzy ordering via a finite set of $\alpha$-cuts (say r). This approach results in solving an appropriate multi-objective linear programming problem for the given fuzzy linear programming problem. Recently Aliev et al. [3] presented ranking on Z-numbers and its application in decision making.

Compared to the ordering methodologies discussed above, the conceptual frame work of Tanaka and Asai [30] for comparing fuzzy numbers is totally different and seems to be very natural. For this, Tanaka and Asai [30] introduced a very basic notion of 'almost positive TFN' and used the same to transform a fuzzy number linear programming problem to an appropriate crisp optimization problem.

Recently Atanassov's I-fuzzy set based linear programming problems have become popular because of their wide applications in areas like clustering [31], medical diagnosis [14], pattern recognition [24, 31]. Angelov [4] extended Zimmermann's approach [34] to solve linear programming problems with I-fuzzy goals. Later Aggarwal et al. [1] studied I-fuzzy linear programming problems with application in I-fuzzy matrix games. Motivated by the works of Hurwicz [20] and Yager [33], Dubey et al. [17] studied an I-fuzzy optimization problem with Atanassov's I-fuzzy goals, by resolving the indeterminacy factor before applying the Bellman and Zadeh extension principle [10] to it. Further Dubey et al. [18] studied multiobjective flexible linear programming in a bipolar frame work and developed an algorithm for computing a Pareto-optimal solution.

As regards the case of linear programming with I-fuzzy parameters, there are contributions of Dubey and Mehra [18], Li [26] and Li et al.[27] where various indices based ranking function approaches have been employed. Aggarwal et al. [2] have extended Clemente et al. [12] approach to I-fuzzy parameter scenario and showed its application in a variety of I-fuzzy matrix games. To the best of our knowledge, Tanaka and Asai's [30] approach has not yet been attempted to study I-fuzzy linear programming problems and that is what we wish to study in the present paper.

In a recent work Inuiguchi and Lodwick [23] presented an overview of the contributions of Tanaka and Asai [30] approach for solving fuzzy number and other related linear programming problems. This contribution of Inuguichi and Lodwick [23] has created renewed interest in Tanaka and Asai's [30] approach so as to explore its application in more general type of fuzzy linear programming problems. Here we attempt to study Atanassov's I-fuzzy linear programming problem via Tanaka and Asai's [30] approach. Thus we are able to solve an I-fuzzy number linear programming problem without using any defuzzification function and/or any indices based ranking function approach.

The paper is organized as follows. Section 2 presents the basic definitions corresponding to I-fuzzy set and I-fuzzy number. Section 3, first presents the main concept, 'an almost positive' symmetric triangular fuzzy number, of Tanaka and Asai [30] and then, the generalization of this notion to I-fuzzy scenario so as to understand the meaning of 'an almost positive' symmetric triangular I-fuzzy number. Section 4, starts with the basic formulation of a linear programming problem with I-fuzzy parameters and converts it into a finite system of I-fuzzy inequalities. This later system is solved by solving an appropriate nonlinear programming problem which is constructed by employing the definition of 'an almost positive' symmetric triangular I-fuzzy number. Here a small numerical example is also included as an illustration. The last section, namely Section 5, is the Conclusion section.

# 2 Preliminaries

In this section, we present certain preliminaries with regard to I-fuzzy sets and I-fuzzy numbers. For results related to fuzzy sets and fuzzy numbers we may refer to Wu [32], Bector and Chandra [9] and Zimmermann [34].

**Definition 2.1 (I-fuzzy set).** An I-fuzzy set $\tilde{a}$ in $X$ is described by

$$\tilde{a} = \{\langle x, \mu_{\tilde{a}}(x), \nu_{\tilde{a}}(x)\rangle \mid x \in X, \ \ \mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) \leqq 1\},$$

where $\mu_{\tilde{a}} : X \to [0, 1]$ and $\nu_{\tilde{a}} : X \to [0, 1]$ define, respectively, the membership function and the nonmembership function.

If $\mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) = 1$, for all $x \in X$, then $\tilde{a}$ degenerates to the standard fuzzy set.

We now take $X = \mathbb{R}$, the real Euclidean space, and recall an I-fuzzy number.

**Definition 2.2 ((I-fuzzy number (Li [25], Nehi [28]))).** An I-fuzzy number $\tilde{a}$ is an I-fuzzy set over $\mathbb{R}$ whose membership function $\mu_{\tilde{a}} : \mathbb{R} \to [0, 1]$ and $\nu_{\tilde{a}} : \mathbb{R} \to [0, 1]$ satisfy the following conditions

(i) there are real numbers $c$ and $d$ such that $\mu_{\tilde{a}}(c) = 1$ and $\nu_{\tilde{a}}(d) = 1$;

(ii) $\mu_{\tilde{a}}$ is quasi concave and $\nu_{\tilde{a}}$ is quasi convex on $\mathbb{R}$;

(iii) $\mu_{\tilde{a}}$ is upper semi-continuous and $\nu_{\tilde{a}}$ is lower semi-continuous;

(iv) the support sets $\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) > 0\}$ and $\{x \in \mathbb{R} \mid \nu_{\tilde{a}}(x) < 1\}$ are bounded.

We denote the set of I-fuzzy numbers by $IFN(\mathbb{R})$.

From above definition we get at once that for any I-fuzzy number $\tilde{a}$ there exists eight numbers $a_1, a_2, a_3, a_4, c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $c_1 \leqq a_1 \leqq c_2 \leqq a_2 \leqq a_3 \leqq c_3 \leqq a_4 \leqq c_4$ and four functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to [0, 1]$, called the sides of a I-fuzzy number, where $f_1$ and $f_4$ are nondecreasing and $f_2$ and $f_3$ are non increasing functions. The membership function $\mu_{\tilde{a}}$ of an I-fuzzy number $\tilde{a}$ can be specified as

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x < a_1, \\ f_1(x), & a_1 \leqq x < a_2, \\ 1, & a_2 \leqq x \leqq a_3, \\ f_2(x), & a_3 < x \leqq a_4, \\ 0, & x > a_4, \end{cases}$$

while the nonmembership function $\nu_{\tilde{a}}$ has the following form

$$\nu_{\tilde{a}}(x) = \begin{cases} 1, & x < c_1, \\ f_3(x), & c_1 \leqq x < c_2, \\ 0, & c_2 \leqq x \leqq c_3, \\ f_4(x), & c_3 < x \leqq c_4, \\ 1, & x > c_4. \end{cases}$$

It is worth noting that each I-fuzzy number $\tilde{a}$ is a conjunction of two fuzzy numbers, the membership function of one is $\mu_{\tilde{a}}$ and that of the other is $1 - \nu_{\tilde{a}}$.

In particular, if the nondecreasing functions $f_1$ and $f_4$ and non increasing functions $f_2$ and $f_3$ are linear and $a_2 = c_2, \ a_3 = c_3$, then the given I-fuzzy number is a trapezoidal I-fuzzy number.

The membership function and nonmembership function for the trapezoidal I-fuzzy number are as follows

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leqq x < a_2, \\ 1, & a_2 \leqq x \leqq a_3, \\ \dfrac{a_3 - x}{a_4 - a_3}, & a_3 < x \leqq a_4, \\ 0, & x > a_4, \end{cases}$$

and

$$\nu_{\tilde{a}}(x) = \begin{cases} 1, & x < c_1, \\ \dfrac{x - c_2}{c_1 - c_2}, & c_1 \leqq x < c_2, \\ 0, & c_2 \leqq x \leqq c_3, \\ \dfrac{x - c_3}{c_4 - c_3},, & c_3 < x \leqq c_4, \\ 1, & x > c_4. \end{cases}$$

We can represent a trapezoidal I-fuzzy number $(TrIFN)$ by

$$\tilde{a} = \langle [a_1, a_2, a_3, a_4], [c_1, c_2, c_3, c_4] \rangle$$

with $a_2 = c_2,\ a_3 = c_3$. Now if, $a_2 = a_3 = c_2 = c_3 = a(\texttt{say})$ then the a above given trapezoidal I-fuzzy number is the triangular I-fuzzy number. Similarly, a triangular I-fuzzy number $(TIFN)$ is represented by $\tilde{a} = \langle [a_1, a_2, a_3], [c_1, c_2, c_3] \rangle$ with $a_2 = c_2$. Again if $(a_2 - a_1) = (a_3 - a_2) = p(\text{say})$ and $(c_2 - c_1) = (c_3 - c_2) = q(\text{say})$ then the given $(TIFN)$ will be a symmetric triangular I-fuzzy number. We note that a symmetric triangular I-fuzzy number $\tilde{a}$ may be denoted by $\tilde{a} = \langle [a - p, a, a + p], [a - q, a, a + q] \rangle$.

**Definition 2.3. (I-fuzzy arithmetic (Li [25], Nehi [28])).** Let $\tilde{a} = \langle [a_1, a_2, a_3], [c_1, c_2, c_3] \rangle$ and $\tilde{b} = \langle [b_1, b_2, b_3], [d_1, d_2, d_3] \rangle$ be two Triangular I-fuzzy number and $k$ be a real number. Then the standard addition $\tilde{a} + \tilde{b}$ and substraction $\tilde{a} - \tilde{b}$ are respectively the I-fuzzy numbers defined as

$$\tilde{a} + \tilde{b} = \langle [a_1 + b_1, a_2 + b_2, a_3 + b_3], [c_1 + d_1, c_2 + d_2, c_3 + d_3] \rangle,$$

and

$$\tilde{a} - \tilde{b} = \langle [a_1 - b_3, a_2 - b_2, a_3 - b_1], [c_1 - d_3, c_2 - d_2, c_3 - d_1] \rangle.$$

Further, multiplication with any real number k, $k\tilde{a}$ is an another I-fuzzy number defined as

$$\begin{aligned} k\tilde{a} &= \langle [ka_1, ka_2, ka_3], [kc_1, kc_2, kc_3] \rangle, \quad \text{if} \quad k > 0, \\ k\tilde{a} &= \langle [ka_3, ka_2, ka_1], [kc_3, kc_2, kc_1] \rangle, \quad \text{if} \quad k < 0. \end{aligned}$$

# 3 An almost positive symmetric triangular I-fuzzy number

While discussing solution methodology for fuzzy linear programming problems, Tanaka and Asai [30] introduced the concept of 'an almost positive' symmetric triangular fuzzy number.

Taking motivation from Tanaka and Asai [30], in this section, we conceptualize the notion of 'an almost positive' symmetric triangular I-fuzzy number. Before we formalize our notion for 'an almost positive' symmetric triangular I-fuzzy number, we have the basic definition and related results with regards to an 'almost positive' symmetric triangular fuzzy number.

**Definition 3.1 ('almost positive' symmetric triangular fuzzy number (Tanaka & Asai [30])).** Let $\tilde{a} = \langle [a-p, a, a+p] \rangle$ be a symmetric triangular fuzzy number. Let $0 \leq h \leq 1$. Then $\tilde{a} = \langle [a-p, a, a+p] \rangle$ is said to be almost positive with degree of belief $h$, denoted by $\tilde{a} \gtrsim_h^F 0$, if $a > 0$ and $\mu_{\tilde{a}}(0) \leq (1-h)$.

Figure 1 illustrates $\tilde{a} \gtrsim_h^F 0$ graphically.

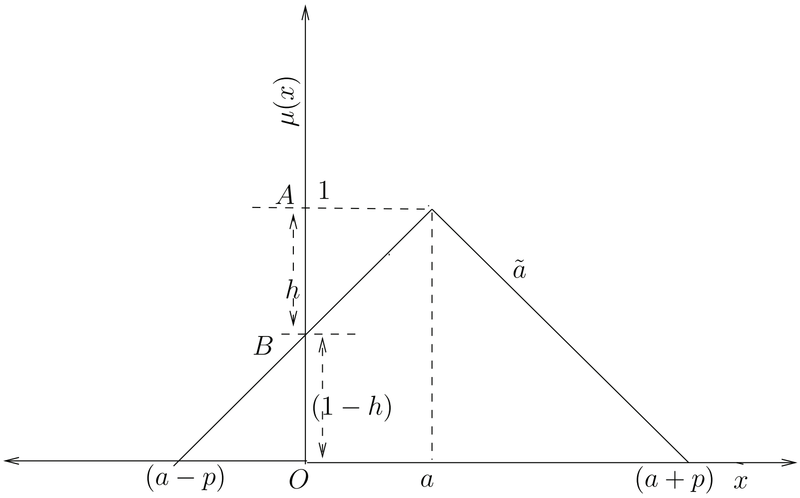


Figure 1: Fuzzy Number $\tilde{a} \gtrsim_h^F 0$

*Remark* 1. Earlier Wu [32] and Dubois and Prade [16] defined a fuzzy number $\widetilde{a}$ to be a 'positive fuzzy number' if $\mu_{\widetilde{a}}(x) = 0, \; \forall x \leq 0$. Therefore, a symmetric triangular fuzzy number is certainly be a 'positive fuzzy number' if it is 'almost positive fuzzy number with degree 1' in the sense of Tanaka and Asai [30]. Thus, the definition of 'almost positive' symmetric triangular fuzzy number is more general than the definition of 'positive fuzzy' number.

*Remark* 2. Tanaka and Asai [30] did not use the term 'degree of belief' but rather used the term 'degree' only. We find the term 'degree of belief' convenient for extending the concept of 'almost positive' to I-fuzzy scenario.

*Remark* 3. As $h$ is the degree of belief for the fuzzy statement '$\tilde{a} \gtrsim_h^F 0$ to be almost positive', the quantity $(1-h)$ may be interpreted as the degree of disbelief for the same fuzzy statement.

*Remark* 4. As $\tilde{a}$ is a symmetric triangular fuzzy number, maximizing the degree of belief $h$ automatically minimizes the degree of disbelief $(1-h)$ for the fuzzy statement $\tilde{a} \gtrsim_h^F 0$.

The concept of $\tilde{a} \gtrsim_h^F 0$ can now be utilized to compare two symmetric triangular fuzzy numbers $\tilde{a}$ and $\tilde{b}$. We say that $\tilde{a} \gtrsim_h^F \tilde{b}$ if $(\tilde{a} - \tilde{b}) \gtrsim_h^F 0$. Thus a fuzzy number $\tilde{a}$ is more than or equal to another fuzzy number $\tilde{b}$, if the fuzzy number $(\tilde{a} - \tilde{b})$ is almost positive with degree of belief $h$. Further, the inequality $\tilde{a} \lesssim_h^F \tilde{b}$ is to be understood in the sense that the fuzzy number $(\tilde{b} - \tilde{a})$ is almost positive with degree of belief $h$.

We now present the definition of 'an almost positive' symmetric triangular I-fuzzy number.

**Definition 3.2 (Almost positive symmetric triangular I-fuzzy number).** Let $\tilde{a} = \langle[a-p, a, a+p][a-q, a, a+q]\rangle$ be a symmetric triangular I-fuzzy number. Let $0 \leq h_1 \leq 1$ and $0 \leq h_2 \leq 1$. We say that $\tilde{a}$ is almost positive, denoted by $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$, if $a > 0$ and

(i) $\mu_{\tilde{a}}(0) \leq (1 - h_1)$, and

(ii) $\nu_{\tilde{a}}(0) \geq h_2$,

where, $h_1$ and $(1 - h_2)$ respectively are interpreted as the 'degree of belief' and the 'degree of disbelief' in making the statement '$\tilde{a}$ is almost positive'.

The below given figure, Figure 2 depicts the meaning of $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$.

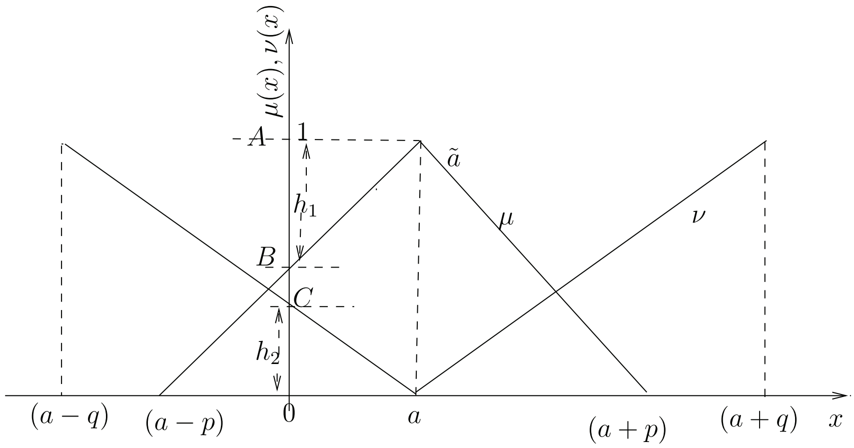


Figure 2: Meaning of $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$

We next define the 'belief score' of the I-fuzzy statement $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$.

**Definition 3.3 (Belief score).** Let $\tilde{a}$ be almost positive with degree of belief $h_1$ and degree of disbelief $(1 - h_1)$. Then the difference $h_1 - (1 - h_2)$, i.e., $(h_1 + h_2 - 1)$ is called the belief score of the I-fuzzy statement $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$.

*Remark* 5. As $\tilde{a}$ is a I-fuzzy number, $\mu_{\tilde{a}}(0) + \nu_{\tilde{a}}(0) \leq 1$ i.e., $(1 - h_1) + h_2 \leq 1$. This gives $h_1 \geq h_2$. Also for a meaningful decision we expect that the degree of belief is greater than or equal to the degree of disbelief. Thus $h_1 \geq (1 - h_2)$, i.e., $h_1 + h_2 \geq 1$, which means that the belief score is always expected to be non-negative.

Now the concept of $\tilde{a} \gtrsim^{IF}_{h_1,h_2} 0$ can be utilized to compare two symmetric triangular I-fuzzy numbers $\tilde{a}$ and $\tilde{b}$. Therefore, an I-fuzzy number $\tilde{a}$ is 'more than or equal to' another I-fuzzy number $\tilde{b}$, denoted by $\tilde{a} \gtrsim^{IF}_{h_1,h_2} \tilde{b}$ if the triangular I-fuzzy number $(\tilde{a} - \tilde{b})$ is almost positive, i.e., $(\tilde{a} - \tilde{b}) \gtrsim^{IF}_{h_1,h_2} 0$. Further, an I-fuzzy number $\tilde{a}$ is 'less than or equal to' another I-fuzzy number $\tilde{b}$, denoted by $\tilde{a} \lesssim^{IF}_{h_1,h_2} \tilde{b}$, if the symmetric triangular I-fuzzy number $(\tilde{b} - \tilde{a})$ is almost positive, i.e., $(\tilde{b} - \tilde{a}) \gtrsim^{IF}_{h_1,h_2} 0$.

One may check that an I-fuzzy number $\tilde{a} = \langle [175, 180, 185], [170, 180, 190] \rangle$ is 'more than or equal to' $178$ with degree of belief $0.4$ and degree of disbelief $0.8$. Further $\tilde{a}$ is 'more than or equal to' $172$ with degree of belief $1$ and degree of disbelief $0.2$. Here a real number $r$ is interpreted as the degenerate triangular I-fuzzy number $\langle [r, r, r], [r, r, r] \rangle$.

# 4 Linear programming with I-fuzzy parameters: proposed approach

In this section, we discuss our proposed approach to solve I-fuzzy number linear programming problems. For this we consider a general model of an I-fuzzy linear programming problem (IFLP).

$$
\text{(IFLP)} \qquad \begin{array}{ll} \text{Maximize} & \tilde{a}_0^T x \\ \text{subject to} & \\ & \tilde{A}^T x \quad \lesssim^{IF} \quad \tilde{b}, \\ & x \geq 0. \end{array}
$$

Here $\tilde{a}_0 = (\tilde{a}_{01}, \tilde{a}_{02}, \cdots, \tilde{a}_{0n})^T$ and $\tilde{b} = (\tilde{a}_{10}, \tilde{a}_{20}, \cdots, \tilde{a}_{m0})^T$ respectively are the n-dimensional and m-dimensional vectors with entries as symmetric triangular I-fuzzy numbers. Further let $\tilde{A} = [\tilde{a}_{ij}]$ be an $(m \times n)$ matrix with $\tilde{a}_{ij}, \ \ (i = 1, \ldots, m, \ j = 1, \ldots, n)$ as symmetric triangular I-fuzzy numbers. Thus $\tilde{a}_{ij} = \langle [a_{ij} - p_{ij}, a_{ij}, a_{ij} + p_{ij}], [a_{ij} - q_{ij}, a_{ij}, a_{ij} + q_{ij}] \rangle \ (i = 0, \ldots, m, \ j = 0, \ldots, n)$ and the inequality between two I-fuzzy numbers is, to compare them in the sense of almost positive.

Let $\tilde{a}_{00}$ be the aspiration level for the objective function (I-fuzzy goal), indicated by the decision maker. Therefore, $\tilde{a}_{00} = \langle [a_{00} - p_{00}, a_{00}, a_{00} + p_{00}], [a_{00} - q_{00}, a_{00}, a_{00} + q_{00}] \rangle$. Now solving the above I-fuzzy optimization problem is equivalent to solve the following system of I-fuzzy linear inequalities (IFLI).

$$
\text{(IFLI)} \qquad \text{Find } \ x \in \mathbf{R}^n \ \text{ such that} \qquad \begin{array}{rcl} \tilde{a}_0^T x & \gtrsim^{IF}_{h_1, h_2} & \tilde{a}_{00}, \\ \tilde{A} x & \lesssim^{IF}_{h_1, h_2} & \tilde{b}, \\ x & \geq & 0. \end{array}
$$

Here $\gtrsim^{IF}$ and $\lesssim^{IF}$ are the I-fuzzy inequalities between two I-fuzzy numbers as introduced above. Without loss of generality the above system of I-fuzzy inequalities can be equivalently written as (EIFLI)

$$
\text{(EIFLI)} \qquad \begin{array}{rcll} \tilde{y_0} & = & -\tilde{a}_{00} x_0 + \tilde{a}_{01} x_1 + \ldots + \tilde{a}_{0n} x_n & \gtrsim^{IF}_{h_1, h_2} 0, \\ \tilde{y_1} & = & \tilde{a}_{10} x_0 - \tilde{a}_{11} x_1 - \ldots - \tilde{a}_{1n} x_n & \gtrsim^{IF}_{h_1, h_2} 0, \\ \vdots & = & \vdots & \vdots \\ \tilde{y_m} & = & \tilde{a}_{m0} x_0 - \tilde{a}_{m1} x_1 - \ldots - \tilde{a}_{mn} x_n & \gtrsim^{IF}_{h_1, h_2} 0, \end{array}
$$

for

$$x_0 = 1, \quad x_j \geq 0, \ \ j = 1, \cdots, n.$$

Here $\tilde{y}_i \gtrsim^{IF}_{h_1,h_2} 0$ is to be understood in the sense of being 'almost positive' in I-fuzzy environment as per the Definition 3.2.

As $y_0 = -a_{00}x_0 + \sum_{j=1}^{n} a_{0j}x_j$, the membership and non-membership functions for $\tilde{y_0}$ are

$$\mu_{\tilde{y}_0}(y) = \begin{cases} 1 + \dfrac{y - y_0}{\sum_{j=0}^{n} p_{0j}x_j}, & y_0 - \sum_{j=0}^{n} p_{0j}x_j < y < y_0, \\ 1, \quad y = y_0, & \\ 1 - \dfrac{y - y_0}{\sum_{j=0}^{n} p_{0j}x_j}, & y_0 < y < y_0 + \sum_{j=0}^{n} p_{0j}x_j, \\ 0, \quad otherwise. & \end{cases}$$

and

$$\nu_{\tilde{y}_0}(y) = \begin{cases} \dfrac{-y + y_0}{\sum_{j=0}^{n} q_{0j}x_j}, & y_0 - \sum_{j=0}^{n} q_{0j}x_j < y < y_0, \\ 0, \quad y = y_0, & \\ \dfrac{y - y_0}{\sum_{j=0}^{n} q_{0j}x_j}, & y_0 < y < y_0 + \sum_{j=0}^{n} q_{0j}x_j, \\ 1, \quad otherwise. & \end{cases}$$

respectively.

Similarly, for $y_i = a_{i0}x_0 - \sum_{j=1}^{n} a_{ij}x_j$, the membership and non-membership functions for $\tilde{y}_i, \ \ i = 1, \ldots, m$ are

$$\mu_{\tilde{y}_i}(y) = \begin{cases} 1 + \dfrac{y - y_i}{\sum_{j=0}^{n} p_{ij}x_j}, & y_i - \sum_{j=0}^{n} p_{ij}x_j < y < y_i, \\ 1, \quad y = y_i, & \\ 1 - \dfrac{y - y_i}{\sum_{j=0}^{n} p_{ij}x_j}, & y_i < y < y_i + \sum_{j=0}^{n} p_{ij}x_j, \\ 0, \quad otherwise. & \end{cases}$$

and

$$\nu_{\tilde{y}_i}(y) = \begin{cases} \dfrac{-y+y_i}{\sum\limits_{j=0}^{n} q_{ij}x_j}, & y_i - \sum\limits_{j=0}^{n} q_{ij}x_j < y < y_i, \\ 0, \quad y = y_i, \\ \dfrac{y-y_i}{\sum\limits_{j=0}^{n} q_{ij}x_j}, & y_i < y < y_i + \sum\limits_{j=0}^{n} q_{ij}x_j, \\ 1 \quad otherwise. \end{cases}$$

respectively.

Since with any fuzzy inequality, there is a degree of belief and also a degree of disbelief associated with it, a decision maker will like to choose a solution for which the belief score is maximum. In view of Remark 5, it is assumed here that $h_{\geq}h_2$ and $h_1+h_2 \geq 1$. Therefore to solve (EIFLI) we need to solve the following equivalent nonlinear programming problem (ENLP).

(ENLP) Max $h_1 + h_2 - 1$

subject to

$$\begin{aligned} \mu_{\tilde{Y}_i}(0) &\leq 1 - h_1, \ \ (i = 0, 1, \cdots, m), \\ \nu_{\tilde{Y}_i}(0) &\geq h_2, \ \ (i = 0, 1, \cdots, m), \\ x_0 &= 1, \ \ x_j \geq 0, \ \ (j = 1, 2, \cdots, n), \\ h_1 &\geq h_2, \\ h_1 + h_2 &\geq 1, \\ 0 \leq h_1, \, h_2 &\leq 1. \end{aligned}$$

Again on substituting the values of $\mu_{\tilde{Y}_i}(0)$ and $\nu_{\tilde{Y}_i}(0), \ \ (i = 0, 1, \cdots, m)$, in (ENLP), we get

(ENLP) max $h_1 + h_2 - 1$

subject to

$$\begin{aligned} -a_{00}x_0 + \sum_{j=1}^{n} a_{0j}x_j &\geq h_1\big(\textstyle\sum_{j=0}^{n} p_{0j}x_j\big), \\ a_{i0}x_0 - \sum_{j=1}^{n} a_{ij}x_j &\geq h_1\Big(\sum_{j=0}^{n} p_{ij}x_j\Big), \\ & \quad i = 1, 2, \cdots, m \\ -a_{00}x_0 + \sum_{j=1}^{n} a_{0j}x_j &\geq h_2 \sum_{j=0}^{n} q_{0j}x_j, \\ a_{i0}x_0 - \sum_{j=1}^{n} a_{ij}x_j &\geq h_2 \sum_{j=0}^{n} q_{ij}x_j, \\ i = & \ \ 1, 2, \cdots, m \\ x_0 &= 1, \ \ x_j \geq 0, \\ j = & \ \ 1, 2, \cdots, n, \\ h_1 + h_2 &\geq 1, \\ 0 \leq h_1, h_2 &\leq 1. \end{aligned}$$

Let $(x^*, h_1^*, h_2^*)$ be an optimal solution of (ENLP). Then we say that $x^*$ is a solution of the system of linear I-fuzzy inequalities (EIFLI) with degree of belief $h_1^*$ and the degree of disbelief $(1 - h_2^*)$. The quantity $(h_1^* + h_2^* - 1)$ is refereed as the belief score of the solution $x^*$.

*Remark* 6. The class of fuzzy linear programming problems can be classified into symmetric and non-symmetric forms. In the symmetric form, an aspiration level is set for the objective function of an optimization problem, which in turn led the objective function to be subsequently treated as a fuzzy inequality along with the other fuzzy inequality constraints, while in the non-symmetric form, a distinction is maintained between the fuzzy objective function and the fuzzy inequality constraints of an optimization problem and some appropriately designed ranking function is used to define optimal solution. Moreover, if the parameters involved in an optimization problem are fuzzy or generalized fuzzy numbers, then some defuzzification function is usually applied to solve such problems.

Different approaches have been suggested in the literature to handle fuzzy linear programming problems in symmetric and non-symmetric forms. For more on the treatment meted out to the two forms, one can refer to Bector et al. [11] and Tanaka and Asai [30] for the symmetric form, and Bector et al. [11] and Clemente et al. [12] for the non-symmetric form. Here it is to be noted that the approaches to handle these two forms of fuzzy optimization problems are non comparable for they employ various indices based ranking functions. There is no research work available in the literature which study (IFLP) in the symmetric form having TIFNs involved in it. Consequently, comparison of our present approach with any existing approach for I-fuzzy linear programming problems is not meaningful.

## 4.1 Numerical example

Suppose a company is into manufacturing three different products $P_1, P_2$ and $P_3$. Three constraints, pertaining to the total demand, availability of raw material, and total available labour hours, have been imposed during the course of production of these products. However, the company do not supply the precise values of the requisite parameters namely, profit per unit per product, demand per unit per product, required raw material per unit per product, and labour hours per unit per product. In order to provide the optimal production plan to the company, a dialogue is initiated with the company. In this context, the company provide the estimated interpretation of the aforemention parameters. For instance, the company says that a profit per unit of product P1 is preferably between Rs. 4.5 and Rs. 5.5 with most likely value Rs. 5 and it can surely be not below Rs. 4 and above Rs. 6. This information has been formulated as TIFN $\tilde{5} = \langle [4.5, 5, 5.5], [4, 5, 6] \rangle$. Similarly, information on other parameters are provided by the company in linguistic form which are translated into TIFNs. So assume that the profits per unit of products $P_2$ and $P_3$ are Rs. $\tilde{3}$ and Rs. $\tilde{2}$, respectively. The daily demand for each product is in the ratio of $\tilde{4} : \tilde{3} : \tilde{1}$ with the total demand $\tilde{12}$ units. The raw material consumed by each product is in the ratio of $\tilde{1} : \tilde{4} : \tilde{2}$ and total availability of raw material is $\tilde{6}$ units per day. Further, the labour hours consumed by per unit of each product is in the ratio of $\tilde{1} : \tilde{2} : \tilde{4}$ with the total available hours per day $\tilde{10}$. The problem is to find how many units of each product $P_1,\ P_2$ and $P_3$ should be made per day to maximize the total profit.

Let $x_i$ denote the number of units of products $P_i,\ i = 1, 2, 3$, respectively to be made in a day. Suppose the decision maker takes a stand that his profit should be more than Rs. $\tilde{12}$. In this scenario, the I-fuzzy optimization problem (IFLP) becomes as follows.

$$
\begin{array}{ll}
Max & \tilde{5}x_1 + \tilde{3}x_2 + 2x_3 \\
\text{subject to} &
\end{array}
$$

$$
\begin{aligned}
\tilde{4}x_1 + \tilde{3}x_2 + x_3 &\lesssim^{IF} \tilde{12}, \\
\tilde{1}x_1 + \tilde{4}x_2 + 2x_3 &\lesssim^{IF} \tilde{6}, \\
\tilde{1}x_1 + \tilde{2}x_2 + 4x_3 &\lesssim^{IF} \tilde{10}, \\
x_1, x_2, x_3 &\geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{a}_{01} = \tilde{5} = \langle[4.5, 5, 5.5], [4, 5, 6]\rangle, \tilde{a}_{02} = \tilde{3} = \langle[2, 3, 4], [1, 3, 5]\rangle, \\
\tilde{a}_{03} = \tilde{2} = \langle[2, 2, 2], [2, 2, 2]\rangle,\ \tilde{a}_{11} = \tilde{4} = \langle[3.5, 4, 4.5], [3, 4, 5]\rangle, \\
\tilde{a}_{12} = \tilde{3} = \langle[1, 3, 5], [0, 3, 6]\rangle,\ \tilde{a}_{13} = \tilde{1} = \langle[1, 1, 1], [1, 1, 1]\rangle, \\
\tilde{a}_{21} = \tilde{1} = \langle[0, 1, 2], [0, 1, 2]\rangle,\ \tilde{a}_{22} = \tilde{4} = \langle[3, 4, 5], [2.5, 4, 5.5]\rangle, \\
\tilde{a}_{23} = \tilde{2} = \langle[2, 2, 2], [2, 2, 2]\rangle,\ \tilde{a}_{31} = \tilde{1} = \langle[0.5, 1, 1.5], [0, 1, 2]\rangle, \\
\tilde{a}_{32} = \tilde{2} = \langle[1, 2, 3], [0.5, 2, 3.5]\rangle,\ \tilde{a}_{33} = \tilde{4} = \langle[4, 4, 4], [4, 4, 4]\rangle, \\
\tilde{a}_{10} = \tilde{12} = \langle[11, 12, 13], [8, 12, 15]\rangle,\ \tilde{a}_{20} = \tilde{6} = \langle[5, 6, 7], [3.5, 6, 8.5]\rangle, \\
\tilde{a}_{30} = \tilde{10} = \langle[8, 10, 12], [7, 10, 13]\rangle,
\end{aligned}
$$

with

$$
\tilde{a}_{00} = \tilde{12} = \langle[11.8, 12, 12.2], [11.5, 12, 12.5]\rangle.
$$

Solving the above problem is equivalent to solve the following system of I-fuzzy linear inequalities.

(IFLI) Find $x \in \mathbf{R}^2$ such that

$$
\begin{aligned}
\tilde{5}x_1 + \tilde{3}x_2 + 2x_3 &\gtrsim^{IF}_{h_1,h_2} \tilde{12}, \\
\tilde{4}x_1 + \tilde{3}x_2 + x_3 &\lesssim^{IF}_{h_1,h_2} \tilde{12}, \\
\tilde{1}x_1 + \tilde{4}x_2 + 2x_3 &\lesssim^{IF}_{h_1,h_2} \tilde{6}, \\
\tilde{1}x_1 + \tilde{2}x_2 + 4x_3 &\lesssim^{IF}_{h_1,h_2} \tilde{10}, \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

The above system of I-fuzzy linear inequalities can be rewritten as follows.

$$\text{(EIFLI)} \qquad \begin{aligned} \tilde{y}_0 &= -\tilde{12}x_0 + \tilde{5}x_1 + \tilde{3}x_2 + 2x_3 \gtrsim^{IF}_{h_1,h_2} 0, \\ \tilde{y}_1 &= \tilde{12}x_0 - \tilde{4}x_1 - \tilde{3}x_2 - x_3 \gtrsim^{IF}_{h_1,h_2} 0, \\ \tilde{y}_2 &= \tilde{6}x_0 - \tilde{1}x_1 - \tilde{4}x_2 - 2x_3 \gtrsim^{IF}_{h_1,h_2} 0, \\ \tilde{y}_3 &= \tilde{10}x_0 - \tilde{1}x_1 - \tilde{2}x_2 - 4x_3 \gtrsim^{IF}_{h_1,h_2} 0, \\ x_0 &= 1,\ x_1, x_2, x_3 \geq 0. \end{aligned}$$

For $y_0 = -12 + 5x_1 + 3x_2 + 2x_3$, the membership and non-membership functions are defined as.

$$\mu_{\tilde{y}_0}(y) = \begin{cases} 1 + \dfrac{y - y_0}{(0.2 + 0.5x_1 + x_2)}, & (y_0 - 0.2 - 0.5x_1 - x_2) \leq y \leq y_0, \\ 1, & y = y_0, \\ 1 + \dfrac{y_0 - y}{(0.2 + 0.5x_1 + x_2)}, & y_0 \leq y \leq (y_0 + 0.2 + 0.5x_1 + x_2), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\nu_{\tilde{y}_0}(y) = \begin{cases} \dfrac{-y + y_0}{(0.5 + x_1 + 2x_2)}, & (y_0 - 0.5 - x_1 - 2x_2) \leq y \leq y_0, \\ 0, & y = y_0, \\ \dfrac{y - y_0}{(0.5 + x_1 + 2x_2)}, & y_0 \leq y \leq (y_0 + 0.5 + x_1 + 2x_2), \\ 1, & \text{otherwise}, \end{cases}$$

Similarly, for $y_1 = 12 - 4x_1 - 3x_2 - x_3$ ; $y_2 = 6 - x_1 - 4x_2 - 2x_3$ and $y_3 = 10 - x_1 - 2x_2 - 4x_3$, the membership and non-membership functions are defined as.

$$\mu_{\tilde{y}_1}(y) = \begin{cases} 1 + \dfrac{y - y_1}{(1 + 0.5x_1 + 2x_2)}, & (y_1 - 1 - 0.5x_1 - 2x_2) \leq y \leq y_1, \\ 1, & y = y_1, \\ 1 + \dfrac{y_1 - y}{(1 + 0.5x_1 + 2x_2)}, & y_1 \leq y \leq (y_1 + 1 + 0.5x_1 + 2x_2), \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu_{\tilde{y}_1}(y) = \begin{cases} -\dfrac{y - y_1}{(3 + x_1 + 3x_2)}, & (y_1 - 3 - x_1 - 3x_2) \leq y \leq y_1, \\ 0, & y = y_1, \\ \dfrac{y - y_1}{(3 + x_1 + 3x_2)}, & y_1 \leq y \leq (y_1 + 3 + x_1 + 3x_2), \\ 1, & \text{otherwise}, \end{cases}$$

and

$$\mu_{\tilde{y}_2}(y) = \begin{cases} 1 + \dfrac{y - y_2}{(1 + x_1 + x_2)}, & (y_2 - 1 - x_1 - x_2) \leq y \leq y_2, \\ 1, & y = y_2, \\ 1 - \dfrac{y_2 - y}{(1 + x_1 + x_2)}, & y_2 \leq y \leq (y_2 + 1 + x_1 + x_2), \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu_{\tilde{y_2}}(y) = \begin{cases} -\dfrac{y-y_2}{(2.5+x_1+1.5x_2)}, & (y_2-2.5-x_1-1.5x_2) \leq y \leq y_2, \\ 0, & y = y_2, \\ \dfrac{y-y_2}{(2.5+x_1+1.5x_2)}, & y_2 \leq y \leq (y_2+2.5+x_1+1.5x_2), \\ 1, & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{y_3}}(y) = \begin{cases} 1+\dfrac{y-y_3}{(2+0.5x_1+x_2)}, & (y_3-2-0.5x_1-x_2) \leq y \leq y_3, \\ 1, & y = y_3, \\ 1-\dfrac{y_3-y}{(2+0.5x_1+x_2)}, & y_3 \leq y \leq (y_3+2+0.5x_1+x_2), \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{\tilde{y_3}}(y) = \begin{cases} -\dfrac{y-y_3}{(3+x_1+1.5x_2)}, & (y_3-3-x_1-1.5x_2) \leq y \leq y_3, \\ 0, & y = y_3, \\ \dfrac{y-y_3}{(3+x_1+1.5x_2)}, & y_3 \leq y \leq (y_3+3+x_1+1.5x_2), \\ 1, & \text{otherwise.} \end{cases}$$

In this scenario the equivalent non linear programming problem (ENLP) becomes as follows

(ENLP) Max $h_1 + h_2 - 1$
subject to

$$\begin{aligned} 12+0.2h_1-5x_1+0.5h_1x_1-3x_2+h_1x_2-2x_3 &\leq 0, \\ -12+h_1+4x_1+0.5h_1x_1+3x_2+2h_1x_2+x_3 &\leq 0, \\ -6+h_1+x_1+h_1x_1+4x_2+h_1x_2+2x_3 &\leq 0, \\ -10+2h_1+x_1+0.5h_1x_1+2x_2+h_1x_2+4x_3 &\leq 0, \\ 12+0.5h_2-5x_1+h_2x_1-3x_2+2h_2x_2-2x_3 &\leq 0, \\ -12+3h_2+4x_1+h_2x_1+3x_2+3h_2x_2+x_3 &\leq 0, \\ -6+2.5h_2+x_1+h_2x_1+4x_2+1.5h_2x_2+2x_3 &\leq 0, \\ -10+3h_1+x_1+h_1x_1+2x_2+1.5h_1x_2+4x_3 &\leq 0, \\ h_1 &\geq h_2, \\ h_1+h_2 &\geq 1, \\ 0 \leq h_1,\ h_2 &\leq 1, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

An optimal solution of the above problem is $(x_1^* = 2.459, x_2^* = 0,\ x_3^* = 0.4127,\ h_1^* = 0.7848,\ h_2^* = 0.3205)$. Therefore, the degree of belief and degree of disbelief in making the statement that 'the system of linear inequalities is almost positive' is $0.7848$ and $0.6795$, respectively. Further the belief score of the solution $x^*$ is $0.1053$.

*Remark* 7. If the decision maker is not satisfied with the belief score as obtained by solving the problem (ENLP), it will indicate that the underlying model does not represent the reality as perceived by the decision maker. This will require changing of the parameters appropriately and re-applying the whole methodology to the new model.

# 5 Conclusions

This work extends Tanaka and Asai [30] approach to study I-fuzzy linear programming problem in which the problem data is prescribed via symmetric triangular I-fuzzy numbers. Its extension to symmetric Trapezoidal I-fuzzy number is natural and may be worked out.

Here it may be noted that the optimization problem (ENLP) is a non convex optimization problem and this scenario is similar to Tanaka and Asai [30]. As there are some algorithms available for solving non convex optimization problems e.g., ‘ the modified subgradient method’ and ‘the fuzzy decisive set method’ in [21], their utility in solving the resulting optimization problem (ENLP) should be explored. The proposed work further my be enhanced to study fully fuzzy linear programming problem with I-fuzzy numbers on the lines of [13].


# Acknowledgements

Authors would like to thank the Editor and the anonymous reviewers for their valuable suggestions that have improved the presentation of this paper. The first author also acknowledges the research support through Faculty Research Grant Scheme from GGSIP University, Delhi, India.


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