ICIFSS, 8–10 January 2018, Erode, Tamil Nadu, India
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 24, 2018, No. 1, 21–29
DOI: 10.7546/nifs.2018.24.1.21-29

Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space

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Received: 25 October 2017 **Revised:** 28 November 2017 **Accepted:** 2 December 2017

Abstract: In the present paper the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space has been developed with the help of intuitionistic fuzzy basis.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy vector space, Intuitionistic fuzzy dimension.
AMS Classification: 03E72, 15A03.

1 Introduction

The notion of intuitionistic fuzzy set (IFS) was introduced by Atanassov [1, 2, 3, 4] as a generalization of Zadeh's fuzzy set [22]. There are situations where IFS theory is more appropriate to dealt with [7]. IFS theory have successfully been applied in knowledge engineering, medical diagnosis, decision making, career determination, etc., [11, 21, 12]. Several researchers have extended various mathematical aspects such as groups, rings, topological spaces, metric spaces, topological groups, topological vector spaces etc. in IFS [6, 10, 13, 16, 17, 18, 19]. The notion of fuzzy vector subspaces has been introduced by Katsaras [14] and a notion of fuzzy bases and fuzzy dimension was studied by Shi *et al.* [20]. We have introduced a notion of intuitionistic fuzzy vector space and intuitionistic fuzzy basis in [9]. As a continuation of our paper [9], here we introduced the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space with the help of intuitionistic fuzzy basis and studied some of its basic results.

2 Preliminaries

Definition 2.1. *[1] Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X is defined as an object having the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, *where* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non-membership (namely* $\nu_A(x)$*) of each element* $x \in X$ *to the set A, respectively, and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for each* $x \in X$*. For the sake of simplicity we shall use the symbol* $A = (\mu_A, \nu_A)$ *for the intuitionistic fuzzy set* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

In this paper, we use the symbols $a \wedge b = min\{a,b\}$ *and* $a \vee b = max\{a,b\}$.

Definition 2.2. *[1] Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be intuitionistic fuzzy sets of a set X. Then*

(1) $A \subseteq B$ *iff* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$.

(2) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$.

(3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$

(4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

(5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

(6) $\square A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in X\}$, $\lozenge A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3. *[4] Let A be an IFS in a set X. Then for* $\lambda, \xi \in [0,1]$ *with* $\lambda + \xi \leq 1$*, the set* $A^{[\lambda,\xi]} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \xi\}$ *is called* (λ,ξ)*-level subset of A.*

Proposition 2.4. *[4] Let A be an IFS in a set X and* $(\lambda_1,\xi_1),(\lambda_2,\xi_2) \in Im(A)$*. If* $\lambda_1 \leq \lambda_2$ *and* $\xi_1 \geq \xi_2$*, then* $A^{[\lambda_1,\xi_1]} \supseteq A^{[\lambda_2,\xi_2]}$.

Definition 2.5. *[15, 5] Let X be a vector space over the field K, the field of real and complex numbers,* $\alpha \in K$*,* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be two intuitionistic fuzzy sets of X. Then*

(1) the sum of A and B is defined to be the intuitionistic fuzzy set $A + B = (\mu_A + \mu_B, \nu_A + \nu_B)$ *of X given by*

$$\mu_{A+B}(x) = \begin{cases} \sup\limits_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} & if\ x = a+b \\ 0 & otherwise, \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \inf\limits_{x=a+b} \{\nu_A(a) \vee \nu_B(b)\} & if\ x = a+b \\ 1 & otherwise. \end{cases}$$

(2) αA *is defined to be the IFS* $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ *of X, where*

$$\mu_{\alpha A}(x) = \begin{cases} \mu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \sup\limits_{y \in X} \mu_A(y) & if\ \alpha = 0, x = \theta \\ 0 & if\ \alpha = 0, x \neq \theta, \end{cases}$$

$$\nu_{\alpha A}(x) = \begin{cases} \nu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \inf\limits_{y\in X} \nu_A(y) & if\ \alpha = 0, x = \theta \\ 1 & if\ \alpha = 0, x \neq \theta. \end{cases}$$

.

Proposition 2.6. *[9] Let $A, A_1, \ldots, A_n$ be intuitionistic fuzzy sets in a vector space X and $\lambda_1, \ldots, \lambda_n$ be scalars. Then the following assertions are equivalent:*

(1) $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n \subseteq A$.

(2) For all $x_1, x_2, \ldots, x_n$ in X, we have
$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \geq min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}$ *and* $\nu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots, \nu_{A_n}(x_n)\}$.

Definition 2.7. *[9] An IFS $V = (\mu_V, \nu_V)$ of a vector space X over the field K is said to be intuitionistic fuzzy vector space over X if*

(i) $V + V \subseteq V$

(ii) $\alpha V \subseteq V$, *for every scalar* α.

We denote the set of all intuitionistic fuzzy vector spaces over a vector space X by $IFVS(X)$.

Remark 2.8. *[9] Let X be a vector space.*

(1) If μ_V is a fuzzy subspace of X, then $V = (\mu_V, \mu_V^c) \in IFVS(X)$.

(2) If $V \in IFVS(X)$, then μ_V and ν_V^c are fuzzy vector subspace of X.

(3) If $V \in IFVS(X)$, then $\Box V, \Diamond V \in IFVS(X)$.

Lemma 2.9. *[9] Let V be an intuitionistic fuzzy set in a vector space X. Then, the following are equivalent:*

(1) V is an intuitionistic fuzzy vector space over X.

(2) For all scalars α, β, we have $\alpha V + \beta V \subseteq V$.

(3) For all scalars α, β and for all $x, y \in X$, we have
$\mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y)\}$ *and* $\nu_V(\alpha x + \beta y) \leq \nu_V(x) \vee \nu_V(y)$.

Remark 2.10. *[9] Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [19] and [8].*

Proposition 2.11. *[8] If $V, W \in IFVS(X)$, then $V + W \in IFVS(X)$.*

Proposition 2.12. *[9] If $V \in IFVS(X)$ $\alpha \in K$, then $\alpha V \in IFVS(X)$.*

Proposition 2.13. *[8] If $\{V_i\}_{i\in I} \in IFVS(X)$, then $\bigcap\limits_{i\in I} V_i \in IFVS(X)$.*

Proposition 2.14. *[9] Let $V \in IFVS(X)$. Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x), \forall x \in X$.*

Proposition 2.15. *[9] Let $V \in IFVS(X)$. Then for each $(\lambda, \xi) \in [0,1] \times [0,1]$ with $\lambda + \xi \leq 1$, $\lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$, $V^{[\lambda,\xi]}$ is a subspace of the vector space X,*

Definition 2.16. *[9] For any $(a,b),(c,d) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $c+d \leq 1$, we say that:*

(1) $(a,b) \geq (c,d)$ if $a \geq b$ and $c \leq d$.

(2) $(a,b) \leq (c,d)$ if $a \leq b$ and $c \geq d$.

(3) $(a,b) > (c,d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.

(4) $(a,b) < (c,d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.

(5) $(a,b) = (c,d)$ if $a = b$ and $c = d$.

Proposition 2.17. *[9] Let $V \in IFVS(X)$ with $\dim X = m$. Then $Im(V)$ contains at most $m+1$ points of $[0,1] \times [0,1]$.*

Definition 2.18. *[9] Let $V = (\mu_V, \nu_V) \in IFVS(X)$. Then for any $\lambda \in \mu_V(X), \xi \in \nu_V(X)$ we define $\mu_V^{[\lambda]} = \{x \in X : \mu_V(x) \geq \lambda\}$ and $\nu_V^{[\xi]} = \{x \in X : \nu_V(x) \leq \xi\}$, $[\lambda 1_{\mu_V^{[\lambda]}}](x) = \begin{cases} \lambda, & if\ x \in \mu_V^{[\lambda]} \\ 0, & otherwise \end{cases}$, $[\xi 1_{\nu_V^{[\xi]}}](x) = \begin{cases} \xi, & if\ x \in \nu_V^{[\xi]} \\ 1, & otherwise \end{cases}$.*

Theorem 2.19. *[9] (Representation Theorem) Let $V \in IFVS(X)$ with $\dim X = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \ldots (\lambda_k, \xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \ldots > (\lambda_k, \xi_k) \geq (0,1)$. Then there exists nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0,\xi_0]} \subsetneqq V^{[\lambda_1,\xi_1]} \subsetneqq \ldots \subsetneqq V^{[\lambda_k,\xi_k]} = X$ such that $\mu_V = \lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee \ldots \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}$ and $\nu_V = \xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \ldots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}$. Also,*

(1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1})$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta,\rho]} = V^{[\eta,\sigma]} = V^{[\lambda_i,\xi_i]}$.

(2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1}), (\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta,\rho]} \supsetneqq V^{[\eta,\sigma]}$.

Definition 2.20. *[9] Let $V \in IFVS(X)$ with $\dim X = m$. Consider Theorem 2.19. Let B_{V_i} be the basis of $V^{[\lambda_i,\xi_i]}, i = 0,1,..,k$ such that*

$$B_{V_0} \subsetneqq B_{V_1} \subsetneqq \cdots \subsetneqq B_{V_k}. \qquad (*)$$

If $V^{(\lambda_0,\xi_0)} = \{\theta\}$, we start with $V^{(\lambda_1,\xi_1)}$.
Define a map $\mathbb{B}$ from X to $[0,1] \times [0,1]$ by
$\mu_{\mathbb{B}}(x) = \begin{cases} \vee\{\lambda_i : x \in B_{V_i}\} \\ 0,\ otherwise \end{cases}$ *and* $\nu_{\mathbb{B}}(x) = \begin{cases} \wedge\{\xi_i : x \in B_{V_i}\} \\ 1,\ otherwise \end{cases}$.

Let $\mu_{\mathbb{B}}(x)=\lambda_j$. Then $x\in B_{V_j}$ and $x\notin B_{V_{j-1}}$ i.e. $x\in V^{[\lambda_j,\xi_j]}$ and $x\notin V^{[\lambda_{j-1},\xi_{j-1}]}$. Thus $\mu_V(x)\geq\lambda_j$ and $\nu_V(x)\leq\xi_j$. If $\mu_V(x)>\lambda_j$, then $\mu_V(x)=\lambda_l$ for some $l<j$. Then $x\in V^{[\lambda_l,\xi_l]}$ and $\mu_{(B)}(x)=\lambda_l$, which is a contradiction. Therefore $\mu_V(x)=\lambda_j$. Then $\nu_V(x)=\xi_j$ i.e. $\nu_{\mathbb{B}}(x)=\xi_j$. Therefore $\mathbb{B}$ is an intuitionistic fuzzy set and it is called intuitionistic fuzzy basis of V corresponding to $(*)$.

Proposition 2.21. *[9] Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then*

(1) If $(\zeta,\rho),(\eta,\sigma)\in(\lambda_{i+1},\lambda_i]\times[\xi_i,\xi_{i+1})$ with $\zeta+\rho\leq 1,\eta+\sigma\leq 1$, then $\mathbb{B}^{[\zeta,\rho]}=\mathbb{B}^{[\eta,\sigma]}=B_{V_i}$.

(2) If $(\zeta,\rho)\in(\lambda_{i+1},\lambda_i]\times[\xi_i,\xi_{i+1}),(\eta,\sigma)\in(\lambda_i,\lambda_{i-1}]\times[\xi_{i-1},\xi_i)$ with $\zeta+\rho\leq 1,\eta+\sigma\leq 1$, then $\mathbb{B}^{[\zeta,\rho]}\supsetneqq\mathbb{B}^{[\eta,\sigma]}$.

(3) $\mathbb{B}^{[\lambda,\xi]}$ is a basis of $V^{[\lambda,\xi]}$ for $\lambda\in(0,1],\xi\in[0,1)$ with $\lambda+\xi\leq 1$.

Proposition 2.22. *Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then $\mu_{\mathbb{B}}^{[\lambda_i]}=B_{V_i}=\nu_{\mathbb{B}}^{[\xi_i]}$, for $i=0,1,2,..,k$.*

Proof. Let $x\in\mu_{\mathbb{B}}^{[\lambda_i]}\Rightarrow\mu_{\mathbb{B}}(x)\geq\lambda_i$. Let $\mu_{\mathbb{B}}(x)=\lambda_j\Rightarrow x\in B_{V_j}\subset B_{V_i}$.
Thus $\mu_{\mathbb{B}}^{[\lambda_i]}\subseteq B_{V_i}$. Conversely, let $x\in B_{V_i}\Rightarrow\mu_V(x)\geq\lambda_i$.
Let $\mu_V(x)=\lambda_j$. If $\lambda_j>\lambda_i$, then $\mu_{\mathbb{B}}(x)=\lambda_j$.
If $\lambda_j=\lambda_i$, then $\mu_{\mathbb{B}}(x)\geq\lambda_i$. Therefore, in any case $x\in\mu_{\mathbb{B}}^{[\lambda_i]}$.
Thus $B_{V_i}\subseteq\mu_{\mathbb{B}}^{[\lambda_i]}$. Hence $\mu_{\mathbb{B}}^{[\lambda_i]}=B_{V_i}$.
Similarly, it can be proved that $B_{V_i}=\nu_{\mathbb{B}}^{[\xi_i]}$. □

Proposition 2.23. *Let $V\in IFVS(X)$ with $dim X=m$ and $Im(V)=\{(\lambda_0,\xi_0),(\lambda_1,\xi_1),...(\lambda_k,\xi_k)\},k\leq m$ such that $(1,0)\geq(\lambda_0,\xi_0)>(\lambda_1,\xi_1)>...>(\lambda_k,\xi_k)\geq(0,1)$. Then for $i=0,1,..,k$, $V^{[\lambda_i,\xi_i]}=\mu_V^{[\lambda_i]}=\nu_V^{[\xi_i]}$.*

Proof. Obviously, $V^{[\lambda_i,\xi_i]}\subseteq\mu_V^{[\lambda_i]}$.
Let $x\in\mu_V^{[\lambda_i]}$.
$\Rightarrow\mu_V(x)\geq\lambda_i$.
Let $\mu_V(x)=\lambda_j$. Then $\nu_V(x)=\xi_j$.
$\Rightarrow x\in V^{[\lambda_j,\xi_j]}$
$\Rightarrow x\in V^{[\lambda_i,\xi_i]}$ [as either $(\lambda_j,\xi_j)=(\lambda_i,\xi_i)$ or $(\lambda_j,\xi_j)>(\lambda_i,\xi_i)$].
Thus $\mu_{\mathbb{B}}^{[\lambda_i]}\subseteq V^{[\lambda_i,\xi_i]}$. Therefore $V^{[\lambda_i,\xi_i]}=\mu_V^{[\lambda_i]}$.
Similarly we have $V^{[\lambda_i,\xi_i]}=\nu_V^{[\xi_i]}$. □

Proposition 2.24. *Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then $|\mu_{\mathbb{B}}^{[\lambda_i]}|=dim(\mu_V^{[\lambda_i]})$ and $|\nu_{\mathbb{B}}^{[\xi_i]}|=dim(\nu_V^{[\xi_i]})$, for $i=0,1,2,..,k$.*

Proof. $|\mu_{\mathbb{B}}^{[\lambda_i]}|=|B_{V_i}|=dim(V^{[\lambda_i,\xi_i]})=dim(\mu_V^{[\lambda_i]})$ [By Proposition 2.22 and 2.23].
The rest part is similar. □

3 Intuitionistic fuzzy dimension

Definition 3.1. *Let A be an intuitionistic fuzzy set over X. Define a map $\mid A \mid: \mathbb{N} \to [0,1] \times [0,1]$ such that $\forall\, n \in \mathbb{N}$, $\mu_{|A|}(n) = \vee\{a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ with $a+b \leq 1$ and $\mid A^{[a,b]} \mid \geq n\}$ and $\nu_{|A|}(n) = \wedge\{b : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ with $a+b \leq 1$ and $\mid A^{[a,b]} \mid \geq n\}$. Then $\mid A \mid$ is an intuitionistic fuzzy set over $\mathbb{N}$, which is called the cardinality of A.*

Definition 3.2. *For two IFS A,B over X, the addition $\mid A \mid + \mid B \mid$ of $\mid A \mid$ and $\mid B \mid$ is defined as follows: for any $n \in \mathbb{N}$, $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l))$ and $\nu_{(|A|+|B|)}(n) = \wedge_{k+l=n}(\nu_{|A|}(k) \vee \nu_{|B|}(l))$.*

Proposition 3.3. *For two IFS $\mid A \mid, \mid B \mid$ over $\mathbb{N}$ and for any $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{(|A|+|B|)} = \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$ and $\nu^{[b]}_{(|A|+|B|)} = \nu^{[b]}_{|A|} + \nu^{[b]}_{|B|}$.*

Proof. First we prove that $\mu^{[a]}_{(|A|+|B|)} \subseteq \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$.
Let $n \in \mu^{[a]}_{(|A|+|B|)}$. Then $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$.
Hence there exist k,l such that $n = k+l$ and $\mu_{|A|}(k) \wedge \mu_{|B|}(l) \geq a$. Then $k \in \mu^{[a]}_{|A|}$ and $l \in \mu^{[a]}_{|B|}$, i.e., $n = k+l \in \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$. Similarly, it can be proved that $\nu^{[b]}_{(|A|+|B|)} \subseteq \nu^{[a]}_{|A|} + \nu^{[b]}_{|B|}$.
Conversely suppose that $n \in \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$.
Then there exist k,l such that $n = k+l$ with $k \in \mu^{[a]}_{|A|}, l \in \mu^{[a]}_{|B|}$. Then $(\mu_{|A|})(k) \geq a, (\mu_{|B|})(l) \geq a$.
Therefore $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$. Thus $n \in \mu^{[a]}_{(|A|+|B|)}$.
Hence $\mu^{[a]}_{|A|} + \mu^{[a]}_{|B|} \subseteq \mu^{[a]}_{(|A|+|B|)}$.
Similarly, we have $\nu^{[a]}_{|A|} + \nu^{[b]}_{|B|} \subseteq \nu^{[b]}_{(|A|+|B|)}$. Hence proved. □

Definition 3.4. *Let $V \in IFVS(X)$ with an intuitionistic fuzzy basis $\mathbb{B}$. Define $dim(V) = \mid \mathbb{B} \mid$. Then $dim(V)$ is called intuitionistic fuzzy dimension of V.*

Proposition 3.5. *Let $\mathbb{B}$ and $\mathbb{B}'$ be two intuitionistic fuzzy bases of an intuitionistic fuzzy vector space $V \in IFVS(X)$. Then $\mid \mathbb{B} \mid = \mid \mathbb{B}' \mid$.*

Proof. By Proposition 2.21, $\mathbb{B}^{[a,b]}$ and $\mathbb{B}'^{[a,b]}$ are bases of $V^{[a,b]}$ for $a \in (0,1], b \in [0,1)$ with $a+b \leq 1$. Then $\mid \mathbb{B}^{[a,b]} \mid = \mid \mathbb{B}'^{[a,b]} \mid$.
Hence for any $n \in \mathbb{N}$,
$\mu_{|\mathbb{B}|}(n) = \vee\{a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ *with* $a+b \leq 1$ *and* $\mid \mathbb{B}^{[a,b]} \mid \geq n\}$
$= \vee\{a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ *with* $a+b \leq 1$ *and* $\mid \mathbb{B}'^{[a,b]} \mid \geq n\}$
$= \mu_{|\mathbb{B}'|}(n)$. Similarly, for any $n \in \mathbb{N}$, $\nu_{|\mathbb{B}|}(n) = \nu_{|\mathbb{B}'|}(n)$. Hence proved. □

Remark 3.6. *Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space is independent of intuitionistic fuzzy basis.*

Proposition 3.7. *Let X be a vector space with $dimX = m$ and $V \in IFVS(X)$. Then for any $(a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ with $a+b \leq 1$ and $n \in \mathbb{N}$, $n \in \mu^{[a]}_{dim(V)} \Leftrightarrow n \leq dim(\mu^{[a]}_V)$ and $n \in \nu^{[b]}_{dim(V)} \Leftrightarrow n \leq dim(\nu^{[b]}_V)$.*

Proof. Suppose that $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), ...(\lambda_k, \xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > ... > (\lambda_k, \xi_k) \geq (0,1)$. Then there exists a nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0,\xi_0]} \subsetneqq V^{[\lambda_1,\xi_1]} \subsetneqq ... \subsetneqq V^{[\lambda_k,\xi_k]} = X$.
Let B_{V_i} be the basis of $V^{[\lambda_i,\xi_i]}, i = 0,1,..,k$ such that $B_{V_0} \subsetneqq B_{V_1} \subsetneqq .... \subsetneqq B_{V_k}$.......(*).
Let $\mathbb{B}$ be an intuitionistic fuzzy basis corresponding to (*) defined as in Definition 2.20. Let $n \in \mu^{[a]}_{dim(V)} \Rightarrow \mu_{dim(V)}(n) \geq a \Rightarrow \vee\{c : (c,d) \in (0,1] \times [0,1) \; with \; c+d \leq 1 \; and \; | \mathbb{B}^{[c,d]} | \geq n\} \geq a$.
Then there exists $(c,d) \in [0,1] \times [0,1] \setminus \{(0,1)\} \; with \; c+d \leq 1$ such that $c \geq a$ and $| \mathbb{B}^{[c,d]} | \geq n$.
Now $dim(\mu^{[a]}_V) = | \mu^{[a]}_{\mathbb{B}} | \geq | \mu^{[c]}_{\mathbb{B}} | \geq | \mathbb{B}^{[c,d]} | \geq n$.
Conversely suppose that $n \leq dim(\mu^{[a]}_V) = | \mu^{[a]}_{\mathbb{B}} |$. Now $a \in (\lambda_{i+1}, \lambda_i]$, for some i. Hence $| \mu^{[a]}_{\mathbb{B}} | = | \mu^{[\lambda_i]}_{\mathbb{B}} | = | B_{V_i} | = | \mathbb{B}^{[\lambda_i,\xi_i]} |$. Then $\mu_{dim(V)}(n) = \vee\{c : (c,d) \in [0,1] \times [0,1] \setminus \{(0,1)\} \; with \; c+d \leq 1 \; and \; | \mathbb{B}^{[c,d]} | \geq n\} \geq \lambda_i \geq a \Rightarrow n \in \mu^{[a]}_{dim(V)}$. Hence $n \in \mu^{[a]}_{(dim(V))} \Leftrightarrow n \leq dim(\mu^{[a]}_V)$.

Similarly it can be proved that $n \in \nu^{[b]}_{dim(V)} \Leftrightarrow n \leq dim(\nu^{[b]}_V)$. □

Proposition 3.8. *Let X be a vector space with $dimX = m$ and $V_1, V_2 \in IFVS(X)$. Then we have the following results:*

(1) For all $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{V_1 \cap V_2} = \mu^{[a]}_{V_1} \cap \mu^{[a]}_{V_2}$ and $\nu^{[b]}_{V_1 \cap V_2} = \nu^{[b]}_{V_1} \cap \nu^{[b]}_{V_2}$.

(2) For all $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{(V_1+V_2)} = \mu^{[a]}_{V_1} + \mu^{[a]}_{V_2}$ and $\nu^{[b]}_{(V_1+V_2)} = \nu^{[b]}_{V_1} + \nu^{[b]}_{V_2}$.

Proof. We only give the proof of (2). For any $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, we have
$x \in \mu^{[a]}_{(V_1+V_2)} \Leftrightarrow \underset{x=x_1+x_2}{sup} \{\mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2)\} \geq a$
$\Leftrightarrow$ there exist x_1, x_2 such that $x_1 + x_2 = x$ and $\mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2) \geq a$
$\Leftrightarrow$ there exist x_1, x_2 such that $x_1 + x_2 = x$ and $x_1 \in \mu^{[a]}_{V_1}$ and $x_2 \in \mu^{[a]}_{V_2}$.
Similarly it can be proved that $\nu^{[b]}_{(V_1+V_2)} = \nu^{[b]}_{V_1} + \nu^{[b]}_{V_2}$. □

Proposition 3.9. *Let X be a vector space with $dimX = m$ and $V_1, V_2 \in IFVS(X)$. Then $dim(V_1+V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$.*

Proof. For any $(a,b) \in [0,1] \times [0,1] \; with \; a+b \leq 1$, let $n \in \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Then there exist k,l such that $n = k+l$ and $k \in \mu^{[a]}_{(dim(V_1+V_2))}$ and $l \in \mu^{[a]}_{(dim(V_1 \cap V_2))}$. Then by Proposition 3.7, $k \leq dim(\mu^{[a]}_{(V_1+V_2)}) = dim(\mu^{[a]}_{(V_1)} + \mu^{[a]}_{(V_2)})$ and $l \leq dim(\mu^{[a]}_{(V_1 \cap V_2)}) = dim(\mu^{[a]}_{(V_1)} \cap \mu^{[a]}_{(V_2)})$. Thus $n \leq dim(\mu^{[a]}_{(V_1)} + \mu^{[a]}_{(V_2)}) + dim(\mu^{[a]}_{(V_1)} \cap \mu^{[a]}_{(V_2)}) = dim(\mu^{[a]}_{(V_1)}) + dim(\mu^{[a]}_{(V_2)})$.
Then there exist k' and l' such that $n = k'+l'$ and $k' \leq dim(\mu^{[a]}_{(V_1)})$ and $l' \leq dim(\mu^{[a]}_{(V_2)})$. Now by Proposition 3.7, $k' \in \mu^{[a]}_{dim(V_1)}$ and $l' \in \mu^{[a]}_{dim(V_2)}$. Therefore $n = k'+l' \in \mu^{[a]}_{dim(V_1)} + \mu^{[a]}_{dim(V_2)} = \mu^{[a]}_{(dim(V_1)+dim(V_2))}$. Hence $\mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))} \subseteq \mu^{[a]}_{(dim(V_1)+dim(V_2))}$.
Similarly, $\nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))} \subseteq \nu^{[b]}_{(dim(V_1)+dim(V_2))}$.
Also, it can be proved that for any $(a,b) \in [0,1] \times [0,1] \; with \; a+b \leq 1$, $\mu^{[a]}_{(dim(V_1)+(dim(V_2)} \subseteq \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$ and $\nu^{[b]}_{(dim(V_1)+(dim(V_2)} \subseteq \nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Thus for any $(a,b) \in [0,1] \times [0,1] \; with \; a+b \leq 1$, $\mu^{[a]}_{(dim(V_1)+(dim(V_2)} = \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$ and $\nu^{[b]}_{(dim(V_1)+(dim(V_2)} = \nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Hence $dim(V_1+V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$. □

Acknowledgements

The authors express their sincere gratitude to the anonymous referees for their valuable and constructive suggestions which have improved the presentation.

The research of the first author is supported by UGC (University Grants Commission), India under Junior Research Fellowship in Science, Humanities and Social Sciences. The research of the second author is partially supported by the Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F 510/3/DRS-III/2015 (SAP-I)].

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