Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy modes, medians and mean elements

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Abstract: The concepts of intuitionistic fuzzy mode, median and mean are introduced. Sets of intuitionistic fuzzy modes, intuitionistic fuzzy medians and intuitionistic fuzzy mean values are defined.
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To Janusz!

1 Introduction

The Intuitionistic Fuzzy Pair (IFP, see [3]) is an object of the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$ (see Fig. 1). It is used for an intuitionistic fuzzy evaluation of some object or process and its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. We define the relations ("iff" is an abbreviation of "if and only if")

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x > y & \text{iff} & a > c \text{ and } b < d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x = y & \text{iff} & a = c \text{ and } b = d \end{array}.$$

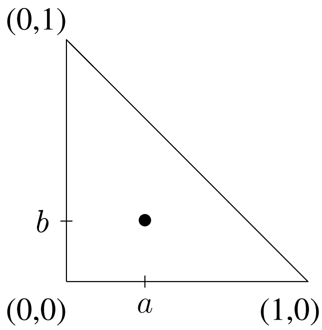

Figure 1. Visual representation of an intuitionistic fuzzy pair

Now, using IFPs as tools for evaluation, we introduce two new concepts, related to mathematical statistics.

Let everywhere below, $X = \{x_1, x_2, ..., x_n\}$ be a finite set with real numbers. Let its elements be well (lineary) ordered, i.e., $x_1 \leq x_2 \leq ... \leq x_n$ and $x_1 < x_n$.

2 Definitions of an intuitionistic fuzzy mode and a set of intuitionistic fuzzy modes

Let $0 < \delta < 1$ be fixed. Let for every two elements of X x_i, x_j: $\nu(x_i, x_j)$ be the number of elements $x_k \in X$, so that $x_i \leq x_k \leq x_j$. Then, we determine the natural numbers a and b so that:

$$x_b - x_a \leq \delta(x_n - x_1),$$

$$\nu(x_a, x_b) = \max\{\nu(x_i, x_j) : x_i, x_j \in X \;\&\; x_j - x_i \leq \delta(x_n - x_1)\}.$$

The δ-Intuitionistic Fuzzy Mode of X (δ-IF-Mode of X) is defined by IFP

$$mo_{X,\delta} = \left\langle \frac{x_a - x_1}{x_n - x_1}, \frac{x_n - x_b}{x_n - x_1} \right\rangle.$$

It is possible that there are more pairs, i.e. $(x_{a_1}, x_{b_1}), ..., (x_{a_s}, x_{b_s})$, such that

$$x_{b_k} - x_{a_k} \leq \delta(x_n - x_1),$$

$$\nu(x_{a_k}, x_{b_k}) = \max\{\nu(x_i, x_j) : x_i, x_j \in X \;\&\; x_j - x_i \le \delta(x_n - x_1)\}$$

for $1 \le k \le s$. In this case we will say that X has a collection of δ-IF-Modes. Obviously, this collection is an Intuitionistic Fuzzy Set (IFS, see [1, 2]).

The set of δ-IF-Mode of X is defined by

$$Mo_{X,\delta} = \bigcup_{i=1}^{s} \{x : x \in X \;\&\; x_{a_i} \le x \le x_{b_i}\}.$$

If for all i and j, $x_j - x_i > \delta(x_n - x_1)$, then the set X has no δ -IF-Mode.

The procedure for obtaining of the (collection of) δ-IF-mode(s) of X and of the set of δ-IF-Mode of X is the following:

1. Let $i = 1$, $Max = 0$, $M = \{\}$.
2. Let $a = i$ and $b = i + 1$.
3. If $x_b - x_a > \delta(x_n - x_1)$ and $i < n - 1$, then $i = i + 1$ and go to 2.
4. If $x_b - x_a \le \delta(x_n - x_1)$ and $Max < \nu(x_a, x_b)$, then $Max = \nu(x_a, x_b)$, $M = \{(x_a, x_b)\}$ and go to 6.
5. If $x_b - x_a \le \delta$ and $Max = \nu(x_a, x_b)$, then $M = M \cup \{(x_a, x_b)\}$.
6. If $b < n$, then $b = b + 1$ and go to 3.
7. For every pair $(x_a, x_b) \in M$ evaluate the mode $mo_{x,\delta} = \langle \frac{x_a - x_1}{x_n - x_1}, \frac{x_n - x_b}{x_n - x_1} \rangle$.

We give two examples to illustrate the new concept.

Example 1. Let us consider the set $X = \{1, 1.5, 2, 2, 3, 4, 4.5, 5, 5.5, 6\}$, $\delta = 0.1$ and $\delta(x_n - x_1) = 0.1(6 - 1) = 0.5$. Then

$$\max\{\nu(x_i, x_j) : x_i, x_j \in X \,\&\, x_j - x_i \le 0.5\} = \nu(1.5, 2) = 3$$

and $2 - 1.5 = 0.5 \le \delta$. That is $x_a = 1.5$ and $x_b = 2$. Substituting these values in the formula for the δ-IF-mode we obtain:

$$mo_{X,\delta} = \left\langle \frac{1.5 - 1}{6 - 1}, \frac{6 - 2}{6 - 1} \right\rangle = \left\langle \frac{0.5}{5}, \frac{4}{5} \right\rangle = \langle 0.1, 0.8 \rangle .$$

The set of δ-IF-Modes of the set X is

$$Mo_{X,\delta} = \{1.5, 2\} .$$

Example 2. Let $X = \{1, 1.5, 2, 2, 3, 4, 4.5, 4.5, 5.5, 6\}$, $\delta = 0.5$ and $\delta(x_n - x_1) = 0.1(6 - 1) = 0.5$. Now we have

$$\max\{\nu(x_i, x_j) : x_i, x_j \in X \,\&\, x_j - x_i \le 0.5\} = \nu(1.5, 2) = \nu(4, 4.5) = 3$$

and $2 - 1.5 = 0.5 \leq \delta$, $4.5 - 4 = 0.4 \leq \delta$. When $x_a = 1.5$ and $x_b = 2$ as in Example 1 we obtain $mo_{X,\delta} = \langle 0.1, 0.8 \rangle$.

When $x_a = 4$ and $x_b = 4.5$ after substitution in the formula for the δ-IF-mode we obtain:

$$mo_{X,\delta} = \left\langle \frac{4-1}{6-1}, \frac{6-4.5}{6-1} \right\rangle = \langle 0.6, 0.3 \rangle \ .$$

The set of δ-IF-Modes of the set X is

$$Mo_{X,\delta} = \{1.5, 2, 4, 4.5\} .$$

In the case when $\delta = 0$ we have the standard mode.

3 Definitions of an intuitionistic fuzzy median and a set of intuitionistic fuzzy medians

Let $0 < \delta < 1$ be fixed again. For n there are two cases that we discuss separately.

Case 1: $n = 2m$, where m is a natural number. Then we determine the natural numbers a and b so that:

$$x_a = \min \left\{ x_i : \frac{x_m + x_{m+1}}{2} - x_i \leq \frac{\delta}{2}(x_n - x_1) \right\},$$

$$x_b = \max \left\{ x_j : x_j - \frac{x_m + x_{m+1}}{2} \leq \frac{\delta}{2}(x_n - x_1) \right\}.$$

If one of the sets above (or both of them) is empty, then we use $\frac{x_m + x_{m+1}}{2}$ as a value of x_a or x_b respectively.

Case 2: $n = 2m + 1$, where m is a natural number. Then we determine the natural numbers a and b so that:

$$x_a = \min \left\{ x_i : x_{m+1} - x_i \leq \frac{\delta}{2}(x_n - x_1) \right\},$$

$$x_b = \max \left\{ x_j : x_j - x_{m+1} \leq \frac{\delta}{2}(x_n - x_1) \right\}.$$

If one of the sets above (or both of them) is empty, then we use x_{m+1} as a value of x_a or x_b respectively. In both cases, the δ-Intuitionistic Fuzzy Median of X (δ-IF-Median of X) is defined as the IFP

$$me_{X,\delta} = \left\langle \frac{x_a - x_1}{x_n - x_1}, \frac{x_n - x_b}{x_n - x_1} \right\rangle.$$

The procedure for evaluation of the δ-IF-Median in Case 1 is the following:

1. $m = \frac{n}{2}$, $x_a = x_b = \frac{x_m + x_{m+1}}{2}$
2. Let $i = m$ and $j = m + 1$.
3. If $\frac{x_m + x_{m+1}}{2} - x_i > \frac{\delta}{2}(x_n - x_1)$, go to 6.
4. $x_a = x_i$.

5. If $i > 1$, then $i = i - 1$ and go to 3.

6. If $x_j - \frac{x_m + x_{m+1}}{2} > \frac{\delta}{2}(x_n - x_1)$ go to 9.

7. $x_b = x_j$.

8. If $j < n$, then $j = j + 1$ and go to 6.

9. Determine δ-IF-Median using the formula

$$me_{X,\delta} = \left\langle \frac{x_a - x_1}{x_n - x_1}, \frac{x_n - x_b}{x_n - x_1} \right\rangle .$$

The set of the δ-IF-Median(s) is

$$Me_{X,\delta} = \{x : x \in X \;\&\; x_a \le x \le x_b\}.$$

To illustrate the new concept we provide two examples.

Example 3. Let us consider the set $X = \{1, 2, 3.8, 4, 5, 6\}$, $\delta = 0.16$ and $\frac{\delta}{2}(x_n - x_1) = 0.4$. For this set $n = 6 = 2 * 3$ and $m = 3$ – Case 1. Then we obtain:

$$x_a = \min\left\{x_i : \frac{3.8 + 4}{2} - x_i \le 0.4\right\} = \min\{x_i : 3.9 - x_i \le 0.4\} = 3.8\,,$$

$$x_b = \max\left\{x_j : x_j - \frac{3.8 + 4}{2} \le 0.4\right\} = \max\{x_j : x_j - 3.9 \le 0.4\} = 4\,.$$

Substituting these values in the formula for the δ-IF-Median we obtain:

$$me_{X,\delta} = \left\langle \frac{3.8 - 1}{6 - 1}, \frac{6 - 4}{6 - 1} \right\rangle = \left\langle \frac{14}{25}, \frac{2}{5} \right\rangle .$$

The set of δ-IF-Medians of the set X is

$$Me_{X,\delta} = \{3.8, 4\}\,.$$

Example 4. Let us consider the set $X = \{1, 2, 3, 3.2, 3.4, 4, 6\}$, $\delta = 0.12$ and $\frac{\delta}{2}(x_n - x_1) = 0.3$. Now $n = 7 = 2 * 3 + 1$ and $m = 3$ – Case 2. Then we obtain

$$x_a = \min\{x_i : 3.2 - x_i \le 0.3\} = 3\,,$$

$$x_b = \max\{x_j : x_j - 3.2 \le 0.3\} = 3.4.$$

Substituting these values in the formula for δ-IF-Median we obtain

$$me_{X,\delta} = \left\langle \frac{3 - 1}{6 - 1}, \frac{6 - 3.4}{6 - 1} \right\rangle = \left\langle \frac{2}{5}, \frac{13}{25} \right\rangle .$$

The set of δ-IF-Medians of the set X is

$$Me_{X,\delta} = \{3, 3.2, 3.4\}\,.$$

4 Definitions of an intuitionistic fuzzy mean element and a set of intuitionistic fuzzy mean elements

Let $0 < \delta < 1$ be fixed. First, we determine the arithmetic mean denoted by $\bar{x}$ of the set X:

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}.$$

Let $x_a = \min\{x_i : x_i \in X \,\&\, \bar{x} - x_i \leq \frac{\delta}{2}(x_n - x_1)\}$ and $x_b = \max\{x_j : x \in X \,\&\, x_j - \bar{x} \leq \frac{\delta}{2}(x_n - x_1)\}$. If one of the sets (or both of them) is empty, we use $\bar{x}$ as a value of x_a or x_b respectively. Then the δ-Intuitionistic Fuzzy Mean Element (δ-IF-ME) of the set X is the IFP

$$\bar{x}_{X,\delta} = \left\langle \frac{x_a - x_1}{x_n - x_1}, \frac{x_n - x_b}{x_n - x_1} \right\rangle .$$

The set of all δ-IF-MEs is

$$\bar{X}_{X,\delta} = \left\{ x : x \in X \,\&\, \bar{x} - \frac{\delta}{2}(x_n - x_1) \leq x \leq \bar{x} + \frac{\delta}{2}(x_n - x_1) \right\}.$$

Example 5. Let $X = \{1, 2, 2.5, 3.5, 4, 5, 5.5, 6\}$, $\delta = 0.16$ and $\frac{\delta}{2}(x_n - x_1) = 0.08 * 5 = 0.4$. Then $\bar{x} = 3.6875$, $\bar{x} - \frac{\delta}{2}(x_n - x_1) = 3.6875 - 0.4 = 3.2875$ and $\bar{x} + \frac{\delta}{2}(x_n - x_1) = 3.6875 + 0.4 = 4.0875$. Then $x_a = 3.5$ and $x_b = 4$. The δ-IF-Mean of X is

$$\bar{x}_{X,\delta} = \left\langle \frac{3.5 - 1}{6 - 1}, \frac{6 - 4}{6 - 1} \right\rangle = \langle 0.5, 0.4 \rangle .$$

The set of all δ-IF-MEs is

$$\bar{X}_{X,\delta} = \{3.5, 4\} .$$

5 Conclusion

In our future work, we will study the properties of the δ-IF Mode, Median and Mean. Also, we intend to propose analogous definitions for other statistical concepts and to extend the definition of IF Mode, Median and Mean over sets of intuitionistic fuzzy pairs.

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