Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 3, 1–4

Properties of the intuitionistic fuzzy extended modal operators

Krassimir T. Atanassov
Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia–1113, Bulgaria,
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Burgas–8010, Bulgaria
e-mail: krat@bas.bg

Abstract: Some basic properties of the intuitionistic fuzzy extended modal operators are discussed.
Keywords: Intuitionistic fuzzy modal operator, Intuitionistic fuzzy operation.
AMS Classification: 03E72.

1 Introduction

In 1988, the first intuitionistic fuzzy extended modal operators D_α and $F_{\alpha,\beta}$ were introduced and some of their propertie were studied. In particular, it was proved that there are representations of the expressions $D_\alpha(D_\beta(A))$ and $F_{\alpha,\beta}(F_{\gamma,\delta}(A))$ in the form of D_ε and $F_{\zeta,\eta}$, respectively (see [1]). These properties were discussed also in [2], where new five operators ($G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}$) were introduced, but in that moment, the author had not received similar formulas for these operators. During the next years, some times he tried to find such representations without success and even now, some formulas are introduced for a first time.

2 Preliminary results

Let a set E be fixed. The Intuitionistic Fuzzy Set (IFS) A in E is defined by (see, e.g., [3, 4]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let $\alpha, \beta \in [0,1]$. We will define (see, e.g., [3, 4]) seven operators over a given IFS A by:

$$\begin{array}{ll} D_\alpha(A) & = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle | x \in E\}, \\ F_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle | x \in E\}, \\ & \text{where } \alpha + \beta \leq 1 \\ G_{\alpha,\beta}(A) & = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in E\}, \\ H_{\alpha,\beta}(A) & = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle | x \in E\}, \\ H^*_{\alpha,\beta}(A) & = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle | x \in E\}, \\ J_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x)\rangle | x \in E\}, \\ J^*_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle | x \in E\}, \end{array}$$

3 Main results

In [1] it is proved that for every IFS A and for every $\alpha, \beta \in [0,1]$:

$$D_\alpha(D_\beta(A)) = D_\beta(A),$$

$$F_{\alpha,\beta}(F_{\gamma,\delta}(A)) = F_{\alpha+\gamma-\alpha\gamma-\alpha\delta,\beta+\delta-\beta\gamma-\beta\delta}(A),$$

where $\alpha + \beta \leq 1$.

Here we formulate and prove two theorems.

Theorem 1. Let $\alpha, \beta, \gamma, \delta \in [0,1]$, such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and $\beta + \delta > 0$. Let A be an IFS. Then

(a) $H_{\alpha,\beta}(H_{\gamma,\delta}(A)) = G_{\frac{\beta+\delta-\beta\delta}{\beta\gamma+\delta-\beta\delta}.\alpha\gamma,1}(H^*_{\frac{\beta\gamma+\delta-\beta\delta}{\beta+\delta-\beta\delta},\beta+\delta-\beta\delta}(A)),$

(b) $H_{\alpha,\beta}(H^*_{\gamma,\delta}(A)) = G_{\alpha,1}(H^*_{\gamma,\beta+\delta-\beta\delta}(A)),$

(c) $H^*_{\alpha,\beta}(H_{\gamma,\delta}(A)) = G_{\frac{\beta+\delta-\beta\delta}{\alpha\beta\gamma+\delta-\beta\delta}.\alpha\gamma,1}(H^*_{\frac{\alpha\beta\gamma+\delta-\beta\delta}{\beta+\delta-\beta\delta},\beta+\delta-\beta\delta}(A)),$

(d) $H^*_{\alpha,\beta}(H^*_{\gamma,\delta}(A)) = G_{\frac{\beta+\delta-\beta\delta}{\alpha\beta+\delta-\beta\delta}.\alpha,1}(H^*_{\frac{\alpha\beta+\delta-\beta\delta}{\beta+\delta-\beta\delta}.\gamma,\beta+\delta-\beta\delta}(A)).$

Proof: Let A be an IFS. Let $\alpha, \beta, \gamma, \delta \in [0,1]$, such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and $\beta + \delta > 0$.

Directly it can be seen that $\beta\gamma + \delta - \beta\delta \leq \beta + \delta - \beta\delta$, i.e.

$$\frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta} \leq 1$$

and

$$\frac{\beta + \delta - \beta\delta}{\beta\gamma + \delta - \beta\delta}.\alpha\gamma \leq 1,$$

because

$$\beta\gamma + \delta - \beta\delta - \alpha\beta\gamma - \alpha\gamma\delta + \alpha\beta\gamma\delta \geq \delta - \beta\delta - \alpha\gamma\delta$$
$$= \delta(1 - \beta - \alpha\gamma) \geq \delta(1 - \beta - \alpha) \geq 0.$$

We prove statement (a):

$$H_{\alpha,\beta}(H_{\gamma,\delta}(A)) = H_{\alpha,\beta}(\{\langle x, \gamma.\mu_A(x), \nu_A(x) + \delta.\pi_A(x)\rangle | x \in E\})$$
$$= \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + \delta\pi_A(x) + \beta(1 - \gamma\mu_A(x) - \nu_A(x) - \delta\pi_A(x))\rangle | x \in E\}$$
$$= \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + \delta - \delta\mu_A(x) - \delta\nu_A(x) + \beta - \beta\gamma\mu_A(x) - \beta\nu_A(x) - \beta\delta$$
$$+\beta\delta\mu_A(x) + \beta\delta\nu_A(x))\rangle | x \in E\}$$
$$= \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + (\beta + \delta - \beta\delta) - (\delta + \beta\gamma - \beta\delta)\mu_A(x) - (\beta + \delta - \beta\delta)\nu_A(x)\rangle | x \in E\}$$
$$= \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + (\beta + \delta - \beta\delta)\left(1 - \frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta}\mu_A(x) - \nu_A(x)\right)\rangle | x \in E\}$$
$$= \{\langle x, \left(\frac{\beta + \delta - \beta\delta}{\beta\gamma + \delta - \beta\delta}.\alpha\gamma\right).\frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta}\mu_A(x),$$
$$\nu_A(x) + (\beta + \delta - \beta\delta)\left(1 - \frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta}\mu_A(x) - \nu_A(x)\right)\rangle | x \in E\}$$
$$= G_{\frac{\beta+\delta-\beta\delta}{\beta\gamma+\delta-\beta\delta}.\alpha\gamma,1}(\{\langle x, \frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta}\mu_A(x),$$
$$\nu_A(x) + (\beta + \delta - \beta\delta)\left(1 - \frac{\beta\gamma + \delta - \beta\delta}{\beta + \delta - \beta\delta}\mu_A(x) - \nu_A(x)\right)\rangle | x \in E\})$$
$$= G_{\frac{\beta+\delta-\beta\delta}{\beta\gamma+\delta-\beta\delta}.\alpha\gamma,1}(H^*_{\frac{\beta\gamma+\delta-\beta\delta}{\beta+\delta-\beta\delta},\beta+\delta-\beta\delta}(A)).$$

Statements (b) – (d) are proved by analogy. □

Theorem 2. Let $\alpha, \beta, \gamma, \delta \in [0, 1]$, such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and $\alpha + \gamma > 0$. Let A be an IFS. Then

(a) $J_{\alpha,\beta}(J_{\gamma,\delta}(A)) = G_{1,\frac{\alpha+\gamma-\alpha\gamma}{\alpha\delta+\gamma-\alpha\gamma}.\beta\delta}(J^*_{\alpha+\gamma-\alpha\gamma,\frac{\alpha\delta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}}(A))$,

(b) $J_{\alpha,\beta}(J^*_{\gamma,\delta}(A)) = G_{1,\beta}(J^*_{\alpha+\gamma-\alpha\gamma,\delta}(A))$,

(c) $J^*_{\alpha,\beta}(J_{\gamma,\delta}(A)) = G_{1,\frac{\alpha+\gamma-\alpha\gamma}{\alpha\beta\delta+\gamma-\alpha\gamma}.\beta\delta}(J^*_{\alpha+\gamma-\alpha\gamma,\frac{\alpha\beta\delta+\gamma-\alpha\delta}{\alpha+\gamma-\alpha\gamma}}(A))$,

(d) $J^*_{\alpha,\beta}(J^*_{\gamma,\delta}(A)) = G_{1,\frac{\alpha+\gamma-\alpha\gamma}{\alpha\beta+\gamma-\alpha\gamma}.\beta}(J^*_{\alpha+\gamma-\alpha\gamma,\frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta}(A))$.

Proof. Let A be an IFS. Let $\alpha, \beta, \gamma, \delta \in [0, 1]$, such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and $\alpha + \gamma > 0$.

First, we again check that

$$\frac{\alpha + \gamma - \alpha\gamma}{\alpha\beta + \gamma - \alpha\gamma}.\beta \leq 1$$

and

$$\frac{\alpha\beta + \gamma - \alpha\gamma}{\alpha + \gamma - \alpha\gamma}.\delta \leq 1.$$

Now, we prove statement (d), and (a) – (c) can be proved by analogy.

Let us proceed in the opposite direction.

$$G_{1,\frac{\alpha+\gamma-\alpha\gamma}{\alpha\beta+\gamma-\alpha\gamma}.\beta}(J^*_{\alpha+\gamma-\alpha\gamma,\frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta}(A))$$

$$= G_{1,\frac{\alpha+\gamma-\alpha\gamma}{\alpha\beta+\gamma-\alpha\gamma}.\beta}(\{\langle x, \mu_A(x) + (\alpha+\gamma-\alpha\gamma)(1-\mu_A(x) - \frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta\nu_A(x)),$$

$$\frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x) + (\alpha+\gamma-\alpha\gamma)(1-\mu_A(x) - \frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta\nu_A(x)),$$

$$\frac{\alpha+\gamma-\alpha\gamma}{\alpha\beta+\gamma-\alpha\gamma}.\beta.\frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x) + (\alpha+\gamma-\alpha\gamma)(1-\mu_A(x) - \frac{\alpha\beta+\gamma-\alpha\gamma}{\alpha+\gamma-\alpha\gamma}.\delta\nu_A(x)), \beta\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x) + (\alpha+\gamma-\alpha\gamma) - (\alpha+\gamma-\alpha\gamma)\mu_A(x) - (\alpha\beta+\gamma-\alpha\gamma).\delta\nu_A(x)), \beta\delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x) + \gamma(1-\mu_A(x) - \delta\nu_A(x)) + \alpha(1-\mu_A(x) - \gamma(1-\mu_A(x) - \delta\nu_A(x)) - \beta\delta\nu_A(x)),$$

$$\beta\delta\nu_A(x)\rangle | x \in E\}$$

$$= J^*_{\alpha,\beta}(\{\langle x, \mu_A(x) + \gamma(1-\mu_A(x) - \delta\nu_A(x)), \delta\nu_A(x)\rangle | x \in E\}$$

$$= J^*_{\alpha,\beta}(J^*_{\gamma,\delta}(A)).$$

This completes the proof. □

4 Conclusion

Letus finish with the following **Open Problem**: *Can we find other representations of the above discussed compositions of the operators $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$ and $J^*_{\alpha,\beta}$?*

References

[1] Atanassov K. (1988) Two operators on intuitionistic fuzzy sets, *Comptes Rendus de l'Academie bulgare des Sciences*, 41(5), 35–38.

[2] Atanassov, K. (1989) More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33(1), 37–46.

[3] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets*, Springer, Heidelberg.

[4] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.