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Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space

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Abstract: In the present paper the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space has been developed with the help of intuitionistic fuzzy basis.
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1 Introduction

The notion of intuitionistic fuzzy set (IFS) was introduced by Atanassov [1, 2, 3, 4] as a generalization of Zadeh's fuzzy set [22]. There are situations where IFS theory is more appropriate to dealt with [7]. IFS theory have successfully been applied in knowledge engineering, medical diagnosis, decision making, career determination, etc., [11, 21, 12]. Several researchers have extended various mathematical aspects such as groups, rings, topological spaces, metric spaces, topological groups, topological vector spaces etc. in IFS [6, 10, 13, 16, 17, 18, 19]. The notion of fuzzy vector subspaces has been introduced by Katsaras [14] and a notion of fuzzy bases and fuzzy dimension was studied by Shi *et al.* [20]. We have introduced a notion of intuitionistic fuzzy vector space and intuitionistic fuzzy basis in [9]. As a continuation of our paper [9], here we introduced the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space with the help of intuitionistic fuzzy basis and studied some of its basic results.

2 Preliminaries

Definition 2.1. *[1] Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X is defined as an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.*

In this paper, we use the symbols $a \wedge b = min\{a,b\}$ and $a \vee b = max\{a,b\}$.

Definition 2.2. *[1] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a set X. Then*

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$

(4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

(5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

(6) $\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in X\}$, $\Diamond A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3. *[4] Let A be an IFS in a set X. Then for $\lambda, \xi \in [0,1]$ with $\lambda + \xi \leq 1$, the set $A^{[\lambda,\xi]} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \xi\}$ is called (λ, ξ)-level subset of A.*

Proposition 2.4. *[4] Let A be an IFS in a set X and $(\lambda_1, \xi_1), (\lambda_2, \xi_2) \in Im(A)$. If $\lambda_1 \leq \lambda_2$ and $\xi_1 \geq \xi_2$, then $A^{[\lambda_1,\xi_1]} \supseteq A^{[\lambda_2,\xi_2]}$.*

Definition 2.5. *[15, 5] Let X be a vector space over the field K, the field of real and complex numbers, $\alpha \in K$, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of X. Then*

(1) the sum of A and B is defined to be the intuitionistic fuzzy set $A + B = (\mu_A + \mu_B, \nu_A + \nu_B)$ of X given by

$$\mu_{A+B}(x) = \begin{cases} \sup\limits_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} & if\ x = a+b \\ 0 & otherwise, \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \inf\limits_{x=a+b} \{\nu_A(a) \vee \nu_B(b)\} & if\ x = a+b \\ 1 & otherwise. \end{cases}$$

(2) αA is defined to be the IFS $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ of X, where

$$\mu_{\alpha A}(x) = \begin{cases} \mu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \sup\limits_{y \in X} \mu_A(y) & if\ \alpha = 0, x = \theta \\ 0 & if\ \alpha = 0, x \neq \theta, \end{cases}$$

$$\nu_{\alpha A}(x)=\begin{cases}\nu_A(\alpha^{-1}x) & if\ \alpha\neq 0\\ \underset{y\in X}{inf}\,\nu_A(y) & if\ \alpha=0, x=\theta\\ 1 & if\ \alpha=0, x\neq\theta.\end{cases}$$

.

Proposition 2.6. *[9] Let $A, A_1, \ldots, A_n$ be intuitionistic fuzzy sets in a vector space X and $\lambda_1, \ldots, \lambda_n$ be scalars. Then the following assertions are equivalent:*

(1) $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n \subseteq A$.

(2) For all $x_1, x_2, \ldots, x_n$ in X, we have $\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \geq min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}$ *and* $\nu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots, \nu_{A_n}(x_n)\}$.

Definition 2.7. *[9] An IFS $V = (\mu_V, \nu_V)$ of a vector space X over the field K is said to be intuitionistic fuzzy vector space over X if*

(i) $V + V \subseteq V$

(ii) $\alpha V \subseteq V$, *for every scalar* α.

We denote the set of all intuitionistic fuzzy vector spaces over a vector space X by $IFVS(X)$.

Remark 2.8. *[9] Let X be a vector space.*

(1) If μ_V is a fuzzy subspace of X, then $V = (\mu_V, \mu_V^c) \in IFVS(X)$.

(2) If $V \in IFVS(X)$, then μ_V and ν_V^c are fuzzy vector subspace of X.

(3) If $V \in IFVS(X)$, then $\Box V, \Diamond V \in IFVS(X)$.

Lemma 2.9. *[9] Let V be an intuitionistic fuzzy set in a vector space X. Then, the following are equivalent:*

(1) V is an intuitionistic fuzzy vector space over X.

(2) For all scalars α, β, we have $\alpha V + \beta V \subseteq V$.

(3) For all scalars α, β and for all $x, y \in X$, we have $\mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y)\}$ *and* $\nu_V(\alpha x + \beta y) \leq \nu_V(x) \vee \nu_V(y)$.

Remark 2.10. *[9] Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [19] and [8].*

Proposition 2.11. *[8] If $V, W \in IFVS(X)$, then $V + W \in IFVS(X)$.*

Proposition 2.12. *[9] If $V \in IFVS(X)$ $\alpha \in K$, then $\alpha V \in IFVS(X)$.*

Proposition 2.13. *[8] If $\{V_i\}_{i\in I} \in IFVS(X)$, then $\underset{i\in I}{\cap} V_i \in IFVS(X)$.*

Proposition 2.14. *[9] Let $V \in IFVS(X)$. Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x)$, $\forall x \in X$.*

Proposition 2.15. *[9] Let $V \in IFVS(X)$. Then for each $(\lambda, \xi) \in [0,1] \times [0,1]$ with $\lambda + \xi \leq 1$, $\lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$, $V^{[\lambda,\xi]}$ is a subspace of the vector space X,*

Definition 2.16. *[9] For any $(a,b),(c,d) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $c+d \leq 1$, we say that:*

(1) $(a,b) \geq (c,d)$ if $a \geq b$ and $c \leq d$.

(2) $(a,b) \leq (c,d)$ if $a \leq b$ and $c \geq d$.

(3) $(a,b) > (c,d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.

(4) $(a,b) < (c,d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.

(5) $(a,b) = (c,d)$ if $a = b$ and $c = d$.

Proposition 2.17. *[9] Let $V \in IFVS(X)$ with $\dim X = m$. Then $Im(V)$ contains at most $m+1$ points of $[0,1] \times [0,1]$.*

Definition 2.18. *[9] Let $V = (\mu_V, \nu_V) \in IFVS(X)$. Then for any $\lambda \in \mu_V(X), \xi \in \nu_V(X)$ we define $\mu_V^{[\lambda]} = \{x \in X : \mu_V(x) \geq \lambda\}$ and $\nu_V^{[\xi]} = \{x \in X : \nu_V(x) \leq \xi\}$, $[\lambda 1_{\mu_V^{[\lambda]}}](x) = \begin{cases} \lambda, & if\ x \in \mu_V^{[\lambda]} \\ 0, & otherwise \end{cases}$,*

$$[\xi 1_{\nu_V^{[\xi]}}](x) = \begin{cases} \xi, & if\ x \in \nu_V^{[\xi]} \\ 1, & otherwise \end{cases}.$$

Theorem 2.19. *[9] (Representation Theorem) Let $V \in IFVS(X)$ with $\dim X = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), ...(\lambda_k, \xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > ... > (\lambda_k, \xi_k) \geq (0,1)$. Then there exists nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0,\xi_0]} \subsetneqq V^{[\lambda_1,\xi_1]} \subsetneqq ... \subsetneqq V^{[\lambda_k,\xi_k]} = X$ such that $\mu_V = \lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee ... \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}$ and $\nu_V = \xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge ... \wedge \xi_k 1_{\nu_V^{[\xi_k]}}$. Also,*

(1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1})$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta,\rho]} = V^{[\eta,\sigma]} = V^{[\lambda_i,\xi_i]}$.

(2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1}), (\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta,\rho]} \supsetneqq V^{[\eta,\sigma]}$.

Definition 2.20. *[9] Let $V \in IFVS(X)$ with $\dim X = m$. Consider Theorem 2.19. Let B_{V_i} be the basis of $V^{[\lambda_i,\xi_i]}, i = 0,1,..,k$ such that*

$$B_{V_0} \subsetneqq B_{V_1} \subsetneqq \cdots \subsetneqq B_{V_k}. \qquad (*)$$

If $V^{(\lambda_0,\xi_0)} = \{\theta\}$, we start with $V^{(\lambda_1,\xi_1)}$.
Define a map $\mathbb{B}$ from X to $[0,1] \times [0,1]$ by

$$\mu_{\mathbb{B}}(x) = \begin{cases} \vee\{\lambda_i : x \in B_{V_i}\} \\ 0,\ otherwise \end{cases} \quad and\ \nu_{\mathbb{B}}(x) = \begin{cases} \wedge\{\xi_i : x \in B_{V_i}\} \\ 1,\ otherwise \end{cases}.$$

Let $\mu_{\mathbb{B}}(x) = \lambda_j$. Then $x \in B_{V_j}$ and $x \notin B_{V_{j-1}}$ i.e. $x \in V^{[\lambda_j,\xi_j]}$ and $x \notin V^{[\lambda_{j-1},\xi_{j-1}]}$. Thus $\mu_V(x) \geq \lambda_j$ and $\nu_V(x) \leq \xi_j$. If $\mu_V(x) > \lambda_j$, then $\mu_V(x) = \lambda_l$ for some $l < j$. Then $x \in V^{[\lambda_l,\xi_l]}$ and $\mu_{(B)}(x) = \lambda_l$, which is a contradiction. Therefore $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$ i.e. $\nu_{\mathbb{B}}(x) = \xi_j$. Therefore $\mathbb{B}$ is an intuitionistic fuzzy set and it is called intuitionistic fuzzy basis of V corresponding to $(*)$.

Proposition 2.21. *[9] Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then*

(1) If $(\zeta,\rho),(\eta,\sigma) \in (\lambda_{i+1},\lambda_i] \times [\xi_i,\xi_{i+1})$ with $\zeta+\rho \leq 1, \eta+\sigma \leq 1$, then $\mathbb{B}^{[\zeta,\rho]} = \mathbb{B}^{[\eta,\sigma]} = B_{V_i}$.

(2) If $(\zeta,\rho) \in (\lambda_{i+1},\lambda_i] \times [\xi_i,\xi_{i+1}), (\eta,\sigma) \in (\lambda_i,\lambda_{i-1}] \times [\xi_{i-1},\xi_i)$ with $\zeta+\rho \leq 1, \eta+\sigma \leq 1$, then $\mathbb{B}^{[\zeta,\rho]} \supsetneqq \mathbb{B}^{[\eta,\sigma]}$.

(3) $\mathbb{B}^{[\lambda,\xi]}$ is a basis of $V^{[\lambda,\xi]}$ for $\lambda \in (0,1], \xi \in [0,1)$ with $\lambda+\xi \leq 1$.

Proposition 2.22. *Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then $\mu_{\mathbb{B}}^{[\lambda_i]} = B_{V_i} = \nu_{\mathbb{B}}^{[\xi_i]}$, for $i = 0,1,2,..,k$.*

Proof. Let $x \in \mu_{\mathbb{B}}^{[\lambda_i]} \Rightarrow \mu_{\mathbb{B}}(x) \geq \lambda_i$. Let $\mu_{\mathbb{B}}(x) = \lambda_j \Rightarrow x \in B_{V_j} \subset B_{V_i}$.
Thus $\mu_{\mathbb{B}}^{[\lambda_i]} \subseteq B_{V_i}$. Conversely, let $x \in B_{V_i} \Rightarrow \mu_V(x) \geq \lambda_i$.
Let $\mu_V(x) = \lambda_j$. If $\lambda_j > \lambda_i$, then $\mu_{\mathbb{B}}(x) = \lambda_j$.
If $\lambda_j = \lambda_i$, then $\mu_{\mathbb{B}}(x) \geq \lambda_i$. Therefore, in any case $x \in \mu_{\mathbb{B}}^{[\lambda_i]}$.
Thus $B_{V_i} \subseteq \mu_{\mathbb{B}}^{[\lambda_i]}$. Hence $\mu_{\mathbb{B}}^{[\lambda_i]} = B_{V_i}$.
Similarly, it can be proved that $B_{V_i} = \nu_{\mathbb{B}}^{[\xi_i]}$. □

Proposition 2.23. *Let $V \in IFVS(X)$ with $dim X = m$ and $Im(V) = \{(\lambda_0,\xi_0),(\lambda_1,\xi_1),...(\lambda_k,\xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0,\xi_0) > (\lambda_1,\xi_1) > ... > (\lambda_k,\xi_k) \geq (0,1)$. Then for $i = 0,1,..,k$, $V^{[\lambda_i,\xi_i]} = \mu_V^{[\lambda_i]} = \nu_V^{[\xi_i]}$.*

Proof. Obviously, $V^{[\lambda_i,\xi_i]} \subseteq \mu_V^{[\lambda_i]}$.
Let $x \in \mu_V^{[\lambda_i]}$.
$\Rightarrow \mu_V(x) \geq \lambda_i$.
Let $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$.
$\Rightarrow x \in V^{[\lambda_j,\xi_j]}$
$\Rightarrow x \in V^{[\lambda_i,\xi_i]}$ [as either $(\lambda_j,\xi_j) = (\lambda_i,\xi_i)$ or $(\lambda_j,\xi_j) > (\lambda_i,\xi_i)$].
Thus $\mu_{\mathbb{B}}^{[\lambda_i]} \subseteq V^{[\lambda_i,\xi_i]}$. Therefore $V^{[\lambda_i,\xi_i]} = \mu_V^{[\lambda_i]}$.
Similarly we have $V^{[\lambda_i,\xi_i]} = \nu_V^{[\xi_i]}$. □

Proposition 2.24. *Let $\mathbb{B}$ be an intuitionistic fuzzy basis of V corresponding to $(*)$ of Definition* 2.20. *Then $|\ \mu_{\mathbb{B}}^{[\lambda_i]}\ | = dim(\mu_V^{[\lambda_i]})$ and $|\ \nu_{\mathbb{B}}^{[\xi_i]}\ | = dim(\nu_V^{[\xi_i]})$, for $i = 0,1,2,..,k$.*

Proof. $|\ \mu_{\mathbb{B}}^{[\lambda_i]}\ | = |\ B_{V_i}\ | = dim(V^{[\lambda_i,\xi_i]}) = dim(\mu_V^{[\lambda_i]})$ [By Proposition 2.22 and 2.23].
The rest part is similar. □

3 Intuitionistic fuzzy dimension

Definition 3.1. *Let A be an intuitionistic fuzzy set over X. Define a map $|A|: \mathbb{N} \to [0,1]\times[0,1]$ such that $\forall\, n \in \mathbb{N}$, $\mu_{|A|}(n) = \vee\{a : (a,b) \in [0,1]\times[0,1] \setminus \{(0,1)\}$ with $a+b \leq 1$ and $|A^{[a,b]}| \geq n\}$ and $\nu_{|A|}(n) = \wedge\{b : (a,b) \in [0,1]\times[0,1]\setminus\{(0,1)\}$ with $a+b \leq 1$ and $|A^{[a,b]}| \geq n\}$. Then $|A|$ is an intuitionistic fuzzy set over $\mathbb{N}$, which is called the cardinality of A.*

Definition 3.2. *For two IFS A,B over X, the addition $|A|+|B|$ of $|A|$ and $|B|$ is defined as follows: for any $n \in \mathbb{N}$, $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l))$ and $\nu_{(|A|+|B|)}(n) = \wedge_{k+l=n}(\nu_{|A|}(k) \vee \nu_{|B|}(l))$.*

Proposition 3.3. *For two IFS $|A|, |B|$ over $\mathbb{N}$ and for any $(a,b) \in [0,1]\times[0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{(|A|+|B|)} = \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$ and $\nu^{[b]}_{(|A|+|B|)} = \nu^{[b]}_{|A|} + \nu^{[b]}_{|B|}$.*

Proof. First we prove that $\mu^{[a]}_{(|A|+|B|)} \subseteq \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$.
Let $n \in \mu^{[a]}_{(|A|+|B|)}$. Then $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$.
Hence there exist k,l such that $n = k+l$ and $\mu_{|A|}(k) \wedge \mu_{|B|}(l) \geq a$. Then $k \in \mu^{[a]}_{|A|}$ and $l \in \mu^{[a]}_{|B|}$, i.e., $n = k+l \in \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$. Similarly, it can be proved that $\nu^{[b]}_{(|A|+|B|)} \subseteq \nu^{[a]}_{|A|} + \nu^{[b]}_{|B|}$.
Conversely suppose that $n \in \mu^{[a]}_{|A|} + \mu^{[a]}_{|B|}$.
Then there exist k,l such that $n = k+l$ with $k \in \mu^{[a]}_{|A|}, l \in \mu^{[a]}_{|B|}$. Then $(\mu_{|A|})(k) \geq a, (\mu_{|B|})(l) \geq a$.
Therefore $\mu_{(|A|+|B|)}(n) = \vee_{k+l=n}(\mu_{|A|}(k) \wedge \mu_{|B|}(l)) \geq a$. Thus $n \in \mu^{[a]}_{(|A|+|B|)}$.
Hence $\mu^{[a]}_{|A|} + \mu^{[a]}_{|B|} \subseteq \mu^{[a]}_{(|A|+|B|)}$.
Similarly, we have $\nu^{[a]}_{|A|} + \nu^{[b]}_{|B|} \subseteq \nu^{[b]}_{(|A|+|B|)}$. Hence proved. □

Definition 3.4. *Let $V \in IFVS(X)$ with an intuitionistic fuzzy basis $\mathbb{B}$. Define $dim(V) = |\mathbb{B}|$. Then $dim(V)$ is called intuitionistic fuzzy dimension of V.*

Proposition 3.5. *Let $\mathbb{B}$ and $\mathbb{B}'$ be two intuitionistic fuzzy bases of an intuitionistic fuzzy vector space $V \in IFVS(X)$. Then $|\mathbb{B}| = |\mathbb{B}'|$.*

Proof. By Proposition 2.21, $\mathbb{B}^{[a,b]}$ and $\mathbb{B}'^{[a,b]}$ are bases of $V^{[a,b]}$ for $a \in (0,1], b \in [0,1)$ with $a+b \leq 1$. Then $|\mathbb{B}^{[a,b]}| = |\mathbb{B}'^{[a,b]}|$.
Hence for any $n \in \mathbb{N}$,
$\mu_{|\mathbb{B}|}(n) = \vee\{a : (a,b) \in [0,1]\times[0,1]\setminus\{(0,1)\}$ *with* $a+b \leq 1$ *and* $|\mathbb{B}^{[a,b]}| \geq n\}$
$= \vee\{a : (a,b) \in [0,1]\times[0,1]\setminus\{(0,1)\}$ *with* $a+b \leq 1$ *and* $|\mathbb{B}'^{[a,b]}| \geq n\}$
$= \mu_{|\mathbb{B}'|}(n)$. Similarly, for any $n \in \mathbb{N}$, $\nu_{|\mathbb{B}|}(n) = \nu_{|\mathbb{B}'|}(n)$. Hence proved. □

Remark 3.6. *Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space is independent of intuitionistic fuzzy basis.*

Proposition 3.7. *Let X be a vector space with $dimX = m$ and $V \in IFVS(X)$. Then for any $(a,b) \in [0,1]\times[0,1]\setminus\{(0,1)\}$ with $a+b \leq 1$ and $n \in \mathbb{N}$, $n \in \mu^{[a]}_{dim(V)} \Leftrightarrow n \leq dim(\mu^{[a]}_V)$ and $n \in \nu^{[b]}_{dim(V)} \Leftrightarrow n \leq dim(\nu^{[b]}_V)$.*

Proof. Suppose that $Im(V) = \{(\lambda_0,\xi_0), (\lambda_1,\xi_1), ...(\lambda_k,\xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0,\xi_0) > (\lambda_1,\xi_1) > ... > (\lambda_k,\xi_k) \geq (0,1)$. Then there exists a nested collection of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0,\xi_0]} \subsetneqq V^{[\lambda_1,\xi_1]} \subsetneqq ... \subsetneqq V^{[\lambda_k,\xi_k]} = X$.
Let B_{V_i} be the basis of $V^{[\lambda_i,\xi_i]}, i = 0,1,..,k$ such that $B_{V_0} \subsetneqq B_{V_1} \subsetneqq .... \subsetneqq B_{V_k}$.......(*).
Let $\mathbb{B}$ be an intuitionistic fuzzy basis corresponding to (*) defined as in Definition 2.20. Let $n \in \mu^{[a]}_{dim(V)} \Rightarrow \mu_{dim(V)}(n) \geq a \Rightarrow \vee\{c : (c,d) \in (0,1] \times [0,1)\ with\ c+d \leq 1\ and\ |\mathbb{B}^{[c,d]}| \geq n\} \geq a$. Then there exists $(c,d) \in [0,1] \times [0,1] \setminus \{(0,1)\}\ with\ c+d \leq 1$ such that $c \geq a$ and $|\mathbb{B}^{[c,d]}| \geq n$. Now $dim(\mu^{[a]}_V) = |\mu^{[a]}_{\mathbb{B}}| \geq |\mu^{[c]}_{\mathbb{B}}| \geq |\mathbb{B}^{[c,d]}| \geq n$.
Conversely suppose that $n \leq dim(\mu^{[a]}_V) = |\mu^{[a]}_{\mathbb{B}}|$. Now $a \in (\lambda_{i+1},\lambda_i]$, for some i. Hence $|\mu^{[a]}_{\mathbb{B}}| = |\mu^{[\lambda_i]}_{\mathbb{B}}| = |B_{V_i}| = |\mathbb{B}^{[\lambda_i,\xi_i]}|$. Then $\mu_{dim(V)}(n) = \vee\{c : (c,d) \in [0,1] \times [0,1] \setminus \{(0,1)\}\ with\ c+d \leq 1\ and\ |\mathbb{B}^{[c,d]}| \geq n\} \geq \lambda_i \geq a \Rightarrow n \in \mu^{[a]}_{dim(V)}$. Hence $n \in \mu^{[a]}_{(dim(V))} \Leftrightarrow n \leq dim(\mu^{[a]}_V)$.

Similarly it can be proved that $n \in \nu^{[b]}_{dim(V)} \Leftrightarrow n \leq dim(\nu^{[b]}_V)$. □

Proposition 3.8. *Let X be a vector space with $dimX = m$ and $V_1,V_2 \in IFVS(X)$. Then we have the following results:*

(1) For all $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{V_1 \cap V_2} = \mu^{[a]}_{V_1} \cap \mu^{[a]}_{V_2}$ and $\nu^{[b]}_{V_1 \cap V_2} = \nu^{[b]}_{V_1} \cap \nu^{[b]}_{V_2}$.

(2) For all $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu^{[a]}_{(V_1+V_2)} = \mu^{[a]}_{V_1} + \mu^{[a]}_{V_2}$ and $\nu^{[b]}_{(V_1+V_2)} = \nu^{[b]}_{V_1} + \nu^{[b]}_{V_2}$.

Proof. We only give the proof of (2). For any $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, we have
$x \in \mu^{[a]}_{(V_1+V_2)} \Leftrightarrow \sup_{x=x_1+x_2} \{\mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2)\} \geq a$
$\Leftrightarrow$ there exist x_1,x_2 such that $x_1+x_2 = x$ and $\mu_{V_1}(x_1) \wedge \mu_{V_2}(x_2) \geq a$
$\Leftrightarrow$ there exist x_1,x_2 such that $x_1+x_2 = x$ and $x_1 \in \mu^{[a]}_{V_1}$ and $x_2 \in \mu^{[a]}_{V_2}$.
Similarly it can be proved that $\nu^{[b]}_{(V_1+V_2)} = \nu^{[b]}_{V_1} + \nu^{[b]}_{V_2}$. □

Proposition 3.9. *Let X be a vector space with $dimX = m$ and $V_1,V_2 \in IFVS(X)$. Then $dim(V_1+V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$.*

Proof. For any $(a,b) \in [0,1] \times [0,1]\ with\ a+b \leq 1$, let $n \in \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Then there exist k,l such that $n = k+l$ and $k \in \mu^{[a]}_{(dim(V_1+V_2))}$ and $l \in \mu^{[a]}_{(dim(V_1 \cap V_2))}$. Then by Proposition 3.7, $k \leq dim(\mu^{[a]}_{(V_1+V_2)}) = dim(\mu^{[a]}_{(V_1)} + \mu^{[a]}_{(V_2)})$ and $l \leq dim(\mu^{[a]}_{(V_1 \cap V_2)}) = dim(\mu^{[a]}_{(V_1)} \cap \mu^{[a]}_{(V_2)})$. Thus $n \leq dim(\mu^{[a]}_{(V_1)} + \mu^{[a]}_{(V_2)}) + dim(\mu^{[a]}_{(V_1)} \cap \mu^{[a]}_{(V_2)}) = dim(\mu^{[a]}_{(V_1)}) + dim(\mu^{[a]}_{(V_2)})$.
Then there exist k' and l' such that $n = k'+l'$ and $k' \leq dim(\mu^{[a]}_{(V_1)})$ and $l' \leq dim(\mu^{[a]}_{(V_2)})$. Now by Proposition 3.7, $k' \in \mu^{[a]}_{dim(V_1)}$ and $l' \in \mu^{[a]}_{dim(V_2)}$. Therefore $n = k'+l' \in \mu^{[a]}_{dim(V_1)} + \mu^{[a]}_{dim(V_2)} = \mu^{[a]}_{(dim(V_1)+dim(V_2))}$. Hence $\mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))} \subseteq \mu^{[a]}_{(dim(V_1)+dim(V_2))}$.
Similarly, $\nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))} \subseteq \nu^{[b]}_{(dim(V_1)+dim(V_2))}$.
Also, it can be proved that for any $(a,b) \in [0,1] \times [0,1]\ with\ a+b \leq 1$, $\mu^{[a]}_{(dim(V_1)+(dim(V_2)} \subseteq \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$ and $\nu^{[b]}_{(dim(V_1)+(dim(V_2)} \subseteq \nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Thus for any $(a,b) \in [0,1] \times [0,1]\ with\ a+b \leq 1$, $\mu^{[a]}_{(dim(V_1)+(dim(V_2)} = \mu^{[a]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$ and $\nu^{[b]}_{(dim(V_1)+(dim(V_2)} = \nu^{[b]}_{(dim(V_1+V_2)+dim(V_1 \cap V_2))}$. Hence $dim(V_1+V_2) + dim(V_1 \cap V_2) = dim(V_1) + dim(V_2)$. □

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