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Implication-based intuitionistic anti-fuzzy subgroup of a finite group

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Abstract: The concept of implication-based intuitionistic anti-fuzzy subgroup and implication-based intutitionistic anti-fuzzy normal subgroup of a group are introduced using the notion of implication-based anti-fuzzy subgroup. The internal product of implication-based intuitionistic anti-fuzzy subgroups is developed. Few fundamental properties concerning them are proved.
Keywords: Implication-based anti-fuzzy subgroup, Implication-based intuitionistic anti-fuzzy subgroup, Internal product of implication-based intuitionistic anti-fuzzy subgroups.
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1 Introduction

In 1965, the concept of fuzzy set was first put forth by Zadeh [11]. Later in 1969 the notion of fuzzy automata was developed by Wee [6]. In 1971, Rosenfeld [4] applied the concept of fuzzy sets introduced by Zadeh to groups and constituted the elementary theory of groupoids and groups. From then on many research works had been carried out on their algebraic structures. In 1990, Biswas [3] introduced the concept of fuzzy subgroups and anti-fuzzy subgroups. In 1983, Atanassov [1, 2] generalised the concept of fuzzy set given by Zadeh [11] and introduced the notion of intuitionistic fuzzy set. Li Xiaoping [7] introduced the concept of intuitionistic fuzzy subgroup and intuitionistic fuzzy normal subgroup. In 2003, Implication-based fuzzy subgroup was developed by Yuan [10]. In 2015, We [5] studied about implication-based fuzzy normal subgroup over a finite group. In this paper, further research has been done and have introduced the concept of implication-based intuitionistic anti-fuzzy subgroup, inplication-based intuitionitistic anti-fuzzy normal subgroup and also the internal product of implication-based intuitionistic anti-fuzzy subgroups. We also proved few properties concerning them.

2 Preliminaries

Definition 2.1. [4] Let $(G, \cdot)$ be a group. Let a fuzzy set in G be a function A from G to $[0, 1]$. A will be called a fuzzy subgroup of G, if it satisfies the following conditions, that is for all x_1, x_2 in G, $A(x_1x_2) \geq min\big(A(x_1), A(x_2)\big)$ and $A({x_1}^{-1}) \geq A(x_1)$.

Definition 2.2. [3] Let G be a group. A fuzzy subset μ of G is called an anti-fuzzy subgroup of G if for $x, y \in G$ $\mu(xy) \leq \mu(x)\mu(y)$ and $\mu(x^{-1}) \leq \mu(x)$.

Definition 2.3. [2] Let X be a nonempty classical set. The traid formed as
$A = \{\langle x, \mu_A(x),\, \nu_A(x)\rangle | x \in X\}$ on X is called an intuitionistic fuzzy set on X, where the functions $\mu_A : A \to (0, 1)$ and $\nu_A : A \to (0, 1)$ denotes the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$

Definition 2.4. [8] Let G be a classical group, the intuitionistic fuzzy subset
$A = \{\langle x, \mu_A(x),\, \nu_A(x)\rangle | x \in G\}$ is called an intuitionistic fuzzy group on G, if the following conditions are satisfied.

$$\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\},\ \nu_A(xy) \leq max\{\nu_A(x), \nu_A(y)\}, \text{ for all } x, y \in G,$$
$$\mu_A(x^{-1}) \geq \mu_A(x),\ \nu_A(x^{-1}) \leq \nu_A(x), \text{ for all } x \in G.$$

Definition 2.5. [7] Let G be a classical group, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$ be an intuitionistic fuzzy set on G, then A is called intuitionistic fuzzy normal subgroup on G if $\mu_A(xyx^{-1}) \geq \mu_A(y)$, $\nu_A(xyx^{-1}) \leq \nu_A(y)$ for all $x, y \in G$.

Let X be an universe of discourse and $(G, \cdot)$ be a group. In fuzzy logic, $[\alpha]$ is used to denote the truth value of fuzzy proposition α. The fuzzy logical and the corresponding set theoretical notations used in this paper are:

$$(x \in A) = A(x);$$

$$(\alpha \wedge \beta) = min\{[\alpha], [\beta]\};$$

$$(\alpha \to \beta) = min\{1, 1 - [\alpha] + [\beta]\};$$

$$(\forall x\ \alpha(x)) = inf_{x \in X}[\alpha(x)];$$

$$(\exists x\ \alpha(x)) = sup_{x \in X}[\alpha(x)];$$

$\vDash \alpha$ if and only if $[\alpha] = 1$ for all valuations.

Definition 2.6. [10] If a fuzzy subset A of a group G satisfies for any $x_1, x_2 \in G$ $\models (x_1 \in A) \wedge (x_2 \in A) \rightarrow (x_1 x_2 \in A)$ and $\models (x_1 \in A) \rightarrow ({x_1}^{-1} \in A)$. Then A is called a fuzzifying subgroup. The concept of λ-tautology is $\models_\lambda (\alpha)$ if and only if $(\alpha) \geq \lambda$ for all valuation by Ying [9].

Definition 2.7. [10] Let A be a fuzzy subset of a finite group G and $\lambda \in (0, 1]$ is a fixed number. If for any $x_1, x_2 \in G$, $\models_\lambda (x_1 \in A) \wedge (x_2 \in A) \rightarrow (x_1 x_2 \in A)$ and $\models_\lambda (x_1 \in A) \rightarrow ({x_1}^{-1} \in A)$. Then A is called an implication-based fuzzy subgroup of G.

Definition 2.8. [5] An implication-based fuzzy subgroup A of G is called an implication-based fuzzy normal subgroup if $\models_\lambda (xy \in A) \rightarrow (yx \in A) \quad \forall x, y \in G$ where $\lambda \in (0, 1]$ is a fixed number.

Hereafter let G be a finite group with the identity element e.

3 Implication-based intuitionistic anti-fuzzy subgroup of a finite group

Definition 3.1. Let A be a fuzzysubset of a finite group G and $\lambda \in (0, 1]$ is a fixed number. If for any $x, y \in G$, $\models_\lambda (xy \in A) \rightarrow ((x \in A) \vee (y \in A))$ and $\models_\lambda (x^{-1} \in A) \rightarrow (x \in A)$. Then A is called an implication-based anti-fuzzy subgroup of G.

Definition 3.2. An intuitionistic fuzzy subset $\langle g, A, B \rangle$ of a group G is called as an implication-based intuitionistic fuzzy subgroup if it satisfies for any $x, y \in G$,

$$\models_\lambda (xy \in A) \rightarrow ((x \in A) \vee (y \in A)) \text{ and } \models_\lambda ((x \in B) \wedge (y \in B)) \rightarrow (xy \in B)$$
$$\models_\lambda (x^{-1} \in A) \rightarrow (x \in A) \text{ and } \models_\lambda (x \in B) \rightarrow (x^{-1} \in B),$$

where $(x \in A)$ denotes the degree of membership and $(x \in B)$ denoted the degree of non-membership and is denoted by $IAF_{\mathscr{A}} = \langle A, B \rangle$.

Example 3.1. Consider the group $G = \{e, a, b, c, d, f, g, h\}$ whose Cayley's Closure Table is given by

$\cdot$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	d	c	g	f	e	h	b
b	b	h	d	a	g	c	e	f
c	c	b	f	d	h	g	a	e
d	d	f	g	h	e	a	b	c
f	f	e	h	b	a	d	c	g
g	g	c	e	f	b	h	d	a
h	h	g	a	e	c	b	f	d

The membership function $A : G \to [0,1]$ is given by $A(e) = 0.025$, $A(a) = 0.115$, $A(b) = 0.150$, $A(c) = 0.345$, $A(d) = 0.425$, $A(f) = 0.575$, $A(g) = 0.625$ and $A(h) = 0.775$. The non-membership function $B : G \to [0,1]$is given by $B(e) = 0.815$, $B(a) = 0.750$, $B(b) = 0.645$, $B(c) = 0.535$, $B(d) = 0.475$, $B(f) = 0.315$, $B(g) = 0.250$ and $B(h) = 0.125$. The following table gives the $\vee$ of the elements of G.

$\vee$	$[e]$	$[a]$	$[b]$	$[c]$	$[d]$	$[f]$	$[g]$	$[h]$
$[e]$	0.025	0.115	0.150	0.345	0.425	0.575	0.625	0.775
$[a]$	0.115	0.115	0.150	0.345	0.425	0.575	0.625	0.775
$[b]$	0.150	0.150	0.150	0.345	0.425	0.575	0.625	0.775
$[c]$	0.345	0.345	0.345	0.345	0.425	0.575	0.625	0.775
$[d]$	0.425	0.425	0.425	0.425	0.425	0.575	0.625	0.775
$[f]$	0.575	0.575	0.575	0.575	0.575	0.575	0.625	0.775
$[g]$	0.625	0.625	0.625	0.625	0.625	0.625	0.625	0.775
$[h]$	0.775	0.775	0.775	0.775	0.775	0.775	0.775	0.775

The following table gives the $\wedge$ of the elements of G.

$\wedge$	$[e]$	$[a]$	$[b]$	$[c]$	$[d]$	$[f]$	$[g]$	$[h]$
$[e]$	0.815	0.750	0.645	0.535	0.475	0.315	0.250	0.125
$[a]$	0.750	0.750	0.645	0.535	0.475	0.315	0.250	0.125
$[b]$	0.645	0.645	0.645	0.535	0.475	0.315	0.250	0.125
$[c]$	0.535	0.535	0.535	0.535	0.475	0.315	0.250	0.125
$[d]$	0.475	0.475	0.475	0.475	0.475	0.315	0.250	0.125
$[f]$	0.315	0.315	0.315	0.315	0.315	0.315	0.250	0.125
$[g]$	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.125
$[h]$	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

With $\lambda = 0.200$ and the implication operator applied here is that of Kleene–Dienes, then $IAF_{\mathscr{A}} = \langle A, B \rangle$ is an implication-based intuitionistic anti-fuzzy subgroup.

Theorem 3.1. *An intuitionistic fuzzy subset $\mathscr{A} = \langle g, A, B \rangle$ of a group G is said to be an implication-based intuitionistic anti-fuzzy subgroup $IAF_{\mathscr{A}} = \langle A, B \rangle$ if and only if for all $x, y \in G$, $\models_\lambda (xy^{-1} \in A) \to ((x \in A) \vee (y \in A))$ and $\models_\lambda ((x \in B) \wedge (y \in B)) \to (xy^{-1} \in B)$.*

Proof. Assume that the intuitionistic fuzzy subset $\mathscr{A} = \langle g, A, B \rangle$ of a group G be an implication-based intuitionistic anti-fuzzy subgroup $IAF_{\mathscr{A}} = \langle A, B \rangle$.

Let $x, y \in G$

$$\begin{aligned}\models_\lambda (xy^{-1} \in A) &\to ((x \in A) \vee (y^{-1} \in A)) \\ &\to ((x \in A) \vee (y \in A)) \text{ since } \models_\lambda (x^{-1} \in A) \\ &\to (x \in A)\end{aligned}$$

$$\begin{aligned}\models_\lambda ((x \in B) \wedge (y \in B)) &\to ((x \in B) \wedge (y^{-1} \in B)) \text{ since } \models_\lambda (x \in B) \to (x^{-1} \in B) \\ &\to (xy^{-1} \in B)\end{aligned}$$

Conversely, let for all $x, y \in G$

$$\models_\lambda (xy^{-1} \in A) \to ((x \in A) \vee (y \in A)) \quad (1)$$

$\models_\lambda ((x \in B) \wedge (y \in B)) \to (xy^{-1} \in B)$ (2)

Let $y = x$ in equation (1).

$\models_\lambda (xx^{-1} \in A) \to (x \in A) \vee (x \in A))$

$\Rightarrow \models_\lambda (e \in A) \to (x \in A)$

$\models_\lambda (y^{-1} \in A) \to (ey^{-1} \in A)$

$\to ((e \in A) \vee (y \in A))$

$\to (y \in A)$

(i.e) $\models_\lambda (y^{-1} \in A) \to (y \in A) \models_\lambda (xy \in A) \to (x(y^{-1})^{-1} \in A)$

$\to (x \in A) \vee (y^{-1} \in A))$

$\to ((x \in A) \vee (y \in A))$

Let $y = x$ in equation (2).

$\models_\lambda ((x \in B) \wedge (x \in B)) \to (xx^{-1} \in B)$

$\Rightarrow \models_\lambda (x \in B) \to (e \in B)$

$\models_\lambda (y \in B) \to ((e \in B) \wedge (y \in B))$

$\to (ey^{-1} \in B)$

$\to (y^{-1} \in B)$

(i.e) $\models_\lambda (y \in B) \to (y^{-1} \in B)$

$\models_\lambda ((x \in B) \wedge (y \in B)) \to ((x \in B) \wedge (y^{-1} \in B))$

$\to (x(y^{-1})^{-1} \in B)$

$\to (xy \in B)$

$\therefore IAF_{\mathscr{A}} = \langle A, B\rangle$ is an implication-based intuitionistic anti-fuzzy subgroup of G. □

Theorem 3.2. *Let $IAF_{\mathscr{A}} = \langle A, B\rangle$ be an implication-based intuitionistic anti-fuzzy subgroup of a group G then for all $x \in G$ we have $\models_\lambda (e \in A) \to (x \in A)$ and $\models_\lambda (x \in B) \to (e \in B)$.*

Proof. Let $x \in G$ then

$\models_\lambda (e \in A) \to (xx^{-1} \in A)$

$\to ((x \in A) \vee (x^{-1} \in A))$

$\to ((x \in A) \vee (x \in A))$

$\to (x \in A)$

Therefore $\models_\lambda (e \in A) \to (x \in A)$

$\models_\lambda (x \in B) \to ((x \in B) \wedge (x^{-1} \in B))$

$\to (xx^{-1} \in B)$

$\to (e \in B)$

Therefore $\models_\lambda (x \in B) \to (e \in B)$. □

Definition 3.3. An implication-based intuitionistic anti-fuzzy subgroup of G, $IAF_{\mathscr{A}} = \langle A, B\rangle$ is said to be an implication-based intuitionistic anti-fuzzy normal subgroup of G if for all $x, y \in G$, $\models_\lambda (xy \in A) \to (yx \in A)$ and $\models_\lambda (xy \in B) \to (yx \in B)$

Theorem 3.3. *Let $IAF_{\mathscr{A}_1} = \langle A_1, B_1\rangle$ and $IAF_{\mathscr{A}_2} = \langle A_2, B_2\rangle$ be two implication-based intuitiontistic anti-fuzzy subgroups over a group G. Then $IAF_{\mathscr{A}_1\cap\mathscr{A}_2} = \langle A_1 \vee A_2, B_1 \wedge B_2\rangle$ is also an implication-based intuitionistic anti-fuzzy subgroup over the group G.*

Proof. Let $x, y \in G$

$\vDash_\lambda (xy^{-1} \in A_1 \vee A_2)$

$\rightarrow ((xy^{-1} \in A_1) \vee (xy^{-1} \in A_2))$

Since $IAF_{\mathscr{A}_1}$ and $IAF_{\mathscr{A}_2}$ are implication-based intuitionistic anti-fuzzy subgroups

$\rightarrow (((x \in A_1) \vee (y \in A_1)) \vee ((x \in A_2) \vee (y \in A_2)))$

$\rightarrow (((x \in A_1) \vee (x \in A_2)) \vee ((y \in A_1) \vee (y \in A_2)))$

$\rightarrow ((x \in A_1 \vee A_2) \vee (y \in A_1 \vee A_2))$

$\vDash_\lambda ((x \in B_1 \wedge B_2) \wedge (y \in B_1 \wedge B_2))$

$\rightarrow (((x \in B_1) \wedge (x \in B_2)) \wedge ((y \in B_1) \wedge (y \in B_2)))$

$\rightarrow (((x \in B_1) \wedge (y \in B_1)) \wedge ((x \in B_2) \wedge (y \in B_2)))$

Since $IAF_{\mathscr{A}_1}$ and $IAF_{\mathscr{A}_2}$ are implication-based intuitionistic fuzzy subgroups

$\rightarrow ((xy^{-1} \in B_1) \wedge (xy^{-1} \in B_2))$

$\rightarrow (xy^{-1} \in B_1 \wedge B_2)$

By Theorem 3.1, $IAF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is an implication-based intuitionistic anti-fuzzy subgroup over a group G. □

Theorem 3.4. *If* $IAF_{\mathscr{A}_1} = \langle A_1, B_1 \rangle$ *and* $IAF_{\mathscr{A}_2} = \langle A_2, B_2 \rangle$ *are two implication-based intuitiontistic ant-fuzzy normal subgroups over a group* G *then* $IAF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ *is also an implication-based intuitionistic anti-fuzzy normal subgroup of* G.

Proof. By Theorem 3.3, $IAF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is an implication-based intuitionistic anti-fuzzy subgroup over a group G.

Let $x, y \in G$

$\vDash_\lambda (xy \in A_1 \vee A_2) \rightarrow ((xy \in A_1) \vee (xy \in A_2))$

$\rightarrow ((yx \in A_1) \vee (yx \in A_2))$

Since $IAF_{\mathscr{A}_1}$ and $IAF_{\mathscr{A}_2}$ are implication-based intuitionistic anti-fuzzy normal subgroups

$\rightarrow (yx \in A_1 \vee A_2)$

$\vDash_\lambda (xy \in B_1 \wedge B_2) \rightarrow ((xy \in B_1) \wedge (xy \in B_2))$

$\rightarrow ((yx \in B_1) \wedge (yx \in B_2))$

Since $IAF_{\mathscr{A}_1}$ and $IAF_{\mathscr{A}_2}$ are implication-based intuitionistic anti-fuzzy normal subgroups

$\rightarrow (yx \in B_1 \wedge B_2)$

Thus $IAF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is an implication-based intuitionistic anti-fuzzy normal subgroup over a group G. □

Definition 3.4. Let $IAF_{\mathscr{A}_i} = \langle A_i, B_i \rangle$ be a family of implication-based intuitionistic anti-fuzzy normal subgroups over a group G. Then we define $IAF_{\cap \mathscr{A}_j} = \langle \exists j A_j, \forall j B_j \rangle$, by previous theorem $IF_{\cap \mathscr{A}_j}$ is also an implication-based intuitionistic anti-fuzzy normal subgroup over the group G.

Theorem 3.5. *Let* $IAF_{\mathscr{A}} = \langle A, B \rangle$ *be an implication-based intuitionistic anti-fuzzy subgroup of a finite group* G. *Then* $IAF_{\mathscr{A}}$ *is an implication-based intuitionistic anti-fuzzy normal subgroup*

of G if and only if $\models_\lambda (x \in A) \to (y^{-1}xy \in A)$ *and*
$\models_\lambda (y^{-1}xy \in B) \to (x \in B)$ *for all* $x, y \in G$

Proof. Let $IAF_{\mathscr{A}}$ be an implication-based intuitionistic anti-fuzzy normal subgroup of G.
Let $x, y \in G$.
$\models_\lambda (x \in A) \to (xy.y^{-1} \in A) \to (y^{-1}xy \in A)$
$\models_\lambda (y^{-1}xy \in B) \to (xy.y^{-1} \in B) \to (x \in B)$
Conversely assume that
$\models_\lambda (x \in A) \to (y^{-1}xy \in A)$
$\models_\lambda (y^{-1}xy \in B) \to (x \in B)$ for all $x, y \in G$
In (i) put $x = yx$ we get
$\models_\lambda (x \in A) \to (y^{-1}xy \in A)$
$\Rightarrow \models_\lambda (yx \in A) \to (y^{-1}.yx.y \in A)$
$\Rightarrow \models_\lambda (yx \in A) \to (y^{-1}y(x.y) \in A)$
$\Rightarrow \models_\lambda (yx \in A) \to (xy \in A)$
In (ii) put $x = yx$ we get
$\models_\lambda (y^{-1}xy \in B) \to (x \in B)$
$\Rightarrow \models_\lambda (y^{-1}(yx)y \in B) \to (yx \in B)$
$\Rightarrow \models_\lambda (y^{-1}y.xy \in B) \to (yx \in B)$
$\Rightarrow \models_\lambda (xy \in B) \to (yx \in B)$
Therefore $IAF_{\mathscr{A}}$ is an implication-based intuitionistic anti-fuzzy normal subgroup of G. □

Theorem 3.6. *Let* $IAF_{\mathscr{A}} = \langle A, B \rangle$ *be an implication-based intuitionistic anti-fuzzy subgroup of a finite group G. Let* $x \in G$ *then* $\models_\lambda (y \in A) \to (xy \in A)$ *and* $\models_\lambda (xy \in B) \to (y \in B)$ *if and only if* $\models_\lambda (e \in A) \to (x \in A)$ *and* $\models_\lambda (x \in B) \to (e \in B)$ *for all* $y \in G$.

Proof. Let $x \in G$ such that
$\models_\lambda (y \in A) \to (xy \in A)$ and $\models_\lambda (xy \in B) \to (y \in B)$ for all $y \in G$
Put $y = e$ we get
$\models_\lambda (e \in A) \to (xe \in A)$ and $\models_\lambda (xe \in B) \to (e \in B)$
(i.e) $\models_\lambda (e \in A) \to (x \in A)$ and $\models_\lambda (x \in B) \to (e \in B)$
Conversely assume that
$\models_\lambda (e \in A) \to (x \in A)$ and $\models_\lambda (x \in B) \to (e \in B)$ for all $x \in G$
Let $y \in G$
Since $\models_\lambda (e \in A) \to (x \in A)$ for all $x \in G$
This is true in particular for $y \in G$.
$\Rightarrow \models_\lambda (e \in A) \to (y \in A)$
Therefore
$\models_\lambda (x \in A) \to (y \in A)$ (1)
But $\models_\lambda (xy \in A) \to ((x \in A) \vee (y \in A)) \to (y \in A)$
$\Rightarrow \models_\lambda (xy \in A) \to (y \in A)$ by (1) (2)
Now $\models_\lambda (xy \in A) \to (y \in A)$ by (2)
$\to (x^{-1}.xy \in A)$

$\rightarrow ((x^{-1} \in A) \vee (xy \in A))$
$\rightarrow ((x \in A) \vee (xy \in A))$
$\vDash_\lambda ((x \in A) \vee (y \in A)) \rightarrow ((x \in A) \vee (xy \in A))$
(i.e) $\vDash_\lambda (y \in A) \rightarrow (xy \in A)$
Now $\vDash_\lambda (x \in B) \rightarrow (e \in B)$ for all $x \in G$
This is true in particular for $y \in G$
Therefore $\vDash_\lambda (y \in B) \rightarrow (e \in B)$
$\Rightarrow \ \vDash_\lambda (y \in B) \rightarrow (x \in B)$
But $\vDash_\lambda (x \in B) \wedge (y \in B) \rightarrow (xy \in B)$
$\Rightarrow \ \vDash_\lambda (y \in B) \rightarrow (xy \in B)$
$\vDash_\lambda (xy \in B) \rightarrow ((x \in B) \wedge (xy \in B))$
$\rightarrow ((x^{-1} \in B) \wedge (xy \in B))$
$\rightarrow (x^{-1}.xy \in B)$
$\rightarrow (y \in B)$
Therefore $\vDash_\lambda (xy \in B) \rightarrow (y \in B)$ □

Definition 3.5. Let $IAF_{\mathscr{A}_1} = \langle A_1, B_1\rangle$ and $IAF_{\mathscr{A}_2} = \langle A_2, B_2\rangle$ be two implication-based intuitiontistic anti-fuzzy subgroups over a group G. Then the internal product $IAF_{\mathscr{A}_1\cdot\mathscr{A}_2} = \langle A_1 \cdot A_2, B_1 \cdot B_2\rangle$ is defined by for $x \in G$
$\vDash_\lambda (\forall y, z\{(y \in A_1) \vee (z \in A_2)\}; yz = x; y, z \in G) \rightarrow (x \in A_1 \cdot A_2)$
$\vDash_\lambda (\exists y, z\{(y \in B_1) \wedge (z \in B_2)\}; yz = x; y, z \in G) \rightarrow (x \in B_1 \cdot B_2)$

Theorem 3.7. *Let $IAF_{\mathscr{A}_1} = \langle A_1, B_1\rangle$ and $IAF_{\mathscr{A}_2} = \langle A_2, B_2\rangle$ be two implication-based intuitiontistic anti-fuzzy subgroups over a group G then $IAF_{\mathscr{A}_1\cdot\mathscr{A}_2} = IAF_{\mathscr{A}_2\cdot\mathscr{A}_1}$*

Proof. Let $x \in G$
$\vDash_\lambda (x \in A_1 \cdot A_2) \rightarrow (\forall l, m\{(l \in A_1) \vee (m \in A_2)\}; lm = x; l, m \in G)$
$\rightarrow (\forall l, m\{(m \in A_2) \vee (l \in A_1)\}; lm = x; l, m \in G)$
$\rightarrow (\forall l, m\{(m \in A_2) \vee (m^{-1}ml \in A_1)\}; m(m^{-1}lm) = x; l, m \in G)$
$\rightarrow (x \in A_2 \cdot A_1)$
$\vDash_\lambda (x \in B_1 \cdot B_2) \rightarrow (\exists p, q\{(p \in B_1) \wedge (q \in B_2)\}; pq = x; p, q \in G)$
$\rightarrow (\exists p, q\{(q \in B_2) \wedge (p \in B_1)\}; pq = x; p, q \in G)$
$\rightarrow (\exists p, q\{(q \in B_2) \wedge (q^{-1}qp \in B_1)\}; q(q^{-1}pq) = x; p, q \in G)$
$\rightarrow (x \in B_2 \cdot B_1)$
Therefore $\vDash_\lambda (x \in A_1 \cdot A_2) \rightarrow (x \in A_2 \cdot A_1)$ and $\vDash_\lambda (x \in B_1 \cdot B_2) \rightarrow (x \in B_2 \cdot B_1)$
Similarly we can prove that
$\vDash_\lambda (x \in A_2 \cdot A_1) \rightarrow (x \in A_1 \cdot A_2)$ and $\vDash_\lambda (x \in B_2 \cdot B_1) \rightarrow (x \in B_1 \cdot B_2)$
$\Rightarrow \ IAF_{\mathscr{A}_1\cdot\mathscr{A}_2} = IAF_{\mathscr{A}_2\cdot\mathscr{A}_1}$ □

Theorem 3.8. *Let $IAF_{\mathscr{A}_1} = \langle A_1, B_1\rangle$ and $IAF_{\mathscr{A}_2} = \langle A_2, B_2\rangle$ be two implication-based intuitiontistic anti-fuzzy subgroups over a group G then $IAF_{\mathscr{A}_1\cdot\mathscr{A}_2}$ is also an implication-based intuitiontistic anti-fuzzy subgroup of G. If $IAF_{\mathscr{A}_1} = \langle A_1, B_1\rangle$ and $IAF_{\mathscr{A}_2} = \langle A_2, B_2\rangle$ are two implication-based intuitiontistic anti-fuzzy normal subgroups then $IAF_{\mathscr{A}_1\cdot\mathscr{A}_2}$ is also an implication-based intuitiontistic anti-fuzzy normal subgroup of G.*

Proof. Let $x, y \in G$

$\vDash_\lambda (xy \in A_1 \cdot A_2) \to (\forall u, v, l, m\{(ul \in A_1) \vee (vm \in A_2)\}; uvlm = xy; u, v, l, m \in G)$

$\to (\forall u, v, l, m\{((u \in A_1) \vee (l \in A_1)) \vee ((v \in A_2) \vee (m \in A_2))\}; uvlm = xy; u, v, l, m \in G)$

$\to (\forall u, v, l, m\{((u \in A_1) \vee (v \in A_2)) \vee ((l \in A_1) \vee (m \in A_2))\}; uvlm = xy; u, v, l, m \in G)$

$\to (\forall l, m\{\forall u, v\{(u \in A_1) \vee (v \in A_2) \vee (l \in A_1) \vee (m \in A_2)\};$
$uv = x; u, v \in G\}; lm = y; l, m \in G)$

$\to ((\forall u, v\{(u \in A_1) \vee (v \in A_2)\}; uv = x; u, v \in G) \vee$
$(\forall l, m\{(l \in A_1) \vee (m \in A_2)\}; lm = y; l, m \in G))$

$\to ((x \in A_1 \cdot A_2) \vee (y \in A_1 \cdot A_2))$

$\vDash_\lambda (x^{-1} \in A_1 \cdot A_2) \to (x^{-1} \in A_2 \cdot A_1)$

$\to (\forall u^{-1}, v^{-1}\{(v^{-1} \in A_2) \vee (u^{-1} \in A_1)\}; v^{-1}u^{-1} = x^{-1}; u, v \in G)$

$\to (\forall u^{-1}, v^{-1}\{(v \in A_2) \vee (u \in A_1)\}; uv = x; u, v \in G)$

$\to (\forall u, v\{(u \in A_1) \vee (v \in A_2)\}; uv = x; u, v \in G)$

$\to (x \in A_1 \cdot A_2)$

$\vDash_\lambda ((x \in B_1 \cdot B_2) \wedge (y \in B_1 \cdot B_2))$

$\to ((\exists u, v\{(u \in B_1) \wedge (v \in B_2)\}; uv = x; u, v \in G) \wedge$
$(\exists l, m\{(l \in B_1) \wedge (m \in B_2)\}; lm = y; l, m \in G))$

$\to (\exists l, m\{\exists u, v\{(u \in B_1) \wedge (v \in B_2) \wedge (l \in B_1) \wedge (m \in B_2)\};$
$uv = x; u, v \in G\}; lm = y; l, m \in G)$

$\to (\exists u, v, l, m\{(ul \in B_1) \wedge (vm \in B_2)\}; uvlm = xy; u, v, l, m \in G)$

$\to (xy \in B_1 \cdot B_2)$

$\vDash_\lambda (x \in B_1 \cdot B_2)$

$\to (\exists u, v\{(u \in B_1) \wedge (v \in B_2)\}; uv = x; u, v \in G)$

$\to (\exists u, v\{(v \in B_2) \wedge (u \in B_1)\}; uv = x; u, v \in G)$

$\to (\exists u, v\{(v^{-1} \in B_2) \wedge (u^{-1} \in B_1)\}; v^{-1}u^{-1} = x^{-1}; u, v \in G)$

$\to (\exists u^{-1}, v^{-1}\{(v^{-1} \in B_2) \wedge (u^{-1} \in B_1)\}; v^{-1}u^{-1} = x^{-1}; u, v \in G)$

$\to (x^{-1} \in B_2 \cdot B_1)$

$\to (x^{-1} \in B_1 \cdot B_2)$

Therefore $IAF_{\mathscr{A}_1 \cdot \mathscr{A}_2}$ is an implication-based intuitiontistic anti-fuzzy subgroup of G.

$\vDash_\lambda (x \in A_1 \cdot A_2)$

$\to (\forall u, v\{(u \in A_1) \vee (v \in A_2)\}; uv = x; u, v \in G)$

$\to (\forall u, v\{(y^{-1}uy \in A_1) \vee (y^{-1}vy \in A_2)\}; (y^{-1}uy).(y^{-1}vy) = y^{-1}(uv)y = y^{-1}xy; u, v \in G)$

$\to (\forall y^{-1}uy, y^{-1}vy\{(y^{-1}uy \in A_1) \vee (y^{-1}vy \in A_2)\}; (y^{-1}uy).(y^{-1}vy) = y^{-1}xy; u, v \in G)$

$\to (y^{-1}xy \in A_1 \cdot A_2)$

$\vDash_\lambda (y^{-1}xy \in B_1 \cdot B_2)$

$\to (\exists u, v\{(u \in B_1) \wedge (v \in B_2)\}; uv = y^{-1}xy; u, v \in G)$

$\to (\exists yuy^{-1}, yvy^{-1}\{(yuy^{-1} \in B_1) \wedge (yvy^{-1} \in B_2)\};$
$(yuy^{-1})(yvy^{-1}) = y(uv)y^{-1} = y(y^{-1}xy)y^{-1} = x; u, v \in G)$

$\to (x \in B_1 \cdot B_2)$

$\therefore IAF_{\mathscr{A}_1 \cdot \mathscr{A}_2}$ is an implication-based intuitiontistic anti-fuzzy normal subgroup of G. □

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