NIFS 1 (1995) 56-60

AN INTUITIONISTIC FUZZY K-NEAREST NEIGHBORS RULE

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1. Introduction

Intuitionistic fuzzy sets (IFS) have been introduced [1,2] as a more rich alternative of the classical fuzzy sets. An IFS A over an universe U is defined as $A=\{<x,\mu(x),\nu(x)>, x\in U\}$ where the functions $\mu: U \rightarrow [0,1]$ and $\nu: U \rightarrow [0,1]$ stand for the degree of membership and the degree of nonmembership, respectively, of x to A. The following inequality always holds: $\mu(x)+\nu(x)\leq 1$. The degree of indeterminateness $\pi(x)= 1-(\mu(x)+\nu(x))$ is supposed to reflect the "hesitation" in assigning the degrees for x. An ordinary fuzzy set B can be represented in terms of IFS as $B=\{<x,\mu(x),1-\mu(x)>, x\in U\}$

A representation of class membership in pattern classification problems by IFS seems to give a better means for designing a classifier in comparison with the crisp or fuzzy labeling of objects. This expectation is based on the possibility to process simultaneously the positive and the negative strength of evidence expressed by μ and ν, respectively. Along with the fact that more information is included into the classifier design, keeping track on the two processes in parallel seems to be more practically plausible [3,4]. Thus a physician who is working on a diagnostic hypothesis keeps in mind both positive and negative pieces of evidence trying (unconsciously) to weight them with respect to each other. This helpful analogy can be applied in pattern classification using an IFS formulation.

Fuzzy k-Nearest Neighbors rule is one of the most widely exploited classification paradigms due to its theoretical and semantic elegance, and feasibility of implementation [5,6,7]. Its idea is to assign an unknown object (subjected to classification) a membership value obtained as a function of the membership values of its nearest neighbors in the feature space. There are a lot of concrete formulas and generalizations but neither of these considers membership and nonmembership of the object simultaneously.

The paper presents an intuitionistic fuzzy version of the k-NN rule. We believe that due to the additional information included into the rule, this extension can lead to higher classification accuracy and better explanation capabilities.

2. Geometrical interpretation of IFS

An IFS can be visualized in a μ-ν plane as shown in fig 1. Every element of U can be assigned a point in the triangle defined by the points (0,0), (0,1), and (1,0). Ordinary fuzzy sets are represented by the line (1,0), (0,1) (the thick line) and crisp sets, by the point (1,0). Obviously, the representation offered by IFS contains more information, and, therefore, more flexibility in handling membership values and more tools to resemble human insight.

Let us suppose that the values $\mu(x)$ and $\nu(x)$ represent a class membership of the object x. In order to infer a crisp classification decision the following regions are formulated: *Acceptance*, *Rejection*, and *Refuse to decide* by introducing appropriate constants. The rational behind this is that we would like to accept that the class being investigated is true only if the membership value is high enough (above certain μ_a), and the corresponding nonmembership is low enough (below certain ν_a). We would reject the hypothesis that the object belongs to that class if the nonmembership is high enough, and the membership is low enough (μ_r, and ν_r, respectively). Otherwise we cannot believe in either of the crisp decisions and it is better to refuse to decide.

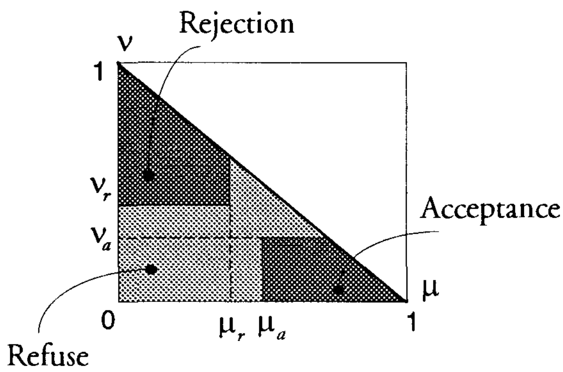

Fig. 1. Geometrical representation of IFS and the regions of Acceptance, Rejection, and Refuse.

Let us define the following function (we will call it "voting function"):

$$\varphi(x) = \begin{cases} \mu(x), \text{ if } \mu(x) \geq \mu_a, \text{ and } \nu(x) \leq \nu_a; \\ -\nu(x), \text{ if } \mu(x) < \mu_r, \text{ and } \nu(x) > \nu_r; \\ 0, \text{ otherwise.} \end{cases}$$

3. Intuitionistic Fuzzy k-nearest neighbors rule

Let $Z = \{Z_1,\ldots,Z_N\}$ be the set of objects generated from M classes $\Omega = \{\omega_1,\ldots,\omega_M\}$. Let us consider a classification problem where the class labels form M IFSs on Z, i.e. every object $Z_j \in Z$ is represented by its respective feature vesctor $[z_{j1},\ldots,z_{jn}]^T$ and by M couples $<\mu_i(x),\nu_i(x)>$, $i = 1,\ldots,M$.

The classical k-NN rule assigns the object the class where belong the majority of its k-nearest neighbors. There are a lot of fuzzy versions, the most trivial among which is to average the membership grades of the k-nearest neighbors of x in respect to a given class in order to obtain the membership degree to be assigned to x. Let $Z^1,\ldots,Z^k$ be the k-nearest neighbors of x among the objects in Z. The membership value in the fuzzy case is computed as:

$$\mu_i(x) = \frac{1}{k}\sum_{t=1}^{k} \mu_i(Z^t) \tag{1}$$

It is obvious that if we put in (1) the 0/1 values for $\mu_i(x)$ and choose the classification rule that assigns to x the class corresponding to the maximal value of $\mu_i(x)$ the fuzzy and the crisp rules coincide.

The representation of class membership in terms of IFS can be used to express also the classical and fuzzy statement:

- for the classical case

  $\mu_i(x) = 1$ and $\nu_i(x) = 0$, if $x \in \omega_i$, and

  $\mu_i(x) = 0$ and $\nu_i(x) = 1$, otherwise

- for the fuzzy case

$$\mu_i(x) = \mu_i(x) \text{ and } \nu_i(x) = 1 - \mu_i(x),\ i = 1,\dots,M$$

For the proposed IFS case the following formula is suggested to determine the membership value to a class :

$$u_i(x) = \sum_{t=1}^{k} \varphi_i(Z^t)$$

$$\mu_i(x) = 0.5\,[1 + \frac{1}{k}\,u_i(x)\,] \tag{2}$$

where the voting function φ_i (.) uses the degrees of membership and nonmembership to class ω_i.

Proposition 1. If $\mu_i(Z^t) = 1$ and $\nu_i(Z^t) = 0$, for $Z^t \in \omega_i$, and $\mu_i(Z^t) = 0$ and $\nu_i(Z^t) = 1$, otherwise, for every $t = 1,\dots,k$, and the decision assigns to x the class with the maximal $\mu_i(x)$, $i = 1,\dots,M$, (2) coincides with the crisp k-NN for every combination of values $\mu_a, \mu_r, \nu_a, \nu_r \in [0,1]$, $\mu_a \geq \mu_r$, $\nu_r \geq \nu_a$.

Proof. Let $\mu_i(Z^t) = 1$ and $\nu_i(Z^t) = 0$ for a given t, i.e. $Z^t \in \omega_i$ Then $\varphi_i(Z^t) = 1$ because the inequalities $\mu_i(Z^t) \geq \mu_a$ and $\nu_i(Z^t) \leq \nu_a$ hold for any μ_a and ν_a from [0,1]. Let, for another t $\mu_i(Z^t) = 0$ and $\nu_i(Z^t) = 1$, i.e. $Z^t \notin \omega_i$. Then $\varphi_i(Z^t) = -1$ because the inequalities dual to the above hold for any μ_r and ν_r from [0,1]. Therefore

$$u_i(x) = 2\,k_i - k$$

where k_i is the number of the neighbors of x (among the k nearest ones) that belong to class ω_i. Therefore

$$\mu_i(x) = \frac{k_i}{k}$$

By using the rule that assigns the class corresponding to the highest value of $\mu_i(x)$, x will obviously be labeled to the class where belong the majority of its nearest neighbors. □

Proposition 2. If $\mu_i(Z^t) = \mu_i(Z^t)$ and $\nu_i(Z^t) = 1 - \mu_i(Z^t)$, $i = 1,\dots,M$, and $t = 1,\dots,k$, and the decision assigns to x the class with the maximal $\mu_i(x)$, $i = 1,\dots,M$, the crisp class label coincides with that obtained by the fuzzy k-NN (1) for any $\mu_a \in [0,1]$, and $\mu_r = \mu_a$ and $\nu_a = \nu_r = 1 - \mu_a = 1 - \mu_r$.

Proof. Without losing generality we can suppose that the neighbors $Z^1,\dots,Z^k$ are ordered in such a way that for a given class ω_i $\mu_i(Z^t) > \mu_a$ for $t = 1,\dots,k_a$-1, and $\mu_i(Z^t) \leq \mu_a$ for $t = k_a,\dots, k$. Then (2) can be rewritten as

$$\mu_i(x) = 0.5 + \frac{1}{2k}\,[\,\sum_{t=1}^{k_a-1} \mu_i(Z^t) - \sum_{t=k_a}^{k} (1-\mu_i(Z^t))\,] =$$

$$= 0.5 + \frac{1}{2k}\,[\,\sum_{t=1}^{k} \mu_i(Z^t) - (\,k - k_a + 1)\,] =$$

$$= \frac{1}{2}\left[\frac{1}{k}\sum_{t=1}^{k} \mu_i(Z^t)\right] + \frac{k_a - 1}{2k} \quad (3)$$

Since (3) is an increasing function of $\frac{1}{k}\sum_{t=1}^{k} \mu_i(Z^t)$, the class corresponding to the maximal value of $\mu_i(x)$ will be the same as the class inferred by the same rule from (2). □

4. An example

In order to show the flexibility of the proposed IFk-NN rule the following example is considered. Let Z^1, ..., Z^5 be the five nearest neighbors of x and let $\Omega = \{\omega_1,\omega_2,\omega_3\}$. The discussion is confined to the case where the degrees of membership for a given reference object sum up to 1, and the crisp label corresponds to the maximal membership value. Since we have no information about the nonmembership we stipulate that $\nu_i(Z^t) = 1 - \mu_i(Z^t)$, $i = 1,2,3$, $t = 1,...,5$. Table 1 shows the degrees of membership and nonmembership (in the parentheses) of the five neighbors and the respective crisp class labels.

Table 1.

	Z^1	Z^2	Z^3	Z^4	Z^5
ω_1	0.2(0.8)	0.2(0.8)	0.9(0.1)	0.3(0.7)	0.2(0.8)
ω_2	0.5(0.5)	0.3(0.7)	0.0(1.0	0.4(0.6)	0.5(0.5)
ω_3	0.3(0.7)	0.5(0.5)	0.1(0.9)	0.3(0.7)	0.3(0.7)
crisp class	ω_2	ω_3	ω_1	ω_2	ω_2

Table 2 contains the inferred degrees of membership by the crisp, fuzzy, and IFk-NN, the latter with three different settings of the parameters μ_a,μ_r,ν_a,ν_r. The notations are as follows:

Table 2.

	μ_1	μ_2	μ_3	μ_4	μ_5
ω_1	0	0.36	0.28	0.35	0.35
ω_2	1	0.34	0.17	0.50	0.40
ω_3	0	0.30	0.15	0.46	0.41
crisp class	ω_2	ω_1	ω_1	ω_2	ω_3

- μ_1 : crisp labeling;
- μ_2 : fuzzy k-NN (1);
- μ_3 : IF k-NN with $\mu_a = 0.79$, $\mu_r = 0.79$, $\nu_a = 0.21$, $\nu_r = 0.21$;
- μ_4 : IF k-NN with $\mu_a = 0.49$, $\mu_r = 0.21$, $\nu_a = 0.51$, $\nu_r = 0.79$;
- μ_5 : IF k-NN with $\mu_a = 0.89$, $\mu_r = 0.21$, $\nu_a = 0.11$, $\nu_r = 0.79$;

Looking at the results in the tables we can conclude that both three hypotheses yielded by the different parameter setting of the IFk-NN rule are plausible because there is no visible dominance of any of the classes to the others. What this example is supposed to emphasize is that the variarion of the threshold values may change the class assignment which can, hopefully lead to a better classification performance. The values of the parameters can be tuned by certain optimization procedure using the data set in the task.

5. Acknowledgments

This work is connected and partially supported by the contract No. TH-468/94 with the National Scientific Fund, Bulgaria.

6. References

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